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ON THE SETWISE CONVERGENCE OF SEQUENCES OF MEASURES

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We consider a sequence $\{\mu_n\}$ of (nonnegative) measures on a general measurable space $(X, \mathfrak{B})$. We establish sufficient conditions for setwise convergence and convergence in total variation.

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1. Introduction

Consider a sequence $\{\mu_n\}$ of (nonnegative) measures on a measurable space $(X, \mathfrak{B})$ where X is some topological space. Setwise convergence of measures, as opposed to weak* or weak convergence, is a highly desirable and strong property. If proved, some important properties can be derived (for instance, the Vitali-Hahn-Saks Theorem) and thus sufficient conditions ensuring this type of convergence are of interest. However, as noted in [2], in contrast to weak or weak* convergence (for instance, in metric spaces), it is in general difficult to exhibit such a property, except if e.g., μ_n is an increasing or decreasing sequence (e.g. [2], [4]).

The present paper provides two simple sufficient conditions. Thus, for instance, in a locally compact Hausdorff space, an **order-bounded** sequence of probability measures that is *vaguely* or *weakly* convergent is in fact **setwise** convergent.

We also establish a sufficient condition for the convergence in total variation norm that is even a stronger property.

2. Notations and Definitions

Let $(X, \mathfrak{B})$ be a measurable space and let $\mathcal{M}b(X)$ denote the Banach space of all bounded measurable real-valued functions on X equipped with the sup-norm. Let S be the positive (convex) cone in $\mathcal{M}b(X)$.

Let $\mathcal{M}b(X)'$ be the (Banach) topological dual of $\mathcal{M}b(X)$ with the duality bracket $\langle \cdot, \cdot \rangle$ between $\mathcal{M}b(X)$ and $\mathcal{M}b(X)'$. $\mathcal{M}b(X)'$ is equipped with the dual norm $|\varphi| := \sup_{\|f\| = 1} |\langle \varphi, f \rangle|$. Let $S' \in \mathcal{M}b(X)'$ be the positive cone in $\mathcal{M}b(X)'$, i.e., the dual cone of $S \in \mathcal{M}b(X)$. Convergence in the weak* topology of $\mathcal{M}b(X)'$ is denoted by $\xrightarrow{w*}$.

If X is a topological space, then $C(X)$ denotes the Banach space of all real-valued bounded continuous functions on X, and if X is locally compact Hausdorff, $C_0(X)$ ($C_c(X)$ resp.) denotes the Banach space of real-valued continuous functions that vanish at infinity (with compact support, resp.).

In the sequel, the term *measure* will stand for a nonnegative σ-additive measure and a set function on $\mathfrak{B}$ with the finite-additivity property (and not necessarily the σ-additivity property) will be referred to as a *finitely additive measure.* Let $M(X)$ be the Banach space of signed measures on $(X, \mathfrak{B})$ equipped with the total variation norm $|\cdot|_{TV}$, simply denoted $|\cdot|$.

Note that $M(X) \subset \mathcal{M}b(X)'$ and for every $f \in \mathcal{M}b(X)$, $\mu \in M(X)$, $\int f d\mu = \langle \mu, f \rangle$ when μ is considered to be an element of $\mathcal{M}b(X)'$.

Also note that any element $\varphi \in S'$ can be associated with a finitely additive nonnegative measure (also denoted φ) $\varphi(A)\colon = \langle \varphi, 1_A \rangle, \forall A \in \mathfrak{B}$, so that $\varphi(A \cup B) = \varphi(A) + \varphi(B)$, $\forall A$, $B \in \mathfrak{B}$ with $A \cap B = \emptyset$. Thus, $\varphi(A) \leq \varphi(X) = |\varphi|$, $\forall A \in \mathfrak{B}$ (see e.g., [3]).

For a topological space X, by analogy with sequences of probability measures in a metric space, a sequence of measures $\{\mu_n\}$ in $M(X)$ is said to converge *weakly* to $\mu \in M(X)$, iff

$$\int f d\mu_n \to \int f d\mu, \quad \forall f \in C(X).$$

This type of convergence is denoted $\mu_n \overset{weakly}{\to} \mu$.

Similarly, and again, by analogy with sequences of probability measures in a metric space, if X is a locally compact Hausdorff space, a sequence of measures $\{\mu_n\}$ in $M(X)$ is said to converge *vaguely* to $\mu \in M(X)$, iff

$$\int f d\mu_n \to \int f d\mu, \quad \forall f \in C_0(X),$$

and this type of convergence is denoted $\mu_n \overset{vaguely}{\to} \mu$. In fact, because the topological dual of $C_0(X)$ is $M(X)$ (see e.g., [1]), the vague convergence is simply the weak* convergence in $M(X)$.

A sequence $\{\mu_n\}$ in $M(X)$ is said to converge *setwise* to $\mu \in M(X)$ iff

$$\mu_n(B) \to \mu(B), \quad \forall B \in \mathfrak{B},$$

and this convergence is denoted $\mu_n \overset{setwise}{\to} \mu$.

Finally, a sequence $\{\mu_n\}$ in $M(X)$ converges to $\mu \in M(X)$ *in total variation* (or convergences strongly (or in norm) to μ) iff $|\mu_n - \mu| \to 0$ as $n \to \infty$. This convergence is denoted by $\mu_n \xrightarrow{TV} \mu$.

3. Preliminaries

In this section, we present some results that we will repeatedly use in the sequel.

For a nonnegative finitely additive measure μ, proceeding as in [6], let:

$$\Gamma(\mu)\colon = \{\nu \in M(X) \mid 0 \leq \nu \leq \mu\}, \quad \Delta(\mu)\colon = \{\nu \in M(X) \mid \mu \leq \nu\},$$

where by $\nu \leq \mu$ we mean $\nu(A) \leq \mu(A)$, $\forall A \in \mathfrak{B}$.

Given two (σ-additive) measures φ and ψ, let

$$\sup(\varphi, \psi) = \varphi \vee \psi \colon = \frac{|\varphi - \psi| + \varphi + \psi}{2}, \ \inf(\varphi, \psi) = \varphi \wedge \psi \colon = \frac{\varphi + \psi - |\varphi - \psi|}{2},$$

where for a signed measure γ, $|\gamma|$ is its corresponding total variation measure. With the partial ordering $\leq$, $M(X)$ is a complete Banach lattice (see e.g., [5]).

Lemma 3.1: *Let μ be (nonnegative) finitely additive measure. Then,*

(i) $\Gamma(\mu)$ has a maximal element $\varphi \in M(X)^+$.

(ii) If $\Delta(\mu) \neq \emptyset$, then μ is σ-additive.

Proof: To prove that $\Gamma(\mu)$ has a maximal element we use arguments similar to those in [6]. Let $\delta \colon = \sup_{\nu \in \Gamma(\mu)} \nu(X)$. Of course, we have $\delta \leq \mu(X) = 1$. Thus, consider a sequence $\{\nu_n\}$ in $M(X)^+$, with $\nu_n(X) \uparrow \delta$. Define

$$\varphi_n \colon = \nu_1 \vee \nu_2 \vee \ldots \vee \nu_n. \tag{3.1}$$

$\{\varphi_n\}$ is an increasing sequence in $\Gamma(\mu)$. Indeed, for any two measures τ and χ in $\Gamma(\mu)$, $(\tau \vee \chi)(A) \leq \mu(A)$ $\forall A \in \mathfrak{B}$.

Since $\varphi_n(A) \leq \mu(A)$, $\forall A \in \mathfrak{B}$, and φ_n is increasing, it converges *setwise* to an element $\varphi \leq \mu$. That φ is a (σ-additive) measure follows from the fact that $\{\varphi_n\}$ is an increasing sequence (see e.g., [2]). It follows that $\varphi \in \Gamma(\mu)$ and $\varphi(X) = \delta$. We now prove that φ is a maximal element of $\Gamma(\mu)$.

Consider any element $\chi \in \Gamma(\mu)$. Assume that there is some $A \in \mathfrak{B}$ such that $\chi(A) > \varphi(A)$. Let $\tau \colon = \chi \vee \varphi$. From the Hahn-Jordan decomposition of $(\chi - \varphi)$, $\exists X_1, X_2$ with $X_1 \cup X_2 = X$, $X_1 \cap X_2 = \emptyset$ so that

$$\tau(A) = \chi(A \cap X_1) + \varphi(A \cap X_2), \quad A \in \mathfrak{B}.$$

Thus, $\chi(X_1) > \varphi(X_1)$ and, therefore,

$$\tau(X) = \chi(X_1) + \varphi(X_2) > \varphi(X_1 \cup X_2) = \delta,$$

is a contradiction with $\tau \in \Gamma(\mu)$ and $\delta = \max\{\nu(X) \mid \nu \in \Gamma(\mu)\}$. Hence, $\chi \leq \varphi$. In fact, φ is the σ-additive part in the decomposition of μ into a σ-additive part μ_c and a purely finitely additive part μ_p, with $\mu = \mu_c + \mu_p$ (see [6]).

To prove *(ii)*, note that if $0 \leq \mu \leq \psi$, where ψ is σ-additive, then μ is σ-additive (see e.g., [3], [6]). Indeed, for any decreasing sequence of sets $\{A_n\}$ in $\mathfrak{B}$ with $A_n \downarrow \emptyset$, we have $0 \leq \mu(A_n) \leq \psi(A_n) \downarrow 0$ which implies $\mu(A_n) \downarrow 0$, i.e., μ is σ-additive. □

Lemma 3.2: *Let $\{\mu_n\}$ be a sequence of (nonnegative) σ-additive measures on $(X, \mathfrak{B})$ with $\sup_n \mu_n(X) < \infty$. Then,*

$$O - \liminf_{n \to \infty} \mu_n \colon = \bigvee_{k \geq 1} \bigwedge_{n \geq k} \mu_n \text{ is a (finite) } \sigma\text{-additive measure.} \tag{3.2}$$

If $\mu_n \leq \nu (\in M(X)) \forall n$, then,

$$O - \limsup_{n \to \infty} \mu_n \colon = \bigwedge_{k \geq 1} \bigvee_{n \geq k} \mu_n \text{ is a (finite) } \sigma\text{-additive measure.} \tag{3.3}$$

Proof: Let $\varphi_{kn} \colon = \mu_k \wedge \mu_{k+1} \wedge \ldots \wedge \mu_n$. $\{\varphi_{kn}\}_{n \geq 0}$ is a decreasing sequence so that it converges setwise to a (finite) σ-additive measure φ_k (see e.g., [4]). In turn, $\{\varphi_k\}$ is an increasing sequence with $\sup_k \varphi_k(X) < \infty$. Hence, φ_k converges setwise to a (finite) σ-additive measure and (3.2) follows from

$$\bigvee_{k\geq 1}\ \bigwedge_{n\geq k}\mu_n=\lim_{k\to\infty}\varphi_k.$$

(3.3) follows from $0\leq\mu_n\leq\nu$ for all n and the fact that $M(X)$ is a complete Banach lattice (see e.g., [5]). □

In the sequel, to avoid confusion, the reader should be careful in distinguishing

$$(O-\liminf_{n\to\infty}\mu_n)(B)\ \text{ from } \liminf_{n\to\infty}\mu_n(B)\qquad B\in\mathfrak{B},$$

for we have in fact,

$$(O-\liminf_{n\to\infty}\mu_n)(B)\leq\liminf_{n\to\infty}\mu_n(B)\quad B\in\mathfrak{B}.$$

4. Setwise Convergence

We now give sufficient conditions for setwise convergence of an F-converging sequence, where F is a subset of $\mathcal{M}b(X)$ separating points of $M(X)$ and such that $\mu\in M(X)$ and $0\leq\int fd\mu\ \forall 0\leq f\in F$ imply $\mu\geq 0$. Typical examples of such F are $C(X)$ for a topological space X and $C_c(X)$ or $C_0(X)$ for a locally compact Hausdorff space X.

Lemma 4.1: *Let F be a subspace of $\mathcal{M}b(X)$ separating points in $M(X)$ and such that for every $\mu\in M(X)$, $0\leq\int fd\mu\quad\forall 0\leq f\in F$ yields that $\mu\geq 0$. Let $\{\mu_n\}$ be a sequence of (nonnegative) measures on $(X,\mathfrak{B})$ with $\sup_n\mu_n(X)<\infty$. Assume that $\mu_n\overset{F}{\to}\mu\in M(X)$, i.e.,*

$$\int fd\mu_n\to\int fd\mu\quad\forall f\in F.\tag{4.1}$$

(i) If $(O-\liminf_n\mu_n)(X)=\mu(X)$ then $\mu_n\overset{setwise}{\to}\mu$.

(ii) If for some $\nu\in M(X)$, $\mu_n\leq\nu\ \forall n$, then $\mu_n\overset{setwise}{\to}\mu$.

(iii) If $(O-\liminf_n\mu_n)(X)=\mu(X)=(O-\limsup_n\mu_n)(X)$ then $\mu_n\overset{TV}{\to}\mu$.

Proof: (i) Since $\sup_n\mu_n(X)<\infty$, the sequence $\{\mu_n\}$ is in a weak* compact set in $\mathcal{M}b(X)'$. Thus, there is a directed set D and a subnet (not a subsequence in general) $\{\mu_{n_\alpha},\alpha\in D\}$ that converges to some $0\leq\varphi$ in $\mathcal{M}b(X)'$ for the weak* topology in $\mathcal{M}b_b(X)'$, and φ is a finitely additive measure. From (3.2) in Lemma 3.2, $O-\liminf_n\mu_n$ exists and $\varphi\geq O-\liminf_n\mu_n$. Now, φ has a unique decomposition into a σ-additive (nonnegative) part φ_c and a purely finitely additive (nonnegative) part φ_p with $\varphi=\varphi_c+\varphi_p$ (see e.g., [6]).

From σ-additivity of $O-\liminf_n\mu_n$ and $O-\liminf_n\mu_n\leq\varphi$, we have that $O-\liminf_n\mu_n\leq\varphi_c$ since φ_c is a maximal element of $\Gamma(\varphi)$ (see the proof of Lemma 3.1). Therefore, $\varphi_c(X)\geq(O-\liminf_n\mu_n)(X)=\mu(X)$. In addition,

$$\langle\mu,f\rangle=\langle\varphi_c,f\rangle+\langle\varphi_pf\rangle\quad\forall f\in F.$$

In particular, for $f\geq 0$ in F,

$$\int fd(\mu-\varphi_c)\geq 0\quad\forall 0\leq f\in F.$$

Thus, $\mu\geq\varphi_c$ and $\mu(X)=\varphi_c(X)$ which in turn implies $\varphi_c=\mu$ and $\varphi_p(X)=0$, i.e., φ is σ-additive and $\varphi=\mu$. As φ was an arbitrary weak* accumulation point, all the weak* accumulation points are identical and equal to μ, i.e., $\mu_n\to\mu$ for the weak* topology in $\mathcal{M}b(X)'$ and, in particular, $\mu_n \overset{setwise}{\rightarrow} \mu$.

(*ii*) As $\mu_n \leq \nu$ for all n, the sequence $\{\mu_n\}$ is in a weakly sequentially compact set of $M(X)$. Indeed, for all n, μ_n are norm-bounded and the σ-additivity of μ_n is uniform in n (if $A_k \downarrow \emptyset$, $\nu(A_k) \rightarrow 0$ so that $\mu_n(A_k)$ ($\leq \nu(A_k)) \rightarrow 0$ uniformly). Therefore, from Theorem 2, p. 306 in [3], $\{\mu_n\}$ forms a weakly sequentially compact set in $M(X)$. Hence, there is a subsequence $\{\mu_{n_k}\}$ that converges weakly to some $\varphi \in M(X)$, and, in particular,

$$\int f d\mu_{n_k} \rightarrow \int f d\varphi \quad \forall f \in \mathcal{M}b(X), \tag{4.2}$$

so that, from the F-convergence of μ_n to μ,

$$\int f d\mu = \int f d\varphi \quad \forall f \in F. \tag{4.3}$$

As both φ and μ are in $M(X)$ and F separates points in $M(X)$, (4.3) implies $\mu = \varphi$. As φ was an arbitrary weak-limit point of $\{\mu_n\}$ in $M(X)$, we also conclude that all the weak-limit points are all equal to μ. In other words, $\mu_n \overset{setwise}{\rightarrow} \mu$.

(*iii*) From (*i*) we conclude that $O-\liminf_n \mu_n = \mu$ and with similar arguments, $O-\limsup_n u_n = \mu$, i.e., the sequence $\{\mu_n\}$ in $M(X)$ has an O-limit μ, or equivalently (see e.g., [5]), there exists $\{w_n\}$ in $M(X)$ such that

$$|\mu_n - \mu| \leq w_n \text{ with } w_n \downarrow 0.$$

Clearly, this implies convergence in total variation since

$$|\mu_n - \mu|(X) \leq w_n(X) \text{ with } w_n(X) \downarrow 0. \qquad \square$$

Lemma 4.1 applies to the following situations

- X is a locally Hausdorff space and $\mu_n \overset{vaguely}{\rightarrow} \mu \in M(X)$, i.e.,

$$\int f d\mu_n \rightarrow \int f d\mu \quad \forall f \in C_0(X) = : F \tag{4.4}$$

or if

$$\int f d\mu_n \rightarrow \int f d\mu \quad \forall f \in C_c(X) = : F \tag{4.5}$$

- X is a topological space and $\mu_n \overset{weakly}{\rightarrow} \mu \in M(X)$, i.e.,

$$\int f d\mu_n \rightarrow \int f d\mu \quad \forall f \in C(X) = : F. \tag{4.6}$$

As a consequence of Lemma 4.1 we also get:

Corollary 4.2: *Let X be a locally compact Hausdorff space and λ a σ-finite measure on $(X, \mathfrak{B})$. Consider a sequence of probability densities $\{f_n\} \in L_1(\lambda)$ with almost everywhere limit $f \in L_1(\lambda)$. Let*

$$\mu(B) := \int_B f d\lambda, \ \mu_n(B) := \int_B f_n d\lambda \quad B \in \mathfrak{B}, \ n = 1, \ldots. \tag{4.7}$$

Assume that $\mu_n \overset{vaguely}{\rightarrow} \mu$. If $\mu_n \leq \nu$ for some $\nu \in M(X)$ then:

$$\int | f_n - f | d\lambda \to 0 \text{ as } n \to \infty \tag{4.8}$$

and $\mu_n \xrightarrow{TV} \mu$. *In addition, if* λ *is finite, the family* $\{f_n\}$ *is uniformly integrable.*

Proof: Since the μ_n's are order-bounded, from Lemma 4.1(ii) with $F: = C_0(X)$, we conclude that $\mu_n \overset{setwise}{\to} \mu$. In particular, this implies that

$$\int f_n d\lambda \to \int f d\lambda \text{ as } n \to \infty,$$

and by Scheffe's Theorem,

$$\int | f_n - f | d\lambda \to 0 \text{ as } n \to \infty,$$

which yields (4.8). That $\mu_n \xrightarrow{TV} \mu$ follows from the L_1 convergence of f_n to f, i.e., (4.8). If λ is finite, the uniform integrability of the family $\{f_n\}$ follows from [2], p. 155. □

Note that if instead of $\mu_n \leq \nu$, we had $| f_n | \leq g \in L_1(\lambda)$ then by the Dominated Convergence Theorem, $\int f_n d\lambda \to \int f d\lambda$ and (4.8) would follow from Scheffe's Theorem. However, note that the condition $\mu_n \leq \nu$ does not require ν to have a density.

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