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A STRONG LAW OF LARGE NUMBERS FOR HARMONIZABLE ISOTROPIC RANDOM FIELDS

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The class of harmonizable fields is a natural extension of the class of stationary fields. This paper considers a strong law of large numbers for the spherical average of a harmonizable isotropic random field.

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1. Introduction

Isotropic random fields play a key role in the statistical theory of turbulence. In addition to the assumption of isotropy, these fields have classically been considered stationary. However, there are applications under which the assumption of stationarity is not physically realistic, e.g. detection of a phase modulated signal. Harmonizable fields provide a natural extension to the stationary class by retaining the powerful Fourier analytic techniques while relaxing the assumption of stationarity.

This paper recalls the necessary theory of harmonizable isotropic random fields and obtains conditions for a strong law of large numbers to be valid for the spherical average of a harmonizable isotropic random field. Yadrenko [14] obtained a similar result for stationary isotropic random fields.

2. Preliminaries

To introduce the desired class of random functions, recall that if a random field $X:\mathbb{R}^n \to L_0^2(P)$ is stationary then it can be expressed as

$$X(t) = \int_{\mathbb{R}^n} e^{i\lambda t} dZ(\lambda), \tag{1}$$

where $Z(\cdot)$ is a σ-additive stochastic measure on the Borel σ-algebra $\mathfrak{B}$ of $\mathbb{R}^n$, with orthogonal values in the complex Hilbert space, $L_0^2(P)$, of centered random variables. The covariance, $r(\cdot,\cdot)$, of the field is

$$r(s,t) = \int\limits_{\mathbb{R}^n} e^{i(s-t)\lambda} dF(\lambda),$$

where $E(Z(A)\overline{Z(B)}) = F(A \cap B)$, F a positive finite Borel measure on $\mathbb{R}^n$. Here $E(\cdot)$ denotes the expectation.

A generalization of the concept of stationarity is given by fields $X: \mathbb{R}^n \to L_0^2(P)$ with covariance $r(\cdot,\cdot)$ expressible as

$$r(s,t) = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{i\lambda s - i\lambda' t} dF(\lambda,\lambda'),$$

where $F(\cdot,\cdot)$ is a complex bimeasure, called the *spectral bimeasure* of the field, of bounded variation in the Vatiali's sense or more inclusively in Fréchet's sense; in which case the integrals are strict Morse-Transue (cf. Rao [5] and Chang and Rao [1]). The covariance as well as the field are termed *strongly* or *weakly harmonizable* respectively. Every weakly or strongly harmonizable field $X: \mathbb{R}^n \to L^2(P)$ has an integral representation given by (1), where $Z: \mathfrak{B} \to L^2(P)$ is a stochastic measure (not necessarily with orthogonal values) and is called the *spectral measure* of the field. Both of these concepts reduce to the stationary case if F concentrates on the diagonal $\lambda = \lambda'$ of $\mathbb{R}^n \times \mathbb{R}^n$.

A subclass of random fields satisfy an additional condition called *isotropy.* Isotropic random fields $X(\cdot)$, have covariance $r(\cdot,\cdot)$ which are invariant under rotation and reflection. Isotropic fields play an important role in the statistical theory of turbulence, where direction in space is unimportant (cf. Yaglom [15]). Swift [10] obtained a representation of a weakly harmonizable isotropic covariance as

$$r(s,t) = 2^{\nu}\Gamma\left(\frac{n}{2}\right)\int_0^{\infty}\int_0^{\infty} \frac{J_{\nu}(\|\lambda s - \lambda' t\|)}{\|\lambda s - \lambda' t\|^{\nu}} dF(\lambda,\lambda') \tag{2}$$

where $J_{\nu}(\cdot)$ is the Bessel function (of the first kind) of order $\nu = (n-2)/2$ and $F(\cdot,\cdot)$ is a complex function of bounded Fréchet variation, with $\|\cdot\|$ denoting the vector norm.

Isotropic covariances $r(s,t)$ are functions of the lengths $\|s\|$, $\|t\|$ of the vectors s,t and of the angle θ between s and t. A representation in spherical-polar form for the covariances of harmonizable isotropic random fields was obtained by Swift as

$$r(s,t) = \alpha_n^2 \sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)} S_m^l(u) S_m^l(v) \int_0^{\infty}\int_0^{\infty} \frac{J_{m+\nu}(\lambda\tau_1)J_{m+\nu}(\lambda'\tau_2)}{(\lambda\tau_1)^{\nu}(\lambda'\tau_2)^{\nu}} dF(\lambda,\lambda') \tag{3}$$

where

(*i*) $s = (\tau_1, u), t = (\tau_2, v)$ are the spherical polar coordinates of s,t in $\mathbb{R}^n$, here $\tau_1 = \|s\|$, $\tau_2 = \|t\|$ and $u = s/\tau_1$, $v = t/\tau_2$ are unit vectors.

(*ii*) $S_m^l(\cdot)$, $1 \le l \le h(m,n) = (2m+2\nu)(m+2\nu-1)!m!$, $m \ge 1$, $S_0^l(u) = 1$ are the spherical harmonics on the unit n-sphere of order m.

(*iii*) $\alpha_n > 0$, $\alpha_n^2 = 2^{2\nu+1}\Gamma(\frac{n}{2})\pi^{\frac{n}{2}}$, with $F(\cdot,\cdot)$ as a complex function of bounded Fréchet variation.

Using (3) and Karhunen's Theorem, the spectral representation for a weakly harmonizable isotropic random field is given as

$$X(t) = \alpha_n \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} S_m^l(u) \int_0^{\infty} \frac{J_{m+\nu}(\lambda\tau)}{(\lambda\tau)^{\nu}} dZ_m^l(\lambda) \tag{4}$$

where $Z_m^l(\cdot)$ satisfies

$$E(Z_m^l(B_1)\overline{Z_{m'}^l(B_2)}) = \delta_{mm'}\delta_{ll'}F(B_1, B_2)$$

with $F(\cdot,\cdot)$ a function of bounded Fréchet variation with the stochastic integral being in the Dunford-Schwartz sense, ([2], IV.10) and with $\delta_{mm'}$ the Kronecker delta.

The theory of harmonizable random fields applicable to the statistical theory of turbulence is being developed by Rao, Swift, and others. The papers of Rao [7, 8] obtain representations for harmonizable isotropic random fields and their application to some sampling and prediction problems. The local behavior of some classes of harmonizable isotropic random fields has been considered by Swift [11-13]. Asymptotic properties of bispectral density estimators have recently been considered by H. Soedjak [9]. The book by Kakihara [3] gives a general treatment of multidimensional second order processes which include the harmonizable class.

3. Laws of Large Numbers

Let $X:\mathbb{R}^n \to L_0^2(P)$ be a strongly harmonizable isotropic random field and let γ_R be the average over $B_R^n = \{t: \|t\| \leq R\}$, the ball of radius R centered at the origin. Thus

$$\begin{aligned}\gamma_R &= \frac{1}{Vol(B_R^n)} \int_{B_R^n} X(t)dt \\ &= \frac{n\alpha_n}{\sqrt{\omega_n}R^n} \int_0^R \tau^{n-1} \int_0^{\infty} \frac{J_{\nu}(\lambda\tau)}{(\lambda\tau)^{\nu}} dZ_0^1(\lambda)d\tau, \end{aligned} \tag{5}$$

where (5) follows from (4) and the orthonormality of the spherical harmonics on B_R^n (cf. Lebedev [4]). To obtain a law of large numbers for γ_R, one must show the variance of γ_R is uniformly bounded, (cf. Rao [6]). Since $X(t)$ has zero mean, the variance of γ_R is given as

$$\begin{aligned}E(\gamma_R^2) &= \frac{n^2\alpha_n^2}{\omega_n R^{2n}} \int_0^R \int_0^R \tau_1^{n-1}\tau_2^{n-1} \int_0^{\infty}\int_0^{\infty} \frac{J_{\nu}(\lambda\tau_1)J_{\nu}(\lambda'\tau_2)}{(\lambda\tau_1)^{\nu}(\lambda'\tau_2)^{\nu}} dF(\lambda,\lambda')d\tau_1 d\tau_2 \\ &= \frac{n^2\alpha_n^2}{\omega_n R^{2n}} \int_0^{\infty}\int_0^{\infty}\int_0^R\int_0^R \tau_1^{n/2}\tau_2^{n/2} \frac{J_{\nu}(\lambda\tau_1)J_{\nu}(\lambda'\tau_2)}{(\lambda\lambda')^{\nu}} d\tau_1 d\tau_2 dF(\lambda,\lambda'). \end{aligned} \tag{6}$$

The relation (cf. Lebedev [4])

$$|J_{\nu}(x)| < \frac{C_{\nu}}{x^{1/2}}, \tag{7}$$

which is valid for all x with C_ν being a constant which only depends upon ν, implies

$$\left| \int_0^R \tau^{n/2} \frac{J_\nu(\lambda\tau)}{\lambda^\nu} d\tau \right| \leq \int_0^R \tau^{n/2} \frac{|J_\nu(\lambda\tau)|}{\lambda^\nu} d\tau$$

$$\leq \int_0^R \tau^{n/2} \frac{C_\nu}{\lambda^\nu (\lambda\tau)^{1/2}} d\tau = \frac{2C_\nu R^{\frac{n+1}{2}}}{(n+1)\lambda^{\frac{n-1}{2}}}.$$

Using this expression together with (6) gives

$$|E(\gamma_R^2)|$$

$$\leq \frac{n^2\alpha_n^2}{\omega_n R^{2n}} \int_0^\infty \int_0^\infty \int_0^R \int_0^R \tau_1^{n/2} \tau_2^{n/2} \frac{|J_\nu(\lambda\tau_1)|\,|J_\nu(\lambda'\tau_2)|}{(\lambda\lambda')^\nu} d\tau_1 d\tau_2 \, |dF(\lambda,\lambda')|$$

$$< \frac{n^2\alpha_n^2}{\omega_n R^{2n}} \int_0^\infty \int_0^\infty \frac{C_\nu D_\nu}{\lambda^{\frac{n-1}{2}} (\lambda')^{\frac{n-1}{2}}} \frac{4R^{\frac{n}{2}} R^{\frac{n}{2}}}{n^2} |dF(\lambda,\lambda')|$$

$$= \frac{4\alpha_n^2 C_\nu D_\nu n^2}{\omega_n R^{n-1}(n+1)^2} \int_0^\infty \int_0^\infty \frac{|dF(\lambda,\lambda')|}{(\lambda\lambda')^{\frac{n-1}{2}}} \tag{8}$$

where C_ν and D_ν are constants given by (7). Thus $\gamma_R \to 0$ in $L^2(P)$ as $R \to \infty$ if and only if

$$\int_0^\infty \int_0^\infty \frac{|dF(\lambda,\lambda')|}{(\lambda\lambda')^{\frac{n-1}{2}}} < \infty.$$

The preceding is summarized in the following proposition.

Proposition 3.1: *The spherical average*

$$\gamma_R = \frac{1}{Vol(B_R^n)} \int_{B_R^n} X(t)dt$$

of a strongly harmonizable isotropic random field $X(\cdot)$ over $B_R^n = \{t\colon \|t\| \leq R\}$, the ball of radius R centered at the origin, satisfies the weak law of large numbers if and only if

$$\int_0^\infty \int_0^\infty \frac{|dF(\lambda,\lambda')|}{(\lambda\lambda')^{\frac{n-1}{2}}} < \infty. \tag{9}$$

There are many harmonizable isotropic random fields which satisfy condition (9), for instance, the field constructed by Swift [10]. In particular, Swift shows that if the spectral bimeasure $F(\cdot,\cdot)$ is absolutely continuous, with density f, having the specific form

$$f(\lambda,\lambda') = \frac{\omega_n \nu \lambda^{m+2\nu+1}(\lambda')^{m+2\nu+1}}{\alpha_n^2(m+\nu)} e^{(-(\lambda^2+(\lambda')^2))} \tag{10}$$

which is positive definite and where as before ω_n is the surface area of the unit sphere and $\nu = (n-2)/2$, then

$$r(\tau_1,\tau_2,\theta) = [1 - 2\tau_1\tau_2\cos\theta + \tau_1^2\tau_2^2]^{-\nu} e^{(-(r_1^2+r_2^2))}$$

is the covariance of a strongly harmonizable isotropic random field. Using (10) with $m = 1$ and simplifying (9) one obtains

$$\int_0^\infty \int_0^\infty \frac{|dF(\lambda,\lambda')|}{(\lambda\lambda')^{\frac{n-1}{2}}} = \left(\Gamma\left(\frac{n+1}{4}\right)\right)^2 < \infty.$$

The condition (9) is also enough to guarantee that $\gamma_R \to \infty$ almost everywhere as $R \to \infty$, so that a strong law of large numbers prevails for the spherical average of a strongly harmonizable isotropic random field. In particular, the following theorem will now be shown.

Theorem 3.1: *If a strongly harmonizable isotropic random field $X: \mathbb{R}^n \to L_0^2(P)$ has a spectral bimeasure $F(\cdot,\cdot)$ which satisfies the inequality (9), then the spherical average γ_R satisfies the strong law of large numbers.*

Proof: The strong law of large numbers will first be shown to hold for a sequence of a radii $R_k = k^\delta$, where $0 < \delta < 1/(3n-1)$. Equation (8) implies that

$$\sum_{k=1}^{\infty} E(\gamma_{R_k}^2) < \frac{4\alpha_n^2 C_\nu D_\nu n^2}{\omega_n(n+1)^2} \sum_{k=1}^{\infty} \frac{1}{k^{n\delta}} < \infty, \tag{11}$$

hence $\gamma_{R_k} \to 0$ almost everywhere as $k \to \infty$.

Now letting

$$Y_m^l(\tau) = \alpha_n \int_0^\infty \frac{J_{m+\nu}(\lambda\tau)}{(\lambda\tau)^\nu} dZ_m^l(\lambda)$$

and

$$d_k = \sup_{R_k \le R \le R_{k+1}} |\gamma_R - \gamma_{R_k}|$$

one obtains

$$d_k \le \frac{n}{\sqrt{\omega_n}} \left[\left(\frac{1}{R^n} - \frac{1}{R_k^n}\right) \int_0^{R_k} \tau^{n-1} |Y_0^1(\tau)| \, d\tau + \frac{1}{R^n} \int_{R_k}^{R} \tau^{n-1} |Y_0^1(\tau)| \, d\tau \right]$$

$$\le \frac{n}{\sqrt{\omega_n}} \left[\frac{R_{k+1}^n - R_k^n}{R_k^{2n}} \int_0^{R_k} \tau^{n-1} |Y_0^1(\tau)| \, d\tau + \frac{1}{R_k^n} \int_{R_k}^{R_{k+1}} \tau^{n-1} |Y_0^1(\tau)| \, d\tau \right].$$

Applying the elementary inequality $(a+b)^2 \le 2(a^2+b^2)$ gives

$$d_k^2 \le \frac{2n^2}{\omega_n^2} \left[\frac{(R_{k+1}^n - R_k^n)^2}{R_k^{4n}} \int_0^{R_k} \int_0^{R_k} \tau_1^{n-1} \tau_2^{n-1} |Y_0^1(\tau_1)| \, |Y_0^1(\tau_2)| \, d\tau_1 d\tau_2 \right.$$

$$+\frac{1}{R_k^{2n}}\int_0^{R_k}\int_0^{R_k}\tau_1^{n-1}\tau_2^{n-1}\,|\,Y_0^1(\tau_1)\,|\,|\,Y_0^1(\tau_2)\,|\,d\tau_1 d\tau_2\Bigg]. \tag{12}$$

Now an application of the Cauchy-Buniakowski-Schwartz inequality yields

$$E\,|\,Y_0^1(\tau_1)\,|\,|\,Y_0^1(\tau_2)\,|\le \alpha_n^2\sqrt{\left|\int_0^{\infty}\int_0^{\infty}\frac{J_\nu(\lambda\tau_1)J_\nu(\lambda'\tau_1)}{(\tau_1^2\lambda\lambda')^\nu}dF(\lambda,\lambda')\right|}$$

$$\times\sqrt{\left|\int_0^{\infty}\int_0^{\infty}\frac{J_\nu(\lambda\tau_2)J_\nu(\lambda'\tau_2)}{(\tau_2^2\lambda\lambda')^\nu}dF(\lambda,\lambda')\right|},$$

but the inequality (7) implies

$$\left|\int_0^{\infty}\int_0^{\infty}\frac{J_\nu(\lambda\tau_1)J_\nu(\lambda'\tau_1)}{(\tau_1^2\lambda\lambda')^\nu}dF(\lambda,\lambda')\right|<\frac{C_\nu D_\nu}{\tau_1^{2\nu+1/4}\tau_2^{\nu+1/4}}\int_0^{\infty}\int_0^{\infty}\frac{|\,dF(\lambda,\lambda')\,|}{(\lambda\lambda')^{\nu+1/2}}.$$

so that

$$E\,|\,Y_0^1(\tau_1)\,|\,|\,Y_0^1(\tau_2)\,|<\frac{\alpha_n^2C_\nu^{1/2}D_\nu^{1/2}}{\tau_1^{\nu+1/4}\tau_2^{\nu+1/4}}\int_0^{\infty}\int_0^{\infty}\frac{|\,dF(\lambda,\lambda')\,|}{(\lambda\lambda')^{\nu+1/2}}.$$

Taking the expectation of (12) gives

$$E(d_k^2)\le\frac{2n^2}{\omega_n^2}\left[\frac{(R_{k+1}^n-R_k^n)^2}{R_k^{4n}}\int_0^{R_k}\int_0^{R_k}\tau_1^{n-1}\tau_2^{n-1}E\,|\,Y_0^1(\tau_1)\,|\,|\,Y_0^1(\tau_2)\,|\,d\tau_1d\tau_2\right.$$

$$\left.+\frac{1}{R_k^{2n}}\int_0^{R_k}\int_0^{R_k}\tau_1^{n-1}\tau_2^{n-1}E\,|\,Y_0^1(\tau_1)\,|\,|\,Y_0^1(\tau_2)\,|\,d\tau_1d\tau_2\right]$$

$$<M\left[\frac{(R_{k+1}^n-R_k^n)^2}{R_k^{4n}}R_k^{3/2}+\frac{(R_{k+1}-R_k)^{3/2}}{R_k^{4n}}\right]$$

where M is the finite constant given by

$$M=\tfrac{2}{3}\alpha_n^2C_\nu^{1/2}D_\nu^{1/2}\int_0^{\infty}\int_0^{\infty}\frac{|\,dF(\lambda,\lambda')\,|}{(\lambda\lambda')^{\nu+1/2}}.$$

Replacing R_k with k^δ and simplifying one obtains

$$E(d_k^2)<\frac{A_\delta}{k^{(n-1)\delta}}+\frac{B_\delta}{k^{(3n-1)\delta}},$$

where A_δ and B_δ are finite constants which do not depend upon k. Hence

$$\sum_{k=1}^{\infty} E(d_k^2) < \infty,$$

so that $d_k \rightarrow 0$ almost everywhere as $k \rightarrow \infty$. But this implies $\gamma_R \rightarrow 0$ almost everywhere, as $R \rightarrow \infty$, proving the assertion. □

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