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A Simple Proof of Suzumura's Extension Theorem for Finite Domains With Applications

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Abstract. In this paper we provide a simple proof of the extension theorem for partial orderings due to Suzumura [1983] when the domain of the partial order is finite. The extension theorem due to Szpilrajn [1930] follows from this theorem. Szpilrajns extension theorem is used to show that an asymmetric binary relation is contained in the asymmetric part of a linear order if and only if it is acyclic. This theorem is then applied to prove three results. Finally we introduce the concept of a threshold choice function, and our third result says that such choice functions are the only ones to satisfy a property called functional acyclicity.

Keywords: Partial Orderings, Extension Theorem, Threshold Choice Function.

1. Introduction

In this paper we provide a simple proof of the extension theorem for partial orderings due to Suzumura [1983] when the domain of the partial order is finite. The extension theorem due to Szpilrajn [1930] follows from this theorem. Szpilrajn's extension theorem is used to show that an asymmetric binary relation is contained in the asymmetric part of a linear order if and only if it is acyclic. This theorem is then applied to prove three results. The first result implied by two theorems in Aizerman and Malishevsky [1981], (see Aizerman and Aleskerov [1995] as well) says that the asymmetric part of a quasi-transitive binary relation can be expressed as the intersection of the asymmetric parts of orders. The well known result due to Dushnik and Miller [1941], which states that any asymmetric and transitive binary relation is the intersection of linear orders follows as an immediate corollary of this result. The second result is a theorem in Lahiri [1999], which says that a choice function is a batch choice function if and only if it satisfies a property called the choice acyclicity property. We provide a new proof

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of this result. The concept of a batch choice function can be found in Aizerman and Aleskerov [1995] and in recent times it has been applied in the study of stable matching problems. Finally we introduce the concept of a threshold choice function, and our third result says that such choice functions are the only ones to satisfy a property called functional acyclicity. This last property can be traced to Aizerman and Aleskerov [1995] as well.

2. The Extension Theorems

Let X be a finite, non-empty set. Given a binary relation R, let P(R) = {(x, y) ∈ R / (y, x) ∉ R} and I(R) = {(x, y) ∈ R / (y, x) ∈ R}. P(R) is called the asymmetric part of R and I(R) is called the symmetric part of R. A binary relation R on X is said to be (a) reflexive if $\forall x \in X : (x, x) \in R$; (b) complete if $\forall x, y \in X$ with $x \neq y$, either $(x, y) \in R$ or $(y, x) \in R$; (c) transitive if $\forall x, y, z \in$ X, [(x, y) ∈ R & (y, z) ∈ R implies (x, z) ∈ R]; (d) asymmetric if ∀ x, y ∈ X : (x, y) ∈ R implies (y, x) ∉ R; (e) quasi-transitive if ∀ x, y, z ∈ X, (x, y) ∈ P (R) and (y, z) ∈ P(R) implies (x, z) ∈ P(R). Given a binary relation R on X a binary relation Q on X is said to extend (be an extension of) R if R ⊂ Q and P(R) ⊂ P(Q).

A binary relation R on X is said to be a partial order if it is reflexive and transitive. It is said to be an order if it is a complete partial order. A binary relation R on X is said to be a linear order if it is an order and further I(R)=Δ_X ≡{(x, x)/x ∈ X}.

Given a binary relation R on X and given any non-empty subset S of X, let M(S, R) denote {x ∈ S/ (y, x) ∈ P(R) implies y ∉ S}.

Given a binary relation R on X define binary relations $T(R)(: T^{\circ}(R))$ on X as follows: (x, y) ∈ $T(R))(: T^{\circ}(R))$ if and only if there exists a positive integer K and x_1, ..., x_K in X with (i) x_1= x, x_K = y : (ii) $(x_i, x_{i+1}) \in R \forall i \in \{1, \ldots, K-1\}$ (:and $(x_i, x_{i+1}) \in P(R)$ for $i \in \{1, \ldots, K-1\}$). T(R) is called the transitive hull of R. Clearly T(R) is always transitive. Further $T(I(R)) \subset I(T(R))$. Note that $T(R) \backslash T(I(R)) \subset T^{\circ}(R)$

A binary relation R on X is said to be acyclic if T(P(R)) is asymmetric. It is said to be consistent if there does not exist any x in X such that (x, x) ∈ $T^{\circ}(R)$.

Theorem 1 **(Suzumura's Extension Theorem):** *If R is a reflexive binary relation on X then it has an extension Q which is an order if and only if R is consistent.*

Proof: Since T(R) is transitive, it is clearly acyclic. Thus whenever S is a non-empty subset of X, M(S, T(R)) is non-empty. Let $A_1 = M(X, T(R))$

and having defined A_n, let $A_{n+1} = M(X\setminus\bigcup_{i=1}^{n} A_i, T(R))$. Since X is finite, there exists a positive integer r such that $A_r \neq \phi$ and $X = \bigcup_{i=1}^{r} A_i$. Further if $i \neq j, then A_i \cap A_j = \phi$. Define f : X → $\Re$ (the set of real numbers) as follows : f(x) = r - i +1 if x ∈ A_i. Suppose (x, y) ∈ P(T(R)). Then x ∈ A_i, y ∈ A_j implies by our method of construction that $i < j$. Thus $f(x) > f(y)$. Now suppose (x, y) ∈ T(R) and towards a contradiction suppose that $f(y) > f(x)$. Hence if y ∈ A_j and x ∈ A_i, clearly $j < i$. Thus, $A_j = M(X\setminus\bigcup_{k=1}^{j-1} A_k, T(R)), X\setminus\bigcup_{k=1}^{j-1} A_j$ is finite and T(R) is transitive implies that there exists z ∈ A_j such that (z, x) ∈ P(T(R)) since x ∈ (X $\setminus\bigcup_{k=1}^{j-1} A_k)\setminus A_j)$. By transitivity of T(R), (z, y) ∈ P(T(R)), contradicting y ∈ A_j. Thus, f(x) ≥ f(y). Let (x, y) ∈ P(R). Thus (x, y) ∈ T(R). If (y, x) ∈ T(R), then along with (x, y) ∈ P(R) it follows that $(y, y) \in T^{\circ}(R)$ contradicting that R is consistent. Thus (x, y) ∈ P(T(R)). Thus $f(x) > f(y)$. Now suppose that (x, y) ∈ R and towards a contradiction suppose that $f(y) > f(x)$. Then as before there exists z∈ X such that $f(z) = f(y).(z, x) \in$ P(T(R)). Thus $(z, y) \in T^{\circ}(R)$. If (y, z) ∈T(R) then $(z, z) \in T(R)$ contradicting the requirement that R is consistent. Thus, (z, y) ∈ P(T(R)). Thus, $f(z) > f(y)$ which contradicts $f(z) = f(y)$. Thus, $(x, y) \in$ R implies $f(x) \geq f(y)$. Let Q = {(x, y) ∈X×X/ f(x) ≥ f(y)}. Thus, Q is an order which extends R. □

Corollary 1 (Szpilrajn's Extension Theorem): *If R is a partial order on X then it has an extension Q which is an order.*

Proof: Follows easily from Suzumura's Extension Theorem by noting that a partial order is always consistent. □

The following lemma proves useful in establishing subsequent results.

LEMMA 1 *Let $f : X \to \Re$ (the set of real numbers) be given. Then, there exists a positive integer n and one to one functions $f_i : X \to N$ (:the set of natural numbers), $i \in \{1, \ldots, n\}$ such that $\{(x, y) \in X \times X/f(x) \geq f(y)\} = \{(x, y) \in X \times X/f_i(x) \geq f_i(y)$ for some $i \in \{1, \ldots, n\}\}$.*

Proof: Let {f(x)/ x ∈X} = {s_1, ..., s_q } where q is a positive integer and $s_j < s_{j+1}$ $\forall j \in \{1, \ldots, q-1\}$. Let $n_j = \{x \in X/f(x) = s_j\}$ and let $n = (n_1)! \times \ldots \times (n_q)!$

Let g: X →N be defined as follows: g(x) = n_1, if f(x) = s_1

g(x) = n_1+...+n_j, if f(x) = s_j.

Clearly, ∀x, y∈X : [f(x) ≥ f(y) if and only if $[g(x) \geq g(y)]$.

A function $\pi : \{1, \ldots, n_1 + \ldots + n_q\} \to X$ is called a restricted permutation if $\forall k \in \{1, \ldots, n_1 + \ldots + n_q\}$: (1) $[\pi(k) \in \{x \in X/f(x) = s_1\}$ if and only $(1 \leq k \leq n_1\ )]$ & (2) $[\pi(k) \in \{x \in X/f(x) = s_i\}$ if and only $(n_{i-1} \leq k \leq n_i$ and $1 < i \leq q)$]. Let Π denote the set of all restricted permutations. Since X is finite so is Π. For $\pi \in \Pi$, define f_π: X→ {1, ..., n_1+ ...+n_q} as follows: ∀x∈X, f_π(x) = k if and only if π (k) =x. It is now easy to verify that, { (x, y) ∈ X x X/f(x) ≥ f(y)} = { (x, y) ∈ X x X/g(x) ≥ g(y)} = { (x, y) ∈ X x X/f_π(x) ≥ f_π(y) for some $\pi \in \Pi$}. This proves the lemma.

□

The following theorem is rather interesting and to an extent original:

THEOREM 2 *Let P be any asymmetric binary relation on X. Then there exists a linear order Q on X such that P ⊂ P(Q) if and only if P is acyclic.*

Proof: Suppose P is an asymmetric binary relation on X and suppose there exists a linear order Q on X such that P ⊂ P(Q). Towards a contradiction suppose P is not acyclic. Then there exists x∈ X such that (x, x) ∈ T(P). Since P ⊂ P(Q), (x, x) ∈ T(P(Q)). Since P(Q) is transitive, (x, x) ∈ P(Q), contradicting the asymmetry of P(Q). Hence P must be acyclic.

Now suppose P is an asymmetric and acyclic binary relation on X. Let R = T(P ∪Δ). Clearly, R is reflexive and transitive. Hence by Szpilrajn's Extension Theorem there exists a reflexive, complete and transitive binary relation L on X such that R ⊂ L and P(R) ⊂ P(L). Since P is asymmetric and acyclic P ⊂ P(R). Hence P ⊂ P(L).

Since L is transitive, it is clearly acyclic. Thus whenever S is a non-empty subset of X, M(S, L) is non-empty. Let A_1 = M(X, L) and having defined A_n, let A_{n+1} = M(X $\setminus \bigcup_{i=1}^{n} A_i, L)$. Since X is finite, there exists a positive integer r such that $A_r \neq \phi$ and X = $\bigcup_{i=1}^{r} A_i$. Further if i ≠ j, then $A_i \cap A_j = \phi$. Define f : X → ℜ (the set of real numbers) as follows : f(x) = r - i+1 if x ∈ A_i. Clearly, L={ (x, y) ∈ X x X/f(x) ≥ f(y)}. By Lemma 1, there exists a positive integer n and one to one functions f_i: X →N, i ∈ {1, ..., n} such that { (x, y) ∈ X x X/f(x) ≥ f(y)} = { (x, y) ∈ X x X/f_i(x) ≥ f_i(y) for some i ∈ {1, ..., n}}. For i ∈ {1, ..., n}, let Q_i = { (x, y) ∈ X x X/f_i(x) ≥ f_i(y)}. Now (x, y) ∈ P(L) implies and is implied by $f(x) > f(y)$ which is equivalent to $f_i(x) > f_i(y)$ for all i ∈ {1, ..., n}}. Thus P(L) =∩ {P(Q_i)/ i ∈ {1, ..., n}}. Thus P ⊂ P(Q_1) where Q_1 is a linear order on X.

□

The following theorem, is really a consequence of two theorems in Aizerman and Malishevsky [1981] and these two theorems have been reproduced in Aizerman and Aleskerov [1995]. It is important enough to merit an independent proof.

THEOREM 3 *If R is a quasi-transitive binary relation then* $P(R) = \cap\{P(Q)/Q \in A\}$ *where* $\phi \neq A \subset \{Q \subset X \times X/Q$ is a linear order$\}$.

Proof: Let P = P(R). P is asymmetric and transitive. Hence by Theorem 2, there exists a linear order R^1 on X such that $P \subset P(R^1)$. Let A = {Q/Q is a linear order on X with $P \subset P(Q)$}. Thus, $P \subset \cap$ {P(Q) / Q ∈ A}.

Now suppose (x, y) ∈ ∩ { P(Q)/ Q∈ A}. Towards a contradiction suppose (x, y) ∉ P. Since (y, x) ∈ P ⊂ ∩{ P(Q)/ Q∈ A} contradicts [(x, y) ∈ P(Q) whenever Q∈ A], clearly (y, x) ∉ P. Further, (x, y) ∈ ∩ { P(Q)/ Q∈ A} implies [(y, x) ∉ P(Q) whenever Q ∈ A].

Let $\overline{P}$ = P ∪{ (y, x)}. Clearly, $\overline{P}$ is asymmetric. Suppose towards a contradiction that (z, z)∈ T($\overline{P}$) for some z ∈X. Thus there exists a positive integer m and elements $z_1, ..., z_m$ in X with $z = z_1 = z_m$ and $(z_i , z_{i+1}) \in$ P ∪{(y, x)} ∀ i ∈ {1, ..., m - 1}. If $(z_i, z_{i+1}) \in P$ ∀ i ∈ {1, ..., m-1}, then we get by transitivity of P, that $(z_1, z_m) \in P(R)$ i.e. (z, z) ∈ P, contradicting asymmetry of P. Hence (z_i, z_{i+1}) = (y, x) for some i ∈ {1, ..., m-1}.

Observe that 'm' is greater than three, for if m ≤ 3, then (z_1, z_2) and (z_2, z_1) belong to P ∪{ (y, x)} which is not possible since by hypothesis x ≠ y and (x, y) does not belong to P(R).

Case 1: Cardinality of {i ∈ {1, ..., m-1}/(z_i, z_{i+1})} = (y, x)} is one.

If (z_1, z_2) = (y, x), then z_m = y implies by transitivity of P that (x, y) ∈ P which is a contradiction.

If i>1, then (z_1, y) ∈ P and (x, z_1) ∈ P by transitivity of P, so that (x, y) ∈ P by transitivity of P which is a contradiction.

Case 2: Cardinality of {i ∈ {1, ..., m-1}/(z_i, z_{i+1}) = (y, x) is greater than one.

Let j = min {i ∈ {1, ..., m-1}/(z_i, z_{i+1}) = (y, x)} and k = min {i ∈ { j+1, ..., m-1}/(z_i, z_{i+1}) = (y, x)}. Thus z_{j+1} = x, z_k = y and by transitivity of P, (x, y) ∈ P which is a contradiction.

Thus (z, z) ∉ T($\overline{P}$) whenever z ∈X. Thus, $\overline{P}$ is acyclic. By Theorem 2, there exists a linear order R° such that $\overline{P} \subset P(R^\circ)$. Thus $P \subset \overline{P} \subset P(R^\circ)$ and hence R° ∈ A. However, (y, x) ∈ $\overline{P}$ implies (y, x) ∈ $P(R^\circ)$. This contradicts (x, y) ∈ ∩ {P(Q)/Q∈ A}. Thus (x, y) ∈ P. Hence the proof is complete. □

The following well known theorem due to Dushnik and Miller [1941] follows as an immediate corollary of Theorem 3:

THEOREM 4 *Let P be any asymmetric and transitive binary relation on X. Then P = ∩ { P(Q)/ Q ∈ B }, where,* $\phi \neq B \subset \{ Q \in X \times X/Q$*is a linear order*}.

3. Batch Choice Functions

Given any non-empty subset S of X, let [S] denote the set of all non-empty subsets of S. Hence in particular, [X] denotes the set of all non-empty subsets of X. A choice function C on X is a function $C : [X] \to [X]$ such that $C(S) \subset S \ \forall S \in [X]$.

A choice function C on X is said to satisfy the Choice Acyclicity Property (CAP) if there does not exist a positive integer K and sets $S_1, ..., S_K \in [X]$ such that : (i) $\forall$ i $\in$ {1, ..., K-1} : $C(S_i) \in [S_{i+1}] \backslash \{C(S_{i+1})\}$; and (ii) $C(S_K) \in [S_1] \backslash \{C(S_1)\}$.

A choice function C on X is said to be a batch choice function if there exists a linear order Q on [X] such that $\forall$ S $\in$ [X], C(S) = {A $\in$[S]/ $\forall$B$\in$[S] : (A, B)$\in$Q}.

THEOREM 5 **(Lahiri [1999])** *C is a batch choice function if and only if C satisfies CAP.*

Proof: If C is a batch choice function it clearly satisfies CAP. Hence suppose C satisfies CAP. If X has just one element then C is obviously a batch choice function. Hence suppose that X has atleast two elements. Let P = { (C(S), A) / A $\in$ [S]\ } C(S)}, S $\in$ [X] and S has atleast two elements}. Clearly P is asymmetric. Further, since C satisfies CAP, P is acyclic. By Theorem 2, there exists a linear order Q on [X] such that P $\subset$ P(Q). Given S $\in$ [X], since (C(S), A) $\in$ P $\forall$ A $\in$ [S]\{C(S)}, C(S) = {A $\in$[S]/ $\forall$B$\in$[S] : (A, B)$\in$Q}. Thus, C is a batch choice function. □

Remark 1 : It is worth observing that there exists a choice function C on X which does not satisfy the CAP and yet there does not exist sets $S_1, S_2 \in [X]$ such that : (i)$C(S_1) \in [S_2] \backslash \{C(S_2)\}$ and (ii) $C(S_2) \in [S_1] \backslash \{C(S_1)\}$.

Example: Let $X = \{x, y, z\}$. Let $C(\{x, y\}) = \{y\}$, $C(\{y, z\}) = \{z\}$, $C(\{x, z\}) = \{z\}$, $C(A) = A$, otherwise. Clearly, there does not exist sets S_1, $S_2 \in [X]$ such that : (i) $C(S_1) \in [S_2] \backslash \{C(S_2)\}$ and (ii) $C(S_2) \subset [S_1] \backslash \{C(S_1)\}$.

However C does not satisfy CAP: $C(\{x, y\}) \in [\{y, z\}] \backslash \{C(\{y, z\})\}$, $C(\{y, z\}) \in [\{x, z\}] \backslash \{C(\{x, z\})\}$ and $C(\{x, z\}) \in [\{x, y\}] \backslash \{C(\{x, y\})\}$. Towards a contradiction suppose there exists an order Q on [X] such that $\forall S \in [X]$, $C(S) = \{A \in [S] / \forall B \in [S] : (A, B) \in Q\}$. Then, $(\{y\}, \{x\}) \in P(Q)$, $(\{x\}, \{z\}) \in P(Q)$ and $(\{z\}, \{y\}) \in P(Q)$ contradicting the assumption that Q is an order on [X]. Thus C is not a batch choice function.

4. Functional Acyclicity

The following property in Aizerman and Aleskerov [1995] known as functional acyclicity implies CAP:

A choice function C on X is said to satisfy Functional Acyclicity (FA) if there does not exist a positive integer K and sets $S_1, \ldots, S_K \in [X]$ such that : (i) $\forall i \in \{1, \ldots, K-1\} : C(S_i) \cap (S_{i+1} \backslash C(S_{i+1})) \neq \phi$; and (ii) $C(S_K) \cap (S_1 \backslash C(S_1)) \neq \phi$. However the following example reveals that the converse need not be true:

Example: Let $X = \{x, y, z\}$. Let $C(X) = \{x, y\}$, $C(\{x, z\}) = \{z\}$ and $C(A) = A$ otherwise. Clearly, C satisfies CAP. However, $(X \backslash C(X)) \cap \{x, z\} \neq \phi$ and $(\{x, z\} \backslash C(\{x, z\})) \cap X \neq \phi$ contradicting FA.

A choice function C is said to be a threshold choice function if there exists a function V : [X] → X and a linear order Q such that : (i)$\forall S \in [X]$: $V(S) \in S$;(ii) $C(S) = \{x \in S/(x, V(S)) \in Q\}$.

The following theorem is equivalent to Theorem 3.15 in Aizerman and Aleskerov [1995] but unlike others we prove it here by appealing to Theorem 2.

THEOREM 6 *A choice correspondence C is a threshold choice function if and only if it satisfies FA.*

Proof: Let C be a threshold choice function. Thus, there exists a function V : [X] → X and a linear order Q such that : (i)$\forall S \in [X]$: $V(S) \in S$;(ii) $C(S) = \{x \in S/(x, V(S)) \in Q\}$. Towards a contradiction suppose that there exists a positive integer K and sets $S_1, \ldots, S_K \in [X]$ such that : (i) $\forall i \in \{1, \ldots, K-1\}$: $C(S_i) \cap (S_{i+1} \backslash C(S_{i+1})) \neq \phi$; and (ii) $C(S_K) \cap (S_1 \backslash C(S_1)) \neq \phi$. Let $x_t \in C(S_t) \cap (S_{t+1} \backslash C(S_{t+1}))$, for t = 1, ..., K-1 and let $x_K \in C(S_K) \cap (S_1 \backslash C(S_1))$. Thus $(x_t, V(S_t)) \in Q$, for t = 1, ..., K, $(V(S_{t+1}), x_t) \in P(Q)$ for t = 1, ..., K-1, and $(V(S_1), x_K) \in P(Q)$. Since Q is transitive we get $(x_K, x_K) \in P(Q)$, contradicting the asymmetry of P(Q). This contradiction implies that C must satisfy FA.

Now suppose that C satisfies FA. Let $P = \{C(S) x (S \backslash C(S)/S \in [X]\}$. P is asymmetric and by Functional Acyclicity P is acyclic. By Theorem 2, there exists a linear order Q on X such that $P \subset P(Q)$.

Given $S \in [X]$, let $\{V(S)\} = \{x \in C(S) \ / \ \forall y \in C(S) : (y, x) \in Q\}$.

Clearly, if $x \in C(S)$ then $(x, V(S)) \in Q$. Now, suppose $x \in S$ and $(x, V(S)) \in Q$ and towards a contradiction suppose $x \notin C(S)$. Thus, $(V(S), x) \in P$. Thus by the above $(V(S), x) \in P(Q)$which contradicts$(x, V(S)) \in Q$. Thus $x \in S$, $(x, V(S)) \in Q$ implies $x \in C(S)$. □

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