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AN ITERATIVE ALGORITHM ON FIXED POINTS OF RELAXED LIPSCHITZ OPERATORS

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Fixed points of Lipschitzian relaxed Lipschitz operators based on a generalized iterative algorithm are approximated.

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1. Introduction

Recently, Wittman [6, Theorem 2], using an iterative procedure

$$x_n = (1 - a_n)x_0 + a_n T x_{n-1} \text{ for } n \geq 1, \tag{1}$$

approximated fixed points of nonexpansive mappings $T: K \rightarrow K$ from a nonempty closed convex subset K of a real Hilbert space H into itself, where x_0 is an element of K and $\{a_n\}$ is an increasing sequence in $[0,1)$ such that

$$\lim_{n \rightarrow \infty} a_n = 1 \text{ and } \sum_{n=1}^{\infty} (1 - a_n) = \infty. \tag{2}$$

This result refines a number of results including [1].

Here our aim is to approximate the fixed points of Lipschitzian relaxed Lipschitz operators in a Hilbert space setting. As such, the iterative algorithm (1) is not suitable for our purpose, so we apply a modified iterative algorithm which reduces to (1).

Let H be a Hilbert space and $\langle u, v \rangle$ and $\| u \|$ denote, respectively, the inner product and norm on H for u, v in H.

An operator $T: H \rightarrow H$ is said to be *relaxed Lipschitz* if, for all u, v in H, there exists a constant $r > 0$ such that

$$\langle Tu - Tv, u - v \rangle \leq -r \| u - v \|^2. \tag{3}$$

The operator T is called *Lipschitz continuous* (or Lipschitzian) if there exists a constant $s > 0$ such that

$$\| Tu - Tv \| \leq s \| u - v \| \text{ for all } u, v \text{ in } H. \tag{4}$$

Next, we consider the main result on the approximation of the fixed points of Lipschitzian relaxed Lipschitz operators using a modified iterative algorithm which contains a number of iterative schemes including those considered by the author [4, 5] as special cases.

2. The Main Result

Theorem 1: *Let H be a real Hilbert space and K be a nonempty closed convex subset of H. Let $T: K \rightarrow K$ be a relaxed Lipschitz and Lipschitz continuous operator on K. Let $r \geq 0$ and $s \geq 1$ be constants for relaxed Lipschitzity and Lipschitz continuity of T, respectively. Let $F = \{x \text{ in } K: Tx = x\}$ be nonempty, and let $\{a_n\}$ be a sequence in $[0,1]$ such that*

$$\sum_{n=0}^{\infty} a_n = \infty \text{ for all } n \geq 0. \tag{5}$$

Then for any x_0 in K the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1 - a_n)x_n + a_n[(1-t)x_n + tTx_n] \text{ for } n \geq 0, \tag{6}$$

$0 < k = ((1-t)^2 - 2t(1-t)r + t^2 s^2)^{1/2} < 1$ *for all t such that* $0 < t < 2(1+r)/(1 + 2r + s^2)$ *and $r \leq s$, converges to an element of F.*

For $\{a_n\} = 1$, Theorem 1 reduces to:

Corollary 1: *Let $T: K \rightarrow K$ be relaxed Lipschitz and Lipschitz continuous. Let $F = \{x \text{ in } K: Tx = x\}$ be a nonempty set. Then, for x_0 in K, the sequence $\{x_n\}$ generated by an iterative algorithm*

$$x_{n+1} = (1-t)x_n + tTx_n \tag{7}$$

for $0 < t < 2(1+r)/(1 + 2r + s^2)$ converges to a unique fixed point of T.

Proof of Theorem 1: For an element z in F, we have

$$\| x_{n+1} - z \| = \| (1-a_n)x_n + a_n[(1-t)x_n + tTx_n] - z \|$$

$$\leq (1 - a_n) \| (x_n - z) \| + a_n \| (1-t)(x_n - z) + t(Tx_n - Tz) \| .$$

Using the relaxed Lipschitzity and Lipschitz continuity of T, we find that

$$\| t(Tx_n - Tz) + (1-t)(x_n - z) \|^2$$

$$= (1-t)^2 \| x_n - z \|^2 + 2t(1-t)\langle Tx_n - z, x_n - z \rangle + t^2 \| Tx_n - z \|^2$$

$$\leq (1-t)^2 \| x_n - z \|^2 - 2t(1-t)r \| x_n - z \|^2 + t^2 s^2 \| x_n - z \|^2$$

$$= ((1-t)^2 - 2t(1-t)r + t^2s^2)\,\|x_n - z\|^2.$$

It follows that

$$\|x_{n+1} - z\| \le (1 - a_n + a_n((1-t)^2 - 2t(1-t)r + t^2s^2)^{1/2})\,\|x_n - z\|$$
$$= (1 - (1-k)a_n)\,\|x_n - z\|$$
$$\le \prod_{j=0}^{n} (1 - (1-k)a_j)\,\|x_0 - z\|,$$

where $0 < k = ((1-t)^2 - 2t(1-t)r + t^2s^2)^{1/2} < 1$ for all t such that $0 < t < 2(1+r)/(1+2r+s^2)$ and $r \le s$.

Since $\sum_{j=0}^{\infty} a_j$ diverges and $k < 1$, $\lim_{n\to\infty} \sum_{j=0}^{n} (1-(1-k)a_j) = 0$ and, as a result, $\{x_n\}$ converges strongly to z. This completes the proof.

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