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REDUCTION OF DIFFERENTIABLE EQUATIONS WITH IMPULSE EFFECT

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We consider a problem of a partial linearization of noninvertible differential equations with impulse effect and establish sufficient conditions for the dynamical equivalence.

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1. Introduction

The theory of differential equations with impulse effect has been developing rapidly in the recent years. These equations are convenient for description of evolutionary processes suddenly changing their state at certain moments. For the general theory of impulse systems, the reader is referred to the monographs by V. Lakshmikantham, D.D. Bainov and P.S. Simeonov [7], D.D. Bainov and P.S. Simeonov [2, 3, 5], D.D. Bainov and V. Covachev [4] and A.M. Samoilenko and N.A. Perestjuk [10]. The classification problems of impulse systems was first considered in [9] and [11]. This article is concerned with a specific aspect of classification, that reduces a noninvertible impulse system to a simpler one. Sufficient conditions for the dynamical equivalence of noninvertible differential systems are established. There are extensive works on classification for ordinary differential equations and maps [1, 6, 8].

2. The Statement of the Problem

Let $\mathbf{X}$ and $\mathbf{Y}$ be Banach spaces. The norms in these spaces are denoted by $|\cdot|$. Consider the following system of differential equations with impulse effect at fixed moments:

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$$\begin{cases} dx/dt = A(t)x + f(t,x,y), \\ dy/dt = B(t)y + g(t,x,y), \\ \Delta x \mid_{t=\tau_i} = C_i x(\tau_i - 0) + I_i(x(\tau_i - 0), y(\tau_i -)), \\ \Delta y \mid_{t=\tau_i} = D_i y(\tau_i - 0) + K_i(x(\tau_i - 0), y(\tau_i - 0)), \end{cases} \tag{1}$$

where:

(i) the maps $A: \mathbf{R} \rightarrow \mathrm{Hom}(\mathbf{X})$, $B: \mathbf{R} \rightarrow \mathrm{Hom}(\mathbf{Y})$ are locally integrable in Bochner's sense, where $\mathrm{Hom}(\mathbf{X})$ is the set of all linear bounded maps from $\mathbf{X}$ to $\mathbf{X}$;

(ii) the maps $f: \mathbf{R} \times \mathbf{X} \times \mathbf{Y} \rightarrow \mathbf{X}$, $g: \mathbf{R} \times \mathbf{X} \times \mathbf{Y} \rightarrow \mathbf{Y}$ are locally integrable in Bochner's sense with respect to t for fixed x and y and they satisfy the Lipschitz conditions with small ϵ uniformly with respect to t also:

$$| f(t,x,y) - f(t,x',y') | \leq \epsilon(| x - x' | + | y - y' |),$$

$$| g(t,x,y) - g(t,x',y') | \leq \epsilon(| x - x' | + | y - y' |),$$

and, in addition

$$\sup_{t,x,y} | f(t,x,y) | < +\infty;$$

(iii) $C_i \in \mathrm{Hom}(\mathbf{X})$, $D_i \in \mathrm{Hom}(\mathbf{Y})$ for $i \in \mathbf{N}$ and the maps $I_i: \mathbf{X} \times \mathbf{Y} \rightarrow \mathbf{X}$, $K_i: \mathbf{X} \times \mathbf{Y} \rightarrow \mathbf{Y}$ satisfy the Lipschitz conditions with small ϵ:

$$| I_i(x,y) - I_i(x',y') | \leq \epsilon(| x - x' | + | y - y' |),$$

$$| K_i(x,y) - K_i(x',y') | \leq \epsilon(| x - x' | + | y - y' |),$$

and, in addition

$$\sup_{i,x,y} | I_i(x,y) | < +\infty;$$

(iv) the moments τ_i of impulse effect form a strictly increasing sequence and

$$\lim_{n \rightarrow \infty} \tau_n = +\infty.$$

Let $\Phi(\cdot, t_0, x_0, y_0): [t_0, +\infty) \rightarrow \mathbf{X} \times \mathbf{Y}$ be a solution of the system (1), where $\Phi(t_0 + 0, t_0, x_0, y_0) = (x_0, y_0)$ and $\Phi(t, t_0, x_0, y_0) = (x(t, t_0, x_0, y_0),\ y(t, t_0, x_0, y_0))$. At the break points, the values are taken at $\tau_i - 0$, unless it is specified otherwise. Let $U(t,\tau)$ and $V(t,\tau)$ be the Cauchy evolution maps of the corresponding linear systems with impulse effect:

$$\begin{cases} dU/dt = A(t)U, \\ \Delta U \mid_{t=\tau_i} = C_i U(\tau_i - 0), \end{cases}$$

$$\begin{cases} dV/dt = B(t)V, \\ \Delta V \mid_{t=\tau_i} = D_i V(\tau_i - 0). \end{cases}$$

In addition, we assume that they satisfy the following inequalities:

$$L = \sup_t \left(\int_t^{+\infty} |U(t,\tau)|\,|V(\tau,t)|\,d\tau + \sum_{t \le \tau_i} |U(t,\tau_i)|\,|V(\tau_i - 0, t)| \right) < \infty, \tag{2}$$

$$M = \sup_t \left(\int_t^{+\infty} |U(t,\tau)|\,d\tau + \sum_{t \le \tau_i} |U(t,\tau_i)| \right) < \infty. \tag{3}$$

Let us consider (1) and the system

$$\begin{cases} dx/dt = A(t)x, \\ dy/dt = B(t)y + g(t, x + v(t,x,y), y), \\ \Delta x \,|_{t=\tau_i} = C_i x(\tau_i - 0), \\ \Delta y \,|_{t=\tau_i} = D_i y(\tau_i - 0) + K_i(x(\tau_i - 0) \\ \quad + v(\tau_i - 0, x(\tau_i - 0), y(\tau_i - 0)), y(\tau_i - 0)). \end{cases} \tag{4}$$

Definition: Two systems of differential equations with impulse effect (1) and (4) are *dynamically equivalent* if there exists a map $H: \mathbf{R} \times \mathbf{X} \times \mathbf{Y} \to \mathbf{X} \times \mathbf{Y}$ with the following properties:

(*i*) $H(t,\cdot,\cdot): \mathbf{X} \times \mathbf{Y} \to \mathbf{X} \times \mathbf{Y}$ is homeomorphism;

(*ii*) $H(t, \Phi(t, t_0, x_0, y_0)) = \Psi(t, t_0, H(t_0, x_0, y_0))$, if $t \ge t_0$, where $\Psi: [t_0, +\infty) \to \mathbf{X} \times \mathbf{Y}$, $\Psi(t, t_0, x_0, y_0) = (x_0(t, t_0, x_0, y_0), y_0(t, t_0, x_0, y_0))$ is solution of the system (4);

(*iii*) if the system (1) is autonomous and without impulses, then H does not depend on t.

3. The Main Theorem

Theorem 1: *Let hypothesis (i)-(iv) be satisfied, and suppose the inequalities* $4L\epsilon \le 1$ *and* $2M\epsilon < 1 + \sqrt{1 - 4L\epsilon}$ *are satisfied, where the constants L and M are specified by formulas* (2) *and* (3).

Then (1) *and* (4) *are dynamically equivalent.*

Proof: Step 1: Let us consider the Banach space $\mathbf{B}_1$ of the bounded maps that are continuous for $(t,x,y) \in (\tau_i, \tau_{i+1}] \times \mathbf{X} \times \mathbf{Y}$ and have first kind breaks for $t = \tau_i$:

$$\mathbf{B}_1 = \left\{ v \mid v: \mathbf{R} \times \mathbf{X} \times \mathbf{Y} \to \mathbf{X} \text{ and } \sup_{t,x,y} |v(t,x,y)| < \infty \right\}$$

with the norm $\|v\| = \sup_{t,x,y} |v(t,x,y)|$.

Let us define the set:

$$\mathbf{G}_1(p) = \left\{ v \in \mathbf{B}_1 \,\middle|\, |v(t,x,z) - v(t,x,z')| \le p\,|z - z'| \right\}. \tag{5}$$

$\mathbf{G}_1(p)$ is a closed subset of the Banach space $\mathbf{B}_1$. In $\mathbf{G}_1(p)$ we consider the functional equation:

$$v_1(t_0, x_0, z) = -\int_{t_0}^{+\infty} U(t_0,\tau) f(\tau, x_0(\tau) + v_1(\tau, x_0(\tau), z(\tau)), z(\tau))\,d\tau$$
$$- \sum_{t_0 \le \tau_1} U(t_0, \tau_i) I_i(x_0(\tau_i - 0) + v_1(\tau_i - 0, x_0(\tau_i - 0), z(\tau_i - 0)), z(\tau_i - 0)) \tag{6}$$

and

$$z(t) = V(t,t_0)z + \int_{t_0}^{t} V(t,\tau)g(\tau, x_0(\tau) + v_1(\tau, x_0(\tau), z(\tau)), z(\tau))d\tau$$

$$+ \sum_{t_0 \leq t} V(t,\tau_i)K_i(x_0(\tau_i - 0) + v_1(\tau_i - 0, x_0(\tau_i - 0), z(\tau_i - 0)), z(\tau_i - 0)),$$

where $x_0(\tau) = U(\tau, t_0)x_0$.

To solve the functional equation (6) we introduce the operator $\mathbf{E}$ from $\mathbf{G}_1(p)$ to $\mathbf{B}_1$ by the formula:

$$\mathbf{E}v_1(t_0, x_0, z) = -\int_{t_0}^{+\infty} U(t_0,\tau)f(\tau, x_0(\tau) + v_1(\tau, x_0(\tau), z(\tau)), z(\tau))d\tau$$

$$- \sum_{t_0 \leq \tau_1} U(t_0,\tau_i)I_i(x_0(\tau_i - 0) + v_1(\tau_i - 0, x_0(\tau_i - 0), z(\tau_i - 0)), z(\tau_i - 0))$$

and

$$z(t) = V(t,t_0)z + \int_{t_0}^{t} V(t,\tau)g(\tau, x_0(\tau) + v_1(\tau, x_0(\tau), z(\tau)), z(\tau))d\tau$$

$$+ \sum_{t_0 \leq t} V(t,\tau_i)K_i(x_0(\tau_i - 0) + v_1(\tau_i - 0, x_0(\tau_i - 0), z(\tau_i - 0)), z(\tau_i - 0)).$$

Next, we determine the difference $\| \mathbf{E}v_1(t_0, x_0, z) - \mathbf{E}v_1'(t_0, x_0, z') \|$, taking into account the properties of f, I_i, v_1. We obtain that:

$$\| \mathbf{E}v_1(t_0, x_0, z) - \mathbf{E}v_1'(t_0, x_0, z') \| \leq \epsilon(p+1)\Bigg(\int_{t_0}^{+\infty} |U(t_0,\tau)| \, |z(\tau) - z'(\tau)| \, d\tau$$

$$+ \sum_{t_0 \leq \tau_i} |U(t_0,\tau_i)| \, |z(\tau_i - 0) - z'(\tau_i - 0)| \Bigg) + \epsilon M \| v_1 - v_1' \|. \tag{7}$$

On the other hand, we estimate the difference $|z(t) - z'(t)|$, taking into consideration the properties of g and K_i:

$$|z(t) - z'(t)| \leq |V(t,t_0)| \, |z - z'| + \epsilon(p+1)\Bigg(\int_{t_0}^{t} |V(t,\tau)| \, |z(\tau) - z'(\tau)| \, d\tau$$

$$+ \sum_{t_0 \leq \tau_i} |V(t,\tau_i)| \, |z(\tau_i - 0) - z'(\tau_i - 0)| \Bigg)$$

$$+ \epsilon \| v_1 - v_1' \| \Bigg(\int_{t_0}^{t} |V(t,\tau)| \, d\tau + \sum_{t_0 \leq t} |V(t,\tau_i)| \Bigg).$$

Multiplying the difference $|z(t) - z'(t)|$ by $|U(t_0,t)|$ and integrating from t_0 to $+\infty$, we obtain:

$$\int_{t_0}^{+\infty} |U(t_0,t)| \, |z(t) - z'(t)| \, dt \leq |z - z'| \int_{t_0}^{+\infty} |U(t_0,t)| \, |V(t,t_0)| \, dt$$

$$+\sup_{\tau}\int_{t_0}^{+\infty}|U(\tau,t)|\,|V(t,\tau)|\,dt\Bigg(\epsilon(p+1)\Bigg(\int_{t_0}^{+\infty}|U(t_0,\tau)|\,|z(\tau)-z'(\tau)|\,d\tau$$

$$+\sum_{t_0\le\tau_i}|U(t_0,\tau_i)|\,|z(\tau_i-0)-z'(\tau_i-0)|\Bigg)$$

$$+\epsilon\,\|v_1-v_1'\|\Bigg(\int_{t_0}^{+\infty}|U(t_0,\tau)|\,d\tau+\sum_{t_0\le\tau_i}|U(t_0,\tau_i)|\Bigg)\Bigg). \tag{8}$$

Multiplying the same difference by $|U(t_0,\tau_i)|$ and summing for all i with respect to $t_0 \le \tau_i$, we get:

$$\sum_{t_0\le\tau_i}|U(t_0,\tau_i)|\,|z(\tau_i-0)-z'(\tau_i-0)|$$

$$\le|z-z'|\sum_{t_0\le\tau_i}|U(t_0,\tau_i)|\,|V(\tau_i-0,t_0)|$$

$$+\sup_{\tau}\sum_{t_0\le\tau_i}|U(\tau,\tau_i)|\,|V(\tau_i-0,\tau)|\Bigg(\epsilon(p+1)\Bigg(\int_{t_0}^{+\infty}|U(t_0,\tau)|\,|z(\tau)-z'(\tau)|\,d\tau$$

$$+\sum_{t_0\le\tau_i}|U(t_0,\tau_i)|\,|z(\tau_i-0)-z'(\tau_i-0)|\Bigg)$$

$$+\epsilon\,\|v_1-v_1'\|\Bigg(\int_{t_0}^{+\infty}|U(t_0,\tau)|\,d\tau+\sum_{t_0\le\tau_i}|U(t_0,\tau_i)|\Bigg)\Bigg). \tag{9}$$

Summing up (8) and (9) we get:

$$\int_{t_0}^{+\infty}|U(t_0,\tau)|\,|z(t)-z'(t)|\,d\tau+\sum_{t_0\le\tau_i}|U(t_0,\tau_i)|\,|z(\tau_i-0)-z'(\tau_i-0)|$$

$$\le L\Bigg(|z-z'|+\epsilon(p+1)\Bigg(\int_{t_0}^{+\infty}|U(t_0,\tau)|\,|z(\tau)-z'(\tau)|\,d\tau$$

$$+\sum_{t_0\le\tau_i}|U(t_0,\tau_i)|\,|z(\tau_i-0)-z'(\tau_i-0)|\Bigg)+\epsilon M\,\|v_1-v_1'\|\Bigg). \tag{10}$$

Let us designate

$$\int_{t_0}^{+\infty}|U(t_0,\tau)|\,|z(\tau)-z'(\tau)|\,dt+\sum_{t_0\le\tau_i}|U(t_0,\tau_i)|\,|z(\tau_i-0)-z'(\tau_i-0)|\;=Q.$$

We suppose that

$$\epsilon(p+1)L<1.$$

Inequality (10) means:

$$Q \leq L(\mid z-z' \mid + \epsilon(p+1)Q + \epsilon M \parallel v_1 - v_1' \parallel).$$

Thus we have shown that

$$Q \leq \frac{L(\mid z-z' \mid + \epsilon M \parallel v_1 - v_1' \parallel)}{1-\epsilon L(p+1)}.$$

Taking into consideration inequalities (7) and (10), we get

$$\parallel \mathbf{E}v_1(t_0,x_0,z) - \mathbf{E}v_1'(t_0,x_0,z') \parallel$$

$$\leq \epsilon(p+1)\frac{L(\mid z-z' \mid + \epsilon M \parallel v_1 - v_1' \parallel)}{1-\epsilon L(p+1)} + \epsilon M \parallel v_1 - v_1' \parallel. \tag{11}$$

If $4L\epsilon \leq 1$, then there is $p>0$ satisfying

$$\frac{\epsilon L(p+1)}{1-\epsilon L(p+1)} \leq p.$$

We choose

$$p = \frac{1-2\epsilon L-\sqrt{1-4\epsilon L}}{2\epsilon L}.$$

Then,

$$p+1 = \frac{1-\sqrt{1-4\epsilon L}}{2\epsilon L}$$

and $2\epsilon L(p+1) \leq 1$. We insert this p into (11) and obtain

$$\parallel \mathbf{E}v_1(t_0,x_0,z) - \mathbf{E}v_1'(t_0,x_0,z') \parallel \leq p \mid z-z' \mid + \epsilon M(p+1) \parallel v_1 - v_1' \parallel.$$

If $2\epsilon M < 1+\sqrt{1-4\epsilon L}$, then $\epsilon M(p+1)<1$. We conclude that $\mathbf{E}$ is a contraction. It shows that there is only one solution in $\mathbf{G}_1(p)$, satisfying the functional equation (6).

Next, we construct the map:

$$H_1(t_0,x_0,z) = (h_1(t_0,x_0,z),z) = (x_0+v_1(t_0,x_0,z),z).$$

It can easily checked that:

$$H_1(t,\Psi(t,t_0,x_0,y_0)) = \Phi(t,t_0,H_1(t_0,x_0,y_0)).$$

Step 2: In the same space $\mathbf{G}_1(p)$, we define the map by the following formula:

$$v_2(t_0,x_0,y_0) = \int_{t_0}^{+\infty} U(t_0,\tau)f(\tau,\Phi(\tau,t_0,x_0,y_0))d\tau$$

$$+ \sum_{t_0 \leq \tau_i} U(t_0,\tau_i)I_i(\Phi(\tau_i-0,t_0,x_0,y_0)).$$

Next, we compute that

$$v_2(t,\Phi(t,t_0,x_0,y_0)) = \int_t^{+\infty} U(t,\tau)f(\tau,\Phi(\tau,t_0,x_0,y_0))d\tau$$

$$+\sum_{t\le\tau_i} U(t,\tau_i)I_i(\Phi(\tau_i-0,t_0,x_0,y_0)). \qquad (12)$$

We construct the map:

$$H_2(t_0,x_0,y_0) = (h_2(t_0,x_0,y_0),y_0) = (x_0+v_2(t_0,x_0,y_0),y_0).$$

Applying (12), we conclude that

$$h_2(t,\Phi(t,t_0,x_0,y_0)) = U(t,t_0)h_2(t_0,x_0,y_0).$$

Step 3: Let us prove that $H_2(t_0,H_1(t_0,x_0,y_0)) = (x_0,y_0)$. It is sufficiently to check that

$$h_2(t_0,H_1(t_0,x_0,y_0)) = h_1(t_0,x_0,y_0)+v_2(t_0,H_1(t_0,x_0,y_0))$$

$$= x_0+v_1(t_0,x_0,y_0)+v_2(t_0,H_1(t_0,x_0,y_0))$$

$$= x_0 - \int_{t_0}^{+\infty} U(t_0,\tau)f(\tau,\Phi(\tau,t_0,H_1(t_0,x_0,y_0)))d\tau$$

$$-\sum_{t_0\le\tau_i} U(t_0,\tau_i)I_i(\Phi(\tau_i-0,t_0,H_1(t_0,x_0,y_0)))$$

$$+\int_{t_0}^{+\infty} U(t_0,\tau)f(\tau,\Phi(\tau,t_0,H_1(t_0,x_0,y_0)))d\tau$$

$$+\sum_{t_0\le\tau_i} U(t_0,\tau_i)I_i(\Phi(\tau_i-0,t_0,H_1(t_0,x_0,y_0))) = x_0.$$

Step 4: Now let us prove $H_1(t_0,H_2(t_0,x_0,y_0)) = (x_0,y_0)$. Let us consider the Banach space $\mathbf{B}_2$ of the bounded maps that are continuous for $(t,x,y,z)\in(\tau_i,\tau_{i+1}]\times\mathbf{X}\times\mathbf{Y}\times\mathbf{Z}$ and have first kind breaks for $t=\tau_i$:

$$\mathbf{B}_2 = \left\{v \mid v:\mathbf{R}\times\mathbf{X}\times\mathbf{Y}\times\mathbf{Z}\to\mathbf{X} \text{ and } \sup_{t,x,y,z} |v(t,x,y,z)| < \infty\right\}$$

with the norm $\|v\| = \sup_{t,x,y,z}|v(t,x,y,z)|$.

Let us introduce the set:

$$\mathbf{G}_2(p) = \{v\in\mathbf{B}_2 \mid |v(t_0,x,y,z)-v(t_0,x,y,z')| \le p\,|z-z'|\}. \qquad (13)$$

In the space $\mathbf{G}_2(p)$ we consider the functional equation:

$$v_3(t_0,x_0,y_0,z) = \int_{t_0}^{+\infty} U(t_0,\tau)(f(\tau,\Phi(\tau))-f(\tau,x(\tau)+v_3(\tau,\Phi(\tau),z(\tau)),z(\tau))d\tau$$

$$+\sum_{t_0<\tau_i} U(t_0,\tau_i)(I_i(\Phi(\tau_i-0))$$

$$-I_i(x(\tau_i-0)+v_3(\tau_i-0,\Phi(\tau_i-0),z(\tau_i-0)),z(\tau_i-0))) \tag{14}$$

and

$$z(t)=V(t,t_0)z+\int_{t_0}^{t} V(t,\tau)g(\tau,x(\tau)+v_3(\tau,\Phi(\tau),z(\tau)),z(\tau))d\tau$$

$$+\sum_{t_0\le\tau_i} V(t,\tau_i)K_i(x(\tau_i-0)+v_3(\tau_i-0,\Phi(\tau_i-0),z(\tau_i-0)),z(\tau_i-0)).$$

We introduce the operator **E**:

$$\mathbf{E}v_3(t_0,x_0,y_0,z)=\int_{t_0}^{+\infty} U(t_0,\tau)(f(\tau,\Phi(\tau))-f(\tau,x(\tau)+v_3(\tau,\Phi(\tau),z(\tau)),z(\tau))d\tau$$

$$+\sum_{\tau_0\le\tau_i} U(t_0,\tau_i)(I_i(\Phi(\tau_i-0))-I_i(x(\tau_i-0)$$

$$+v_3(\tau_i-0,\Phi(\tau_i-0),z(\tau_i-0)),z(\tau_i-0))). \tag{15}$$

In the same manner as we proceeded in the first step, we determine the difference $\| \mathbf{E}v_3-\mathbf{E}v_3' \|$. We make the same decisions and finally obtain that **E** is a contraction in $\mathbf{G}_2(p)$. There is only one solution for the functional equation (14). Next, we construct the map:

$$H_3(t_0,x_0,y_0,z)=(h_3(t_0,x_0,y_0,z),z)=(x_0+v_3(t_0,x_0,y_0,z),z).$$

We notice that the map

$$\alpha(t_0,x_0,y_0,z)=v_2(t_0,x_0,y_0)+v_1(t_0,h_2(t_0,x_0,y_0),z)$$

also satisfies the functional equation (14) and $\alpha\in\mathbf{G}_2(p)$, therefore

$$v_3(t_0,x_0,y_0,z)=v_2(t_0,x_0,y_0)+v_1(t_0,h_2(t_0,x_0,y_0),z).$$

Now we set equal the third and the fourth argument of v_3 and put them into the expression. We obtain $v_3(t_0,x_0,y_0,y_0)=0$. Therefore $H_1(t_0,H_2(t_0,x_0,y_0))=(x_0,y_0)$,

$$h_1(t_0,H_2(t_0,x_0,y_0))=h_2(t_0,x_0,y_0)+v_1(t_0,H_2(t_0,x_0,y_0))$$

$$=x_0+v_2(t_0,x_0,y_0)+v_1(t_0,H_2(t_0,x_0,y_0))=x_0.$$

We get that $H_1(t,\cdot,\cdot)$ is a homeomorphism establishing dynamical equivalence of systems (1) and (4).

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