© Journal of Applied Mathematics & Decision Sciences, 2(2), 147–158 (1998)
Reprints Available directly from the Editor. Printed in New Zealand.

ON THE EQUIPONDERATE EQUATION $x^a + x^b + x = x^c + x^d + 1$ AND A REPRESENTATION OF WEIGHT QUADRUPLETS

BERNARD DE BAETS AND HANS DE MEYER
Department of Applied Mathematics and Computer Science
University of Gent, Krijgslaan 281 (S9), B-9000 Gent, Belgium

Abstract. The equation $x^a + x^b + x = x^c + x^d + 1$ considered in this paper is a particular equiponderate equation. The number and location of the roots (w.r.t. $x = 1$) of this equation are determined in case $(a, b, c, d) \in]0, 1[^4$. Based on these results, it is shown that any weight quadruplet, a basic tool in fuzzy preference modelling, admits an interesting expression in terms of Frank t-norms with reciprocal parameters.

Keywords: Equiponderate equation, Frank t-norm, Weight quadruplet.

1. Introduction

In this paper, we are concerned with an equation of the type

$$\sum_{i=1}^{n} x^{a_i} - \sum_{j=1}^{m} x^{b_j} = 0,$$

where $(a_i)_{i=1}^n$ and $(b_j)_{j=1}^m$ are weights in $[0, 1]$ satisfying

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{m} b_j,$$

whence the name *equiponderate*. We consider the particular case where $n = m = 3$, $a_3 = 1$ and $b_3 = 0$. In other words, we are considering the equation

$$x^a + x^b + x = x^c + x^d + 1$$

constrained by $a + b + 1 = c + d$. This equation is clearly equivalent to the equation

$$x^{a'} + x^{b'} + x = x^{1-c'} + x^{1-d'} + 1,$$

constrained by $a' + b' + c' + d' = 1$. The first formulation, however, seems to be more elegant. We will show that this equation plays an important part in developing a representation of *weight quadruplets* in terms of Frank t-norms. A weight quadruplet is a quadruplet $(a, b, c, d) \in [0, 1]^4$ satisfying $a + b + c + d = 1$. Such quadruplets are the basic building blocks in fuzzy preference modelling.

2. The equaton $x^a + x^b + x = x^c + x^d + 1$

In this section, we consider the equation $x^a + x^b + x = x^c + x^d + 1$ on $]0, +\infty[$. Obviously, $x = 1$ is always a root of this equation. Next, we investigate the cases in which this equation has a unique solution in $]0, 1[\cup]1, +\infty[$.

THEOREM 1 *Consider $(a, b, c, d) \in]0, 1[^4$ such that $a + b + 1 = c + d$. The roots of the equation*

$$x^a + x^b + x = x^c + x^d + 1$$

in $]0, +\infty[$ are located as follows:

(i) *if $a^2 + b^2 + 1 = c^2 + d^2$: a triple root $x = 1$;*

(ii) *if $a^2 + b^2 + 1 > c^2 + d^2$: a double root $x = 1$ and a single root in $]0, 1[$;*

(iii) *if $a^2 + b^2 + 1 < c^2 + d^2$: a double root $x = 1$ and a single root in $]1, +\infty[$.*

Proof: We deliver the proof in a number of steps.

- Consider the $[0, +\infty] \to [0, +\infty]$ mappings f and g defined by

$$f(x) = x^a + x^b + x$$
$$g(x) = x^c + x^d + 1.$$

It trivially holds that $f(0) = 0$ and $g(0) = 1$, and hence $f(0) - g(0) = -1$. On the other hand, one easily verifies that

$$\lim_{x \to +\infty} (f(x) - g(x)) = +\infty,$$

since $(a, b, c, d) \in]0, 1[^4$. Consequently, the equation $f(x) - g(x) = 0$ has an odd number of roots in $]0, +\infty[$, taking into account the multiplicity of the roots.

- The root $x = 1$ has at least multiplicity equal to 2. Indeed,

$$f(1) - g(1) = 3 - 3 = 0$$
$$(f'(x) - g'(x))\,|_{x=1} = \left(ax^{a-1} + bx^{b-1} + 1 - cx^{c-1} - dx^{d-1}\right)|_{x=1}$$
$$= a + b + 1 - c - d = 0.$$

Hence, the equation $f(x) - g(x) = 0$ has at least three roots, always taking into account the multiplicity.

- Without loss of generality, we can assume that $a \geq b$ and $c \geq d$. From $(a, b, c, d) \in]0, 1[^4$ and $a + b + 1 = c + d$ it follows that

$$0 < a + b < 1 < c + d < 2.$$

Next, we consider the 6 possible orderings of a, b, c and d.

(i) The ordering $a \geq b \geq c \geq d$. This case cannot occur since it would imply that $a + b \geq c + d$.

(ii) The ordering $a \geq c \geq b \geq d$. Similar to (i).

(iii) The ordering $a \geq c \geq d \geq b$. It then follows that $2a \geq c + d \geq 2b$, and hence $2a \geq a + b + 1 \geq 2b$. The latter implies that $a \geq b + 1$ which is impossible in view of $(a, b) \in]0, 1[^2$.

(iv) The ordering $c \geq a \geq b \geq d$. It then follows that $2c \geq a + b \geq 2d$, and hence $2c \geq c + d - 1 \geq 2d$. The latter implies that $d \leq c - 1$ which is impossible in view of $(c, d) \in]0, 1[^2$.

(v) The ordering $c \geq a \geq d \geq b$. In this case, it follows that $a + c \geq d + c$, i.e. $a + c \geq a + b + 1$, and hence $c \geq b + 1$. This again contradicts $(b, c) \in]0, 1[^2$.

(vi) The ordering $c \geq d \geq a \geq b$. This case is only possible when $d > a$. Indeed, the assumption $a = d$ leads to $a + c = d + c = a + b + 1$, and therefore $c = b + 1$, again a contradiction.

We conclude that the only possible ordering is given by:

$$c \geq d > a \geq b.$$

- We will now show that the equation $f(x) - g(x) = 0$ has exactly three roots. Assume that it has at least 5 roots in $]0, +\infty[$, then we want to obtain a contradiction. According to Rolle's theorem, the equation $f'(x) - g'(x) = 0$ then has at least 4 roots in $]0, +\infty[$ and the equation $f''(x) - g''(x) = 0$ has at least 3 roots in $]0, +\infty[$. The roots of the equation $f''(x) - g''(x) = 0$ in $]0, +\infty[$ coincide with the roots of

$$x^{2-b} \left(f''(x) - g''(x)\right) = 0,$$

i.e.

$$a(a-1)x^{a-b} + b(b-1) - c(c-1)x^{c-b} - d(d-1)x^{d-b} = 0.$$

Again according to Rolle's theorem, its derivative

$$a(a-1)(a-b)x^{a-b-1} - c(c-1)(c-b)x^{c-b-1} - d(d-1)(d-b)x^{d-b-1} = 0$$

has at least 2 roots. The roots of this equation in $]0, +\infty[$ coincide with the roots of

$$a(a-1)(a-b) - c(c-1)(c-b)x^{c-a} - d(d-1)(d-b)x^{d-a} = 0.$$

Its derivate then has at least one root in $]0, +\infty[$:

$$c(c-1)(c-b)(c-a)x^{c-a-1} + d(d-1)(d-b)(d-a)x^{d-a-1} = 0.$$

This is the longawaited contradiction. Indeed, from $c \geq d > a \geq b$ it follows that

$$c(c-1)(c-b)(c-a) < 0$$
$$d(d-1)(d-b)(d-a) < 0,$$

and hence, for any $x \in]0,+\infty[$:

$$c(c-1)(c-b)(c-a)x^{c-a-1} + d(d-1)(d-b)(d-a)x^{d-a-1} < 0.$$

- Finally, we determine the location of the third root. First, $x = 1$ is a triple root if

$$(f''(x) - g''(x))\,|_{x=1} = 0,$$

i.e. if

$$a(a-1) + b(b-1) = c(c-1) + d(d-1),$$

or also, since $a + b + 1 = c + d$, if

$$a^2 + b^2 + 1 = c^2 + d^2.$$

Hence, if $a^2 + b^2 + 1 \neq c^2 + d^2$, then there is a third root in $]0,+\infty[$. If $a^2 + b^2 + 1 > c^2 + d^2$, then we show that $f(1+\epsilon) > g(1+\epsilon)$, and hence the third root lies in $]0,1[$. Indeed, we can elaborate $f(1+\epsilon) - g(1+\epsilon) =$

$$\begin{aligned}
&= (1+\epsilon)^a + (1+\epsilon)^b + 1 + \epsilon - (1+\epsilon)^c - (1+\epsilon)^d - 1 \\
&\approx \left(1 + a\epsilon + \tfrac{a(a-1)}{2}\epsilon^2\right) + \left(1 + b\epsilon + \tfrac{b(b-1)}{2}\epsilon^2\right) + 1 + \epsilon \\
&\quad - \left(1 + c\epsilon + \tfrac{c(c-1)}{2}\epsilon^2\right) - \left(1 + d\epsilon + \tfrac{d(d-1)}{2}\epsilon^2\right) - 1 \\
&= (a + b + 1 - c - d)\epsilon + (a(a-1) + b(b-1) - c(c-1) - d(d-1))\tfrac{\epsilon^2}{2} \\
&= (a^2 + b^2 + 1 - c^2 - d^2)\tfrac{\epsilon^2}{2}.
\end{aligned}$$

Hence, it effectively holds that $f(1+\epsilon) - g(1+\epsilon) > 0$. Similarly, if $a^2 + b^2 + 1 < c^2 + d^2$, then the third root lies in $]1,+\infty[$. □

In view of the next section, we reformulate the decisive condition $a^2 + b^2 + 1 = c^2 + d^2$.

LEMMA 1 *Consider $(a,b,c,d) \in [0,1]^4$ such that $a+b+1 = c+d$, then the following equivalence holds:*

$$a^2 + b^2 + 1 = c^2 + d^2 \quad - \quad ab = (1-c)(1-d).$$

Proof: By squaring both sides of the equality $a + b + 1 = c + d$, we obtain

$$a^2 + b^2 + 1 + 2ab + 2(a+b) = c^2 + d^2 + 2cd.$$

These observations justify the following notations: $\mathcal{T}^0$ for M, $\mathcal{T}^1$ for P and $\mathcal{T}^{+\infty}$ for W. These notations will be adopted in the sequel. Note also that we will identify $\mathcal{T}^{\frac{1}{+\infty}}$ and $\mathcal{T}^0$, and also $\mathcal{T}^{\frac{1}{0}}$ and $\mathcal{T}^{+\infty}$.

The t-norms $\mathcal{T}^s$ were discovered by Frank in his study of *copulas* in the framework of probabilistic metric spaces, in which he was confronted with the following problem: for which continuous, associative, binary operations F on $[0,1]$ satisfying the boundary conditions $(\forall x \in [0,1])(F(0,x) = F(x,0) = 0 \wedge F(1,x) = F(x,1) = x)$, is the binary operation G on $[0,1]$ defined by

$$G(x,y) = x + y - F(x,y)$$

also associative? Frank [5] showed that the only solutions to this problem are *ordinal sums* (see e.g. [8]) whose summands are continuous Archimedean t-norms belonging to the family $(\mathcal{T}^s)_{s\in\,]0,+\infty]}$. The family $(\mathcal{T}^s)_{s\in[0,+\infty]}$ is called the Frank t-norm family. The dual t-conorm $(\mathcal{T}^s)^*$ is usually denoted by $\mathcal{S}^s$. In particular, it then holds that

$$\mathcal{T}^s(x,y) + \mathcal{S}^s(x,y) = x + y,$$

for any $s \in [0,+\infty]$ and any $(x,y) \in [0,1]^2$.

4. Weight quadruplets

In this section, we show that the Frank t-norm family allows us to easily construct weight quadruplets. Let us formally define a weight quadruplet.

DEFINITION 2 *A weight quadruplet is a quadruplet* $(a,b,c,d) \in [0,1]^4$ *such that*

$$a + b + c + d = 1.$$

Weight quadruplets are used, for instance, in fuzzy preference modelling for expressing the pairwise evaluation of a set of alternatives by a decision maker [1], [9]. Roughly speaking, each pair of alternatives (A,B) is described by a weight quadruplet (p,p',i,j), where p is the degree to which alternative A is strictly preferred to alternative B, p' is the degree to which B is strictly preferred to A, i is the degree to which A and B are indifferent and j is the degree to which A and B are incomparable.

In order to demonstrate the construction procedure of weight quadruplets based on Frank t-norms, we first prove the following proposition connecting Frank t-norms with reciprocal parameters.

PROPOSITION 2 *Consider* $s \in [0,+\infty]$ *and* $(u,v) \in [0,1]^2$. *Then the following equality holds:*

$$\mathcal{T}^s(u,1-v) + \mathcal{T}^{1/s}(u,v) = u.$$

Proof: We first consider the cases $s = 0$, $s = 1$ and $s = +\infty$ separately.

(i) The case $s = 0$. Then

$$\mathcal{T}^0(u, 1 - v) + \mathcal{T}^{+\infty}(u, v) = \min(u, 1 - v) + \max(0, u + v - 1).$$

One easily verifies by considering the cases $u \leq 1 - v$ and $u > 1 - v$ that always $\min(u, 1 - v) + \max(0, u + v - 1) = u$.

(ii) The case $s = 1$. Then

$$\mathcal{T}^1(u, 1 - v) + \mathcal{T}^1(u, v) = u(1 - v) + uv = u.$$

(iii) The case $s = +\infty$. Similar to (i).

(iv) The case $s \in]0, 1[\cup]1, +\infty[$. In this case, the equality

$$\mathcal{T}^s(u, 1 - v) + \mathcal{T}^{1/s}(u, v) = u$$

reads explicitly

$$\log_s \left(1 + \tfrac{(s^u - 1)(s^{1-v} - 1)}{s - 1}\right) + \log_{1/s} \left(1 + \tfrac{((1/s)^u - 1)((1/s)^v - 1)}{(1/s) - 1}\right) = u,$$

and can be rewritten subsequently as

$$\log_s \left(1 + \tfrac{(s^u - 1)(s^{1-v} - 1)}{s - 1}\right) + \log_{1/s} \left(1 - \tfrac{s(s^{-u} - 1)(s^{-v} - 1)}{s - 1}\right) = u$$

and

$$\log_s \left(\tfrac{(s-1) + (s^u - 1)(s^{1-v} - 1)}{s - 1}\right) - \log_s \left(\tfrac{(s-1) - s(s^{-u} - 1)(s^{-v} - 1)}{s - 1}\right) = \log_s s^u.$$

The latter is equivalent to

$$\tfrac{(s-1) + (s^u - 1)(s^{1-v} - 1)}{(s-1) - s(s^{-u} - 1)(s^{-v} - 1)} = s^u,$$

which can be verified easily. □

The following theorem is an important result in fuzzy preference modelling. The first part of it (the construction of weight quadruplets) is a corollary of more general results proven in [3], [4], the second part (the additional properties) is proven in [9]. For both parts, we provide here a direct, more transparant alternative proof. The basic inspiration behind this theorem is the observation that the formula

$$(p \wedge \neg q) \vee (\neg p \wedge q) \vee (p \wedge q) \vee (\neg p \wedge \neg q)$$

is a tautology in classical logic.

THEOREM 2 *Consider* $s \in [0, +\infty]$ *and* $(u, v) \in [0, 1]^2$*, then the quadruplet*

$$(a, b, c, d) = (\mathcal{T}^{1/s}(u, 1-v), \mathcal{T}^{1/s}(1-u, v), \mathcal{T}^{s}(u, v), \mathcal{T}^{s}(1-u, 1-v)),$$

is a weight quadruplet. Moreover,

(i) *for* $s = 0$ *it holds that* $\min(a, b) = 0$*;*

(ii) *for* $s = +\infty$ *it holds that* $\min(c, d) = 0$*;*

(iii) *for* $s = 1$ *it holds that* $ab = cd$*;*

(iv) *for* $s \in]0, 1[\cup]1, +\infty[$ *it holds that* $s^a + s^b + s = s^{1-c} + s^{1-d} + 1$.

Proof: With Proposition 2 it follows that

$$a + c = \mathcal{T}^{1/s}(u, 1-v) + \mathcal{T}^{s}(u, v) = u,$$

and

$$b + d = \mathcal{T}^{1/s}(1-u, v) + \mathcal{T}^{s}(1-u, 1-v)) = 1 - u,$$

and hence $a+b+c+d = 1$. Similarly, it follows that $b+c = v$. We now demonstrate the additional properties:

(i) The case $s = 0$. Then it holds that

$$a = \mathcal{T}^{+\infty}(u, 1-v) = \max(u + 1 - v - 1, 0) = \max(u - v, 0),$$

and similarly,

$$b = \max(v - u, 0).$$

Hence, it clearly holds that $\min(a, b) = 0$.

(ii) The case $s = +\infty$. Similar to (i).

(iii) The case $s = 1$. It trivially holds that

$$ab = u(1-v)(1-u)v,$$

and

$$cd = uv(1-u)(1-v).$$

(iv) The case $s \in]0, 1[\cup]1, +\infty[$. Consider

$$c = \log_s \left(1 + \tfrac{(s^{a+c}-1)(s^{b+c}-1)}{s-1}\right),$$

then

$$s^c = 1 + \tfrac{(s^{a+c}-1)(s^{b+c}-1)}{s-1},$$

i.e.

$$(s^c - 1)(s - 1) = (s^{a+c} - 1)(s^{b+c} - 1),$$

and after multiplication of both sides by s^{-c}:

$$s - s^{1-c} - 1 = s^{a+b+c} - s^a - s^b.$$

Since $a + b + c + d = 1$, it then follows that

$$s^a + s^b + s = s^{1-c} + s^{1-d} + 1.$$

This concludes the proof. □

In fuzzy preference modelling [9], considerable attention is given to weight quadruplets satisfying the condition

$$s^{\mathcal{T}^s(a,b)} + s^{-\mathcal{T}^{1/s}(c,d)} = 2,$$

with $s \in]0,1[\cup]1,+\infty[$. The following proposition links this condition to our equiponderate equation.

PROPOSITION 3 *Consider a weight quadruplet* (a,b,c,d) *and* $s \in]0,1[\cup]1,+\infty[$, *then it holds that*

$$s^{\mathcal{T}^s(a,b)} + s^{-\mathcal{T}^{1/s}(c,d)} = 2$$

if and only if

$$s^a + s^b + s = s^{1-c} + s^{1-d} + 1.$$

Proof: The following chain of equivalences is easily established:

$$\begin{array}{ll} & s^{\mathcal{T}^s(a,b)} + s^{-\mathcal{T}^{1/s}(c,d)} = 2 \\ - & s^{\log_s\left(1+\frac{(s^a-1)(s^b-1)}{s-1}\right)} + s^{-\log_{1/s}\left(1+\frac{((1/s)^c-1)((1/s)^d-1)}{(1/s)-1}\right)} = 2 \\ - & 1 + \frac{(s^a-1)(s^b-1)}{s-1} + 1 - \frac{s(s^{-c}-1)(s^{-d}-1)}{s-1} = 2 \\ - & (s^a-1)(s^b-1) = s(s^{-c}-1)(s^{-d}-1) \\ - & s^{a+b} - s^a - s^b + 1 = s^{1-c-d} + s - s^{1-c} - s^{1-d} \\ - & s^a + s^b + s = s^{1-c} + s^{1-d} + 1. \end{array}$$

This concludes the proof. □

The following theorem is quite remarkable. It shows that the above construction technique is characteristic, i.e. any weight quadruplet can be generated in the above way. Again, this result can be found partially in [9], in a less transparant way. This theorem will have major implications in the field of fuzzy preference modelling.

THEOREM 3 *Consider a weight quadruplet* (a,b,c,d). *Let* $u = a + c$ *and* $v = b + c$, *then it holds that*

$$(a,b,c,d) = (\mathcal{T}^{1/s}(u,1-v), \mathcal{T}^{1/s}(1-u,v), \mathcal{T}^s(u,v), \mathcal{T}^s(1-u,1-v)),$$

where

(i) $s = 0$ *if* $\min(a, b) = 0$*;*

(ii) $s = +\infty$ *if* $\min(c, d) = 0$*;*

(iii) $s = 1$ *if* $ab = cd$*;*

(iv) s *is the unique solution in* $]0, 1[\cup]1, +\infty[$ *of the equation*

$$s^a + s^b + s = s^{1-c} + s^{1-d} + 1$$

if none of the above holds.

Proof: In view of Proposition 2 it is sufficient to prove that

$$(a, b) = (\mathcal{T}^{1/s}(u, 1 - v), \mathcal{T}^{1/s}(1 - u, v)).$$

Indeed, then it follows that

$$\mathcal{T}^s(u, v) = u - \mathcal{T}^{1/s}(u, 1 - v) = a + c - a = c$$
$$\mathcal{T}^s(1 - u, 1 - v) = 1 - u - \mathcal{T}^{1/s}(1 - u, v) = 1 - a - c - b = d.$$

We now distinguish four cases:

(i) The case $\min(a, b) = 0$. Assume for instance that $a = 0$, then $u = c$, $v = b + c$ and $b + c + d = 1$. One easily verifies that

$$\mathcal{T}^{+\infty}(u, 1 - v) = \max(c + 1 - b - c - 1, 0) = 0 = a$$
$$\mathcal{T}^{+\infty}(1 - u, v) = \max(1 - c + b + c - 1, 0) = b.$$

(ii) The case $\min(c, d) = 0$. Assume for instance that $c = 0$, then $u = a$, $v = b$ and $a + b + d = 1$. One easily verifies that

$$\mathcal{T}^0(u, 1 - v) = \min(a, 1 - b) = \min(a, a + d) = a$$
$$\mathcal{T}^0(1 - u, v) = \min(1 - a, b) = \min(b + d, b) = b.$$

Next, assume that $d = 0$, then $u = a + c$ and $v = b + c$. Then it holds that

$$\mathcal{T}^0(u, 1 - v) = \min(a + c, 1 - b - c) = \min(a + c, a + d) = \min(a + c, a) = a$$
$$\mathcal{T}^0(1 - u, v) = \min(1 - a - c, b + c) = \min(b + d, b + c) = \min(b, b + c) = b.$$

(iii) The case $ab = cd$. It now follows that

$$\begin{aligned}\mathcal{T}^1(u, 1 - v) &= (a + c)(1 - b - c) = a - ab - ac + c - bc - c^2 \\ &= a - cd - ac + c - bc - c^2 = a + c - c(a + b + c + d) \\ &= a + c - c = a\end{aligned}$$

$$\begin{aligned}\mathcal{T}^1(1 - u, v) &= (1 - a - c)(b + c) = b + c - ab - ac - bc - c^2 \\ &= b + d - cd - ac - bc - c^2 = b + c - c(a + b + c + d) \\ &= b + c - c = b.\end{aligned}$$

(iv) If $\min(a,b) \neq 0$ and $\min(c,d) \neq 0$, then it follows with $a+b+c+d=1$ that $(a,b,c,d) \in]0,1[^4$. If also $ab \neq cd$, then it follows from Lemma 1 and Theorem 1 that the equation $x^a + x^b + x = x^{1-c} + x^{1-d} + 1$ has a unique solution in $]0,1[\cup]1,+\infty[$. Let s be this solution, then the following chain of equivalences holds:

$$
\begin{aligned}
& \mathcal{T}^{1/s}(u, 1-v) = a \\
- \quad & \log_{1/s}\left(1 + \tfrac{((1/s)^u - 1)((1/s)^{1-v} - 1)}{1/s - 1}\right) = a \\
- \quad & 1 - \tfrac{s(s^{-u} - 1)(s^{v-1} - 1)}{s-1} = s^{-a} \\
- \quad & s - 1 - s(s^{-a-c} - 1)(s^{b+c-1} - 1) = (s-1)s^{-a} \\
- \quad & s - 1 - s^{-a+b} + s^{-a-c+1} + s^{b+c} - s = s^{-a+1} - s^{-a} \\
- \quad & -s^a - s^b + s^{1-c} + s^{a+b+c} = s - 1 \\
- \quad & s^a + s^b + s = s^{1-c} + s^{1-d} + 1.
\end{aligned}
$$

Similarly, it follows that $\mathcal{T}^{1/s}(1-u, v) = b$. □

Note that the first three cases in the previous theorem are not mutually exclusive.

5. Conclusion

Based on our knowledge of the roots of the equiponderate equation $x^a + x^b + x = x^c + x^d + 1$, we have developed a representation of weight quadruplets in terms of Frank t-norms with reciprocal parameters. This representation can be stated as follows: a quadruplet $(a,b,c,d) \in [0,1]^4$ is a weight quadruplet if and only if there exists a parameter $s \in [0,+\infty]$ such that it can be written as

$$(\mathcal{T}^{1/s}(a+c, 1-b-c), \mathcal{T}^{1/s}(1-a-c, b+c), \mathcal{T}^s(a+c, b+c), \mathcal{T}^s(1-a-c, 1-b-c)).$$

Acknowledgments

Bernard De Baets is a Post-Doctoral Fellow and Hans De Meyer is a Research Director of the Fund for Scientific Research – Flanders (Belgium).

References

1. B. De Baets and B. Van de Walle, *Minimal definitions of classical and fuzzy preference structures*, Proceedings of the Annual Meeting of the North American Fuzzy Information Processing Society (Syracuse, New York, USA, September 1997), pp. 299–304.
2. B. De Baets, B. Van de Walle and E. Kerre, *The Frank t-norm family in fuzzy preference modelling*, Proceedings of the Fourth European Congress on Intelligent Techniques and Soft Computing (Aachen, Germany, September 1996) (H.-J. Zimmermann, ed.), Vol. 1, ELITE, Aachen, 1996, pp. 78–82.

3. J. Fodor and M. Roubens, *Fuzzy Preference Modelling and Multicriteria Decision Support*, Kluwer Academic Publishers, Dordrecht, 1994.
4. J. Fodor and M. Roubens, *Valued preference structures*, European Journal of Operational Research **79** (1994), 277–286.
5. M. Frank, *On the simultaneous associativity of* $F(x,y)$ *and* $x+y-F(x,y)$, Aequationes Mathematicae **19** (1979), 194–226.
6. M. Navara, *How prominent is the role of Frank t-norms?*, Proceedings of the Seventh International Fuzzy Systems Association World Congress (Prague, Czech Republic, June 1997), Vol. I, Academia, 1997, pp. 291–296.
7. B. Schweizer and A. Sklar, *Associative functions and abstract semi-groups*, Publ. Math. Debrecen **10** (1963), 69–84.
8. B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*, North-Holland, New York, 1983.
9. B. Van de Walle, B. De Baets and E. Kerre, *Characterizable fuzzy preference structures*, Annals of Operations Research, Special Issue "Preference modelling" (D. Bouyssou and Ph. Vincke, eds.) **80** (1998), 105–136.