

# STOCHASTIC APPROXIMATION-SOLVABILITY OF LINEAR RANDOM EQUATIONS INVOLVING NUMERICAL RANGES

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The random (or stochastic) approximation-solvability, based on a projection scheme, of linear random operator equations involving the theory of the numerical range of a bounded linear random operator is considered. The obtained results generalize results with regard to the deterministic approximation-solvability of linear operator equations using the Galerkin convergence method.

**Key words:** Stochastic Projection Scheme, Numerical Range, Approximation-Solvability.

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## 1. Introduction

The theory of random operator equations originated from a desire to develop deterministic operator equations that were more application-oriented, with a special desire to deal with various natural systems in applied mathematics, since the behavior of natural systems is governed by chance. As attempts were made by many scientists and mathematicians to develop and unify the theory of random equations employing concepts and methods of probability theory and functional analysis, the Prague School of probabilists under Spacek initiated a systematic study using probabilistic operator equations as models for various systems. This development was further energized by the survey article by Bharucha-Reid [2] on various treatments of random equations under the framework of functional analysis. For details on random operator equations, consider the work of Bharucha-Reid [1-3], Hans [5], Saaty [8], and others.

Engl and Nashed [4] studying the deterministic projection schemes of the approximation-solvability of linear operator equations, considered a stochastic projection

scheme in a Hilbert space setting, and established the existence of the best-approximate solutions by a selective approach. Our aim has been to apply the theory of a random numerical range to the random (or stochastic) approximation-solvability of linear random operator equations. Among the obtained results, is a generalization of a result of Zarantonello [14] regarding the numerical range of the classical type. Non-linear analogs of the results involving  $A$ -regular random operators using the random version of the Zarantonello numerical range [14] can be discovered. For more information about  $A$ -regular operators, please study [10].

Consider a complete measure space  $(W, F, \mu)$ . Let  $X$  be a separable (real or complex) Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and  $\| \cdot \|$ . Let  $B(X)$  denote the  $\sigma$ -algebra or Borel fields of Borel subsets of  $X$ . Let  $f: W \rightarrow X$  be a mapping such that  $f^{-1}(B)$  is in  $F$  whenever  $B$  is in  $B(X)$ , that is,  $f$  is a random variable in  $X$ . Giving this definition is equivalent to stating that a random variable with values in  $X$  is a Borel measurable function. An operator  $T: W \times X \rightarrow X$  is said to be a *random operator* if  $\{w$  in  $W: T(w, x)$  is in  $B\}$  is in  $F$  for all  $x$  in  $X$  and for all  $B$  in  $B(X)$ . An operator  $T: W \times X \rightarrow X$  is *measurable* if it is measurable with respect to the  $\sigma$ -algebra  $F \times B(X)$ . A random operator  $T$  is *continuous* at  $x$ , if for each  $w$  in  $W$ ,  $T(w, \cdot)$  is continuous at  $x$ , that is, if

$$x_n \rightarrow x \text{ implies } T(w, x_n) \rightarrow T(w, x) \text{ for all } w \text{ in } W.$$

A measurable mapping  $f: W \rightarrow X$  is called a *random fixed point* of a random operator  $T: W \times X \rightarrow X$  if, for all  $w$  in  $W$ ,  $T(w, f(w)) = f(w)$ .

**Definition 1.1:** Let  $T: W \times X \rightarrow X$  be a linear random operator. A *random (stochastic) numerical range* of  $T$ , denoted  $N[T(w)]$ , is defined for all  $w$  in  $W$  and  $u$  in  $X$  by

$$N[T(w)] = \{(T(w, u), u): \|u\| = 1\}.$$

$N[T(w)]$  is a random version of the classical numerical range, and it does have properties similar to those of the classical numerical range.

**Definition 1.2:** (The Moore-Penrose inverse). Let  $T: X \rightarrow Y$  be a bounded linear operator from a Hilbert space  $X$  to another Hilbert space  $Y$ . Let  $T|N_T^\perp$  denote the restriction of  $T$  to the orthogonal complement of the null space of  $T$ . The *generalized inverse* or the *Moore-Penrose inverse* of  $T$ , denoted by  $T^+$ , is the unique linear extension of  $\{T|N_T^\perp\}^{-1}$  so that its domain is  $D_{T^+} = R_T \oplus R_T^\perp$  and its null space is  $N_{T^+} = R_T^\perp$ .

If  $P_M$  denotes the orthogonal projector of  $X$  onto a closed subspace  $M$  of  $X$  and if  $Q$  denotes the orthogonal projector of  $Y$  onto  $R_T^\perp$ , then  $T^+T = P_{N_T^\perp}$  and  $I - TT^+ = Q|D_{T^+}$ . On one hand, when  $R_T$  is not closed,  $D_{T^+}$  is a dense linear manifold of  $Y$  and  $T^+$  is unbounded; and on the other hand, if  $R_T$  is closed, then by the open mapping theorem,  $T^+$  is bounded and  $D_{T^+} = Y$ . For more on generalized inverses, see [6]. Generalized inverses of linear operators seem to have nice applications in analysis, statistics, prediction and control theory. As most of these applications are related to the least-squares property that the generalized inverses possess in Hilbert spaces,  $T^+$  is characterized by the following extremal property. Let  $u$  be defined so that  $u = T^+y$  for  $y$  in  $R_T \oplus R_T^\perp$ . Then  $u$  minimizes  $\|Tx - y\|$  over  $x$  in  $D_T$  and has the smallest norm among all other minimizers.

**Definition 1.3:** (The least-squares solution): Let  $T$  be a linear operator from a Hilbert space  $X$  into another Hilbert space  $Y$ . We call  $u$  in  $X$  a *least-squares*

solution of the operator equation  $Tx = y$  for a fixed  $y$  in  $Y$  if  $\inf\{\|Tx - y\| : x \text{ is in } X\} = \|Tu - y\|$ . If, in addition,  $u$  in  $X$  is a least-squares solution of minimal norm, it is called a *best-approximate solution* of  $Tx = y$ .

Note that when  $T$  is a bounded linear operator,  $Tx = y$  has a best-approximate solution if and only if  $y$  is in  $R_T \oplus R_T^\perp$ . Furthermore,  $T^+y$  for  $y$  in  $D_{T^+} = R_T \oplus R_T^\perp$  is the unique best-approximate solution of minimal norm. As a result, if  $R_T$  is closed, a best-approximate solution exists for every  $y$  in  $Y$ ; however, if  $R_T$  is not closed, a best-approximate solution does not exist if the orthogonal projection of  $y$  onto  $\bar{R}_T$  is not in  $R_T$ .

Now let us recall the random version of a best-approximate solution. For random operator  $T: W \times X \rightarrow Y$ , we consider the operator equation  $T(\cdot, x) = y$ , where  $y$  is a fixed element of  $Y$ ; or more generally, we consider the equation

$$T(\cdot, x(\cdot)) = y(\cdot). \tag{1.1}$$

A mapping  $u: W \rightarrow X$  satisfying the relation

$$\inf\{\|T(w, x(w)) - y(w)\| : x(w) \text{ is in } X\} = \|T(w, u(w)) - y(w)\|$$

for all  $w$  in  $W$ , is said to be a *wide-sense best-approximate solution* of (1.1). A wide-sense best-approximate solution which is also measurable is called a *random best-approximate solution*.

Next, we recall some auxiliary results crucial to the work at hand.

**Lemma 1.1:** [4] *Let  $X, Y$  and  $Z$  be separable Hilbert spaces, and let  $T: W \times X \rightarrow Y$  be a continuous random operator,  $U: W \times Z \rightarrow X$  be a random operator, and  $z: W \rightarrow X$  be measurable. Then*

- (i)  $w \rightarrow T(w, z(w))$  is measurable.
- (ii)  $ToU$  is a random operator on  $W \times Z$  mapping each  $(w, u)$  into  $T(w, U(w, u))$ .

**Lemma 1.2:** [4] *For  $n$  in  $\mathbb{N}$ , let  $e_1, \dots, e_n: W \rightarrow X$  be measurable. For all  $w$  in  $W$ , let  $X_n(w) := \text{span}\{e_1(w), \dots, e_n(w)\}$ . Then  $X_n: W \rightarrow 2^X$  is measurable.*

**Lemma 1.3:** [4] *Let  $S: W \rightarrow 2^X$  and  $P: W \times X \rightarrow X$  be such that for all  $w$  in  $W$ ,  $S(w)$  is a closed subspace of  $X$  and  $P(w, \cdot)$  is the orthogonal projection onto  $S(w)$ . Then the following are equivalent.*

- (i)  $S$  is measurable.
- (ii)  $P$  is a continuous random operator.
- (iii) *There exists a sequence of measurable functions  $u_1, u_2, \dots: W \rightarrow X$ , such that for all  $w \in W$ ,  $\{u_1(w), u_2(w), \dots\}$  is an orthonormal basis of  $S(w)$  (with the understanding that some  $u_i(w)$  may be zero).*
- (iv) *There is a sequence of measurable functions  $e_1, e_2, \dots: W \rightarrow X$  such that for all  $w$  in  $W$ ,*

$$cl\{\text{span}\{e_1(w), e_2(w), \dots\}\} = S(w).$$

**Lemma 1.4:** [7, Theorem 2.2] *Let  $(W, F, \mu)$  be a complete measure space, and let  $X$  and  $Y$  be separable Hilbert spaces. Let  $T$  be a.s. a bounded random linear operator from  $W \times X$  into  $Y$ . Let  $T^+(w)$  denote a.s. the generalized inverse of  $T(w)$ . Then*

- (a) *a.s. for each  $\alpha$ , with  $0 < \alpha < 2/\|T(w)\|^2$ ,  $\sum_{k=0}^n \alpha(I - \alpha T^*(w)T(w))^k T^*(w)y$  converges to  $T^+(w)y$  for each  $y$  in  $D_{T^+(w)}$ ;*
- (b)  *$T^+(w)$  is a random linear operator from  $W \times Y$  into  $X$ ;*
- (c) *for each  $Y$ -valued generalized random variable  $y(w)$*

such that a.s.  $y(w)$  is in  $D_{T^+}(w)$ ,  $T^+(w)y(w)$  is an  $X$ -valued generalized random variable.

**Lemma 1.5:** [15, Theorem 18E]: *Let  $A: X \rightarrow X$  be a continuous linear operator on a Hilbert space  $X$  over the field  $K$  (real or complex). Suppose that there is a constant  $c > 0$  such that*

$$|\langle Au, u \rangle| \geq c \|u\|^2 \text{ for all } u \text{ in } X.$$

*Then, for each given  $f$  in  $X$ , the operator equation  $Au = f$  ( $u$  in  $X$ ) has a unique solution.*

## 2. Stochastic Projection Schemes

In this section we consider stochastic projection schemes based on the deterministic projection schemes in Hilbert spaces. Let  $X$  and  $Y$  be separable Hilbert spaces. The approximation-solvability of a deterministic linear operator equation of the form

$$Tx = y \quad (x \text{ in } X, y \text{ in } Y) \quad (2.1)$$

and corresponding approximate equations of the form

$$T_n x_n = Q_n y, \quad (2.2)$$

where  $T_n := Q_n T$ , is based on a projection scheme  $\Pi_1 = \{X_n, Y_n, P_n, Q_n\}$ . Here  $X_n$  and  $Y_n$  are, respectively, subspaces of  $X$  and  $Y$ ; and  $P_n$  and  $Q_n$  are orthogonal projectors onto  $X_n$  and  $Y_n$  respectively. A very frequent choice of the approximation scheme involves  $Y_n := TX_n$ .

**Definition 2.1:** (Stochastic projection scheme) Let  $T: W \times X \rightarrow Y$  be a linear random operator.  $\Pi_1 = \{X_n, Y_n, P_n, Q_n\}$  is a *stochastic projection scheme* if  $X_n: W \rightarrow 2^X$  and  $Y_n: W \rightarrow 2^Y$  are measurable, if for all  $n$  in  $\mathbb{N}$  and  $w$  in  $W$ ,  $X_n(w)$  and  $Y_n(w)$  are closed subspaces of  $X$  and  $Y$  respectively, and if  $P_n(w)$  and  $Q_n(w)$  are orthogonal projectors onto  $X_n(w)$  and  $Y_n(w)$  respectively.

Furthermore, let  $T_n := Q_n T$ . Note that if  $\Pi_1$  is a stochastic projection scheme and  $T$  is a bounded linear random operator, then  $P_n, Q_n$  and  $T_n$  are random operators.

We need to recall the following lemma [4] regarding the measurability of functions.

**Lemma 2.1:** *Let  $\Pi_1 = \{X_n, Y_n, P_n, Q_n\}$  be a stochastic projection scheme, let  $y: W \rightarrow Y$  be measurable, let  $T: W \times X \rightarrow Y$  be a bounded linear random operator; and let*

$$T_n(w, x) := Q_n(w, T(w, x)). \quad (2.3)$$

*If, for some  $k$  in  $\mathbb{N}$ ,*

$$T_k(w, x) = Q_k(w, y(w)) \quad (2.4)$$

*is solvable in  $X_k(w)$  for all  $w$  in  $W$ , then there is a measurable function  $x_k: W \rightarrow X$  such that, for all  $w$  in  $W$ ,  $x_k(w)$  is in  $X_k(w)$  and  $T_k(w, x_k(w)) = Q_k(w, y(w))$ .*

### 3. Random Operator Equations

As random operator equations differ from their deterministic counterparts only in the aspect of the measurability of solutions, one general approach to establishing the measurability of the solutions is as follows. First of all, represent a solution by a convergent approximation scheme (use iterative or projectional methods), and then establish the measurability of approximations. Having done so, apply the following lemma on limits.

**Lemma 3.1:** *Let  $\{x_n\}$  be sequence of measurable functions from  $W$  to  $X$  converging (weakly or strongly) to  $x$ . Then  $x$  is measurable.*

We remark that, when the equation considered has a nonunique solution, one may not expect measurability of all solutions even in very simple cases. Consider the following example, where a random operator equation has a nonmeasurable solution.

**Example 3.1:** Let  $T:W \times \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $T(w, x) = x^2 - 1$ . Let  $E$  be a non-measurable subset of  $W$ , that is,  $E$  is not in  $F$ . Then the real-valued random variable  $x:W \rightarrow \mathbb{R}$ , defined by

$$x(w) = \begin{cases} 1, & w \text{ in } E \\ -1, & w \text{ in } W - E, \end{cases}$$

is a nonmeasurable solution of  $T(w, x) = 0$ .

Now, we turn our attention to the stochastic approximation-solvability of a linear random operator equation of the form

$$T(w, x) - \lambda(w)x = y(w), \tag{3.1}$$

where  $T:W \times X \rightarrow X$  is a bounded linear random operator on  $W \times X$ ,  $\lambda:W \rightarrow \mathbb{R}^+$  is a random variable, and  $y:W \rightarrow X$  is a measurable function. We let  $T_\lambda := T - \lambda I$ . The symbols “ $\rightarrow$ ” and “ $\xrightarrow{w}$ ” represent strong convergence and weak convergence respectively. First, we consider the result where the strong monotonicity of the operator  $T_\lambda$  is quite restrictive.

**Theorem 3.1:** *Let  $T:W \times X \rightarrow X$  be an everywhere define linear random operator on  $W \times X$ , where  $X$  is a separable Hilbert space. Let a number  $\lambda$  be at a positive distance*

$$d = \inf\{|\lambda - \gamma| : \gamma \text{ in } N[T(w)]\}$$

*from the numerical range  $N[T(w)]$  of  $T$ . Let  $\{e_n\}$  be a sequence of measurable functions from  $W$  to  $X$  such that, for all  $w$  in  $W$ ,  $\{e_n(w)\}$  is linearly independent and complete in  $X$ . Let  $y:W \rightarrow X$  be measurable, and let, for all  $w$  in  $W$  and  $n$  in  $\mathbb{N}$ ,  $X_n(w) := \text{span}\{e_1(w), \dots, e_n(w)\}$  with orthogonal projector  $P_n(w)$ . Then  $\Pi_2 = \{X_n, P_n\}$  is a stochastic projection scheme. Suppose, in addition, that  $T_n := P_n(T - \lambda I)$ , that is,  $T_n(w, x) := P_n(w, T(w, x)) - \lambda P_n(w, x)$ . Then, for an  $r$  in  $\mathbb{N}$  such that, for all  $n \geq r$  and for each corresponding  $x_n$  in  $X_n(w)$ , there exist measurable functions  $x, x_r, x_{r+1}, \dots:W \rightarrow X$  such that, for all  $w$  in  $W$ ,  $x_n(w)$  is the unique solution of  $T_n(w, x) = P_n(w, y(w))$  in  $X_n(w)$ . The sequence  $\{x_n(w)\}$  converges to  $x(w)$ , the unique solution of  $T(w, x) - \lambda x = y(w)$ .*

**Proof:** By Lemma 1.2,  $\Pi_2 = \{X_n, P_n\}$  is a stochastic projection scheme. Since  $T$  is random,  $T_\lambda$  is also random. Since  $d > 0$ , for all  $w$  in  $W$  and  $x \in X$ ,

$$|\langle T_\lambda(w, x), x \rangle| = \left| \frac{\langle T(w, x), x \rangle}{\|x\|^2} - \lambda \right| \|x\|^2 \geq d \|x\|^2. \quad (3.2)$$

This implies that  $T_\lambda(w, \cdot)$  is a strongly monotone random operator, and as a result,  $T_\lambda(w, \cdot)$  is one-to-one. It is easy to see that the range of  $T_\lambda, R_{T_\lambda}$ , is  $X$ ; therefore,  $T_\lambda(w, \cdot)$  is onto. It further follows from (3.2), as  $T_n := P_n(T - \lambda I)$ , that for an  $r$  in  $\mathbb{N}$  and for all  $n \geq r$ , for  $w$  in  $W$ , and for  $x_n$  in  $X_n(w)$ , we have

$$\begin{aligned} |\langle T_n(w, x_n), x_n \rangle| &= |\langle P_n(w)(T - \lambda I)(w)x_n, x_n \rangle| \\ &= |\langle (T - \lambda I)(w, x_n), P_n(w, x_n) \rangle| \geq d \|x_n\|^2. \end{aligned} \quad (3.3)$$

Thus,  $T_n(w, \cdot)$  is a strongly monotone random operator. Inequality (3.3) and Lemma 1.5 guarantee the existence of approximate solutions from the index  $r$  onward for all  $w$  in  $W$ ; and it follows that, for all  $n$  in  $\mathbb{N}$  and  $w$  in  $W$ ,  $x_n(w)$  is the unique solution of the approximate equations of the form:

$$P_n(w, T(w, x)) - \lambda P_n(w, x) = P_n(w, y(w)). \quad (3.4)$$

To this end, we proceed to show that, for all  $w$  in  $W$ ,  $\{x_n(w)\}$  converges to  $x(w)$ , the unique solution of the equation

$$T(w, x) - \lambda x = y(w). \quad (3.5)$$

For all  $n \geq j$ ,  $w$  in  $W$  and  $x_n(w)$  in  $X_n(w)$ , it follows from (3.4) that

$$\langle T(w, x_n(w)) - \lambda x_n(w), e_j(w) \rangle = \langle y(w), e_j(w) \rangle \quad (3.6)$$

and

$$\langle T(w, x_n(w)) - \lambda x_n(w), x_n(w) \rangle = \langle y(w), x_n(w) \rangle. \quad (3.7)$$

By inequality (3.3),

$$d \|x_n(w)\|^2 \leq |\langle y(w), x_n(w) \rangle| \leq \|y(w)\| \|x_n(w)\|.$$

This yields an *a priori estimate*

$$d \|x_n(w)\| \leq \|y(w)\|,$$

and thus,  $\{x_n(w)\}$  is bounded.

Let  $\{x_{n'}(w)\}$  be a weakly convergent sequence with  $x_{n'}(w) \xrightarrow{w} z(w)$  as  $n \rightarrow \infty$ . By (3.6), we find that

$$\langle T(w, x_{n'}(w)) - \lambda x_{n'}(w), e(w) \rangle \rightarrow \langle y(w), e(w) \rangle \text{ as } n \rightarrow \infty,$$

for all  $w$  in  $W$  and  $e(w)$  in  $\bigcup_n X_n(w)$ . Since  $\bigcup_n X_n(w)$  is dense in  $X$  and  $\{(T - \lambda I)(w, x_n(w))\}$  is bounded, we find that (as  $n \rightarrow \infty$ )

$$T(w, x_{n'}(w)) - \lambda x_{n'}(w) \xrightarrow{w} y(w).$$

We also have from the hypothesis that

$$T(w, x_{n'}(w)) - \lambda x_{n'}(w) \xrightarrow{w} T(w, z(w)) - \lambda z(w).$$

It follows that

$$(T - \lambda I)(w, z(w)) = y(w) \text{ and } z(w) = x(w).$$

Since the weak limit  $z(w)$  is the same for all weakly convergent subsequences of  $\{x_n(w)\}$ , we see that for all  $w$  in  $W$ ,

$$x_n(w) \xrightarrow{w} x(w) \text{ as } n \rightarrow \infty.$$

Thus, we have (as  $n \rightarrow \infty$ )

$$\begin{aligned} d \|x_n(w) - x(w)\|^2 &\leq | \langle (T - \lambda I)(w, x_n(w) - x(w)), x_n(w) - x(w) \rangle | \\ &= | \langle (T - \lambda I)(w, x_n(w)) - (T - \lambda I)(w, x(w)), x_n(w) - x(w) \rangle | \\ &= \langle y(w), x_n(w) \rangle - \langle (T - \lambda I)(w, x_n(w)), x(w) \rangle \\ &\quad | - \langle (T - \lambda I)(w, x(w)), x_n(w) - x(w) \rangle | \rightarrow 0. \end{aligned}$$

This implies that  $x_n(w) - x(w) \rightarrow 0$  as  $n \rightarrow \infty$ , that is,  $x_n(w) \rightarrow x(w)$ . It follows that  $(T - \lambda I)(w, x(w)) = y(w)$ .

Finally, since  $x_r, x_{r+1}, \dots: W \rightarrow X$  are measurable functions by Lemma 2.1, and since  $x_n(w) \rightarrow x(w)$  for all  $w$  in  $W$  and  $n$  in  $\mathbb{N}$ , it follows from Lemma 3.1 that the limit  $x(w)$  is measurable. This completes the proof.

We note that  $d$  and  $r$  are independent of  $w$  in Theorem 3.1, and this independence is rather restrictive; but it guarantees the existence of approximate solutions for all  $w$  in  $W$  from the index  $r$  onward. However, in the next theorem we allow  $d$  and  $r$  to depend on  $w$ . As one cannot guarantee, then, the solvability of a projectional equation for all  $w$  in  $W$  for any given  $n$ , we define the functions of the form  $x_n$  as best-approximate solutions of the projectional equation. Once inequality (3.3) is established for specific  $w$ , this best-approximate solution will, in fact, be a solution of the projectional equation for this  $w$ .

**Theorem 3.2:** *Let  $T: W \times X \rightarrow X$  be an everywhere defined linear random operator. Let a real-valued random variable  $\lambda: W \rightarrow \mathbb{R}^+$  (with  $\lambda(w) > 0$ ) be at a positive distance  $d(w)$  from the numerical range  $N[T(w)]$  of  $T$ , where  $d: W \rightarrow \mathbb{R}^+$  is a random variable. Let  $y: W \rightarrow X$  be measurable, and let  $\Pi_2 = \{x_n, P_n\}$  and  $T_n$  as in Theorem 3.1. Then, for all  $w$  in  $W$  and  $r(w)$  in  $\mathbb{N}$  and for all  $n \geq r(w)$  and  $x_n(w)$  in  $X_n(w)$ , we have*

$$| \langle T_n(w, x_n), x_n \rangle | \geq d(w) \|x_n\|^2. \tag{3.8}$$

Furthermore, there exist measurable functions  $x, x_1, x_2, \dots: W \rightarrow X$  with the following properties.

- (i)  $x_n(w)$  is in  $X_n(w)$  for all  $n$  in  $\mathbb{N}$  and  $w$  in  $W$ .
- (ii)  $T_n(w, x_n(w)) = P_n(w, y(w))$  for all  $w$  in  $W$  and  $n \geq r(w)$ .
- (iii) For all  $w$  in  $W$ ,  $x(w)$  is the unique solution of  $T_\lambda(w, x) = y(w)$ .
- (iv) For all  $w$  in  $W$  and  $n$  in  $\mathbb{N}$ ,  $x_n(w)$  is the best-approximate solution of

$$T_n(w)P_n(w, x) = P_n(w, y(w)) \text{ in } X.$$

(v) For all  $w$  in  $W$  and  $n$  in  $\mathbb{N}$ ,  $x_n(w)$  is the best-approximate solution of  $T_n(w, x) = P_n(w, y(w))$  in  $X_n(w)$ .

(vi) For all  $w$  in  $W$ ,  $\{x_n(w)\}$  converges to  $x(w)$ .

**Proof:** Inequality (3.8) is similar to inequality (3.3), and this, in fact, follows from the hypotheses. Suppose that  $w$  in  $W$  and  $n$  in  $\mathbb{N}$  are arbitrary but fixed. We may omit the argument  $w$  during the proof for the sake of brevity. The proof is based on [4]. Since  $P_n$  is the orthogonal projector onto  $X_n$ ,  $R_{T_n P_n} \subseteq X_n$  and  $X_n$  is finite-dimensional, this implies that  $(T_n P_n)^+$  is defined on all of  $X$ ; and so we can let

$$x_n := P_n(T_n P_n)^+ P_n y. \quad (3.9)$$

Now  $x_n$  satisfies (i) and (iv).

To prove (ii), let  $n \geq r(w)$ . It follows from (3.8) that  $T_n x = P_n y$  is solvable in  $X_n$ , and as a result,  $P_n y$  belongs to  $R_{T_n P_n}$ . Since  $T_n P_n (T_n P_n)^+$  projects onto  $R_{T_n P_n}$ , we can conclude that

$$T_n x_n = T_n P_n (T_n P_n)^+ P_n y = P_n y.$$

Proofs of (iii) and (vi) are similar to the proof of Theorem 3.1.

Since  $R_{(T_n P_n)^+} = N_{(T_n P_n)}^\perp \subseteq N_{P_n}^\perp = R_{P_n}$ , we can say  $x_n = (T_n P_n)^+ P_n y$ . Therefore,  $x_n$  minimizes  $\|T_n P_n x - P_n y\|$  over  $X$  and has the minimal norm among all the minimizers. This proves (iv). As a result of this and (i),  $x_n$  minimizes  $\|T_n x - P_n y\|$  over  $X_n$  and has the minimal norm among all the minimizers in  $X_n$ . Thus, (v) holds.

Finally, we establish the measurability of  $x_n$  (and hence of  $x$ ). Since  $P_n$  is a continuous random operator (by Lemma 1.3), it follows from Lemmas 1.1 and 1.4 that each  $x_n$  is measurable. This completes the proof.

## 4. Concluding Remarks

It seems that one can introduce the concept of the numerical range of a nonlinear random operator- a random version of the Zarantonello numerical range. Furthermore, one can extend the stochastic projection schemes to the case of nonlinear random operators, and can consider the general approximation-solvability of nonlinear random operator equations involving  $A$ -regular random operators [10].

For a nonlinear random operator  $T: W \times X \rightarrow X$  and for all  $w$  in  $W$  and  $x, y$  in  $X$ , we define the random numerical range  $N[T(w)]$  of  $T$  by

$$N[T(w)] = \left\{ \frac{\langle T(w, x) - T(w, y), x - y \rangle}{\|x - y\|^2} : x \neq y \right\}.$$

When  $T$  is nonrandom,  $N[T(w)]$  reduces to  $N[T]$ - the Zarantonello classical numerical range [14]. Random numerical  $N[T(w)]$  has the following properties.

Let  $X$  be a Hilbert space,  $S, T: W \times X \rightarrow X$  be random operators, and  $\lambda > 0$ . Then

- (i)  $N[\lambda T(w)] = \lambda N[T(w)]$ ,
- (ii)  $N[S(w) + T(w)] \subseteq N[S(w)] + N[T(w)]$ , and



$$(iii) N[(T - \lambda I)(w)] = N[T(w)] - \{\lambda\}.$$

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