

# GENERALIZED NONLINEAR VARIATIONAL INEQUALITY PROBLEMS INVOLVING MULTIVALUED MAPPINGS

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The solvability of a class of generalized nonlinear variational inequality problems involving multivalued, strongly monotone and strongly Lipschitz (a special type) operators, which are closely associated with generalized nonlinear complementarity problems, is discussed.

**Key words:** Generalized Nonlinear Variational Inequality, Strongly Lipschitz Operator, Strongly Monotone Operator.

**AMS subject classifications:** 47H15.

## 1. Introduction

Variational inequalities and complementarity problems play equally important roles in applied mathematics, physics, control theory and optimization, equilibrium theory of transportation and economics, mechanics, and engineering sciences. These problems, especially variational inequality problems, are studied in convex sets, while complementarity problems are approached in convex cone settings leading to equivalences. The complementarity problem in mathematical programming is based on a special type of variational inequality in finite dimensions that has been central to the development of important algorithms. General variational inequalities can be reduced to this special type by an application of discretization and the introduction of Lagrange multipliers leading to a computational approach. There are situations where computational methods for variational inequalities have an edge over the complementarity. For more details on variational inequalities, we advise the reader to refer to [1, 3-5, 14-16].

### 2. Preliminaries

Let  $H$  be a real Hilbert space and  $H^*$  its dual with the inner product  $\langle u, v \rangle$  and norm  $\|u\|$  for  $u, v$  in  $H$ . Let  $[w, u]$  denote the duality pairing between the element  $w$  in  $H^*$  and the element  $u$  in  $H$ . Let  $f: H^* \rightarrow H$  be a canonical isomorphism from  $H^*$  onto  $H$  defined by

$$[w, x] = \langle f(w), x \rangle \text{ for all } x \text{ in } H \text{ and all } w \text{ in } H^*. \tag{2.1}$$

Thus,  $\|f\| = \|f^{-1}\| = 1$ .

Let  $T, U: H \rightarrow P(H^*)$  be multivalued mappings from  $H$  into the powerset  $P(H^*)$  of  $H^*$ . Let  $K$  be a nonempty, closed, convex subset of  $H$ . Then the problem of determining the elements  $x$  in  $K$ ,  $u$  in  $T(x)$  and  $v$  in  $U(x)$ , such that

$$[u - v, y - x] \geq 0 \text{ for all } y \text{ in } K, \tag{2.2}$$

is called the generalized nonlinear variational inequality (GNVI) problem.

The following presents a class of generalized nonlinear complementarity (GNC) problems corresponding to the GNVI problem (2.2). Find an element  $x$  in  $K$ , an element  $u$  in  $T(x)$  and an element  $v$  in  $U(x)$  such that

$$u - v \text{ is in } K^* \text{ and } [u - v, x] = 0, \tag{2.3}$$

where  $K^* = \{w \text{ in } H^*: [w, z] \geq 0 \text{ for all } z \text{ in } K\}$ .

For  $T: K \rightarrow H$  single-valued and  $U = 0$ , the GNVI problem (2.2) reduces to the variational inequality problem considered by Yao [13]: find an element  $x$  in  $K$  such that

$$\langle x - Tx, y - x \rangle \geq 0 \text{ for all } y \text{ in } K. \tag{2.4}$$

To this end, let us recall some definitions crucial to the work at hand.

**Definition 2.1:** An operator  $T: H \rightarrow P(H^*)$  from a Hilbert space  $H$  into the powerset  $P(H^*)$  of its dual is said to be *strongly monotone* if, for a constant  $r > 0$  and for all  $x, y$  in  $H$ ,

$$[u - v, x - y] \geq r \|x - y\|^2 \text{ for all } u \text{ in } T(x) \text{ and } v \text{ in } T(y). \tag{2.5}$$

Let  $(X, d)$  be a metric space and  $P(X)$  be the powerset of  $X$ . Then for any  $A, B$  in  $P(X)$ , we define

$$\partial(A, B) = \sup\{d(x, y): x \text{ is in } A, \text{ and } y \text{ is in } B\}. \tag{2.6}$$

A mapping  $F: X \rightarrow P(X)$  is said to be an *s-contraction* if

$$\partial(Fx, Fy) \leq s(d(x, y)) \text{ for all } x, y \text{ in } X. \tag{2.7}$$

**Definition 2.2:** An operator  $T: H \rightarrow P(H^*)$  is said to be *Lipschitz continuous* if there is a constant  $s > 0$  such that for all  $x, y$  in  $H$ ,

$$\partial(Tx, Ty) \leq s \|x - y\| \text{ for } s > 0. \tag{2.8}$$

**Definition 2.3:** An operator  $U: H \rightarrow P(H^*)$  is said to be *strongly Lipschitz* if, for  $x, y$  in  $H$  and  $u$  in  $U(x)$  and  $v$  in  $U(y)$ ,

$$[u - v, x - y] \leq -k \|x - y\|^2, \tag{2.9}$$

where  $k \geq 0$  is arbitrary.

**Definition 2.4:** An operator  $T: H \rightarrow H$  is *hemicontinuous* if the real function

$$t \rightarrow \langle T(x + ty), z \rangle$$

is continuous on  $[0, 1]$  for all  $x, y, z$  in  $H$ .

Let us consider an example of a strongly Lipschitz operator where the constant  $k$  is slightly relaxed [13].

**Example 2.5:** Let  $K$  be a nonempty, closed, convex subset of a real Hilbert space  $H$ . Let  $U: K \rightarrow K$  be hemicontinuous and, for all  $x, y$  in  $K$  and for a real number  $k > -1$ ,

$$\langle Ux - Uy, x - y \rangle \leq -k \|x - y\|^2. \tag{2.10}$$

If we define an operator  $V: K \rightarrow K$  by  $V(z) = (I - U)z$  for all  $z$  in  $K$ , then  $V$  is hemicontinuous and strongly monotone with the strong monotonicity constant  $1 + k$ , and as a result,  $U$  has a unique fixed point in  $K$ .

### 3. Auxiliary and Main Results

Before we consider our main result, we need some auxiliary results.

**Lemma 3.1:** ([5]) *Let  $K$  be a nonempty, closed, convex subset of a real Hilbert space  $H$ . Then, for a given element  $z$  in  $H$ ,  $x - P_K z$  iff*

$$\langle x - z, y - x \rangle \geq 0 \text{ for all } y \in K. \tag{3.1}$$

**Lemma 3.2:** *Let  $K$  be a nonempty, closed, convex subset of  $H$ . Then the GNVI problem (2.2) has a solution iff, for a constant  $t > 0$ , the mapping  $F: H \rightarrow P(H)$  defined by*

$$F(x) = \bigcup_{u \in T(x)} \bigcup_{v \in U(x)} [P_K(x - tf(u - v))], \tag{3.2}$$

*has a fixed point.*

**Proof:** The proof is based on [2, Theorem 3.2]. If  $x_1, u_1$  and  $v_1$  form a solution of the GNVI problem (2.2), then  $x_1$  is in  $K$ ,  $u_1$  is in  $T(x_1)$ , and  $v_1$  is in  $U(x_1)$ , such that

$$[u_1 - v_1, y - x_1] \geq 0 \text{ for all } y \text{ in } K. \tag{3.3}$$

This, in turn, implies that for a constant  $t > 0$ ,

$$\langle x_1 - (x_1 - tf(u_1 - v_1)), y - x_1 \rangle \geq 0 \text{ for all } y \text{ in } K. \tag{3.4}$$

It follows from Lemma 3.1 that

$$x_1 = P_k(x_1 - tf(u_1 - v_1)), \tag{3.5}$$

which is in

$$\bigcup_{u_1 \in T(x_1)} \bigcup_{v_1 \in U(x_1)} [P_K(x_1 - tf(u_1 - v_1))] = F(x_1). \tag{3.6}$$

That is,  $x_1$  is a fixed point of  $F$ .

Conversely, if  $x_1$  is a fixed point of  $F$ , then there exist  $u_1$  in  $T(x_1)$  and  $v_1$  in  $U(x_1)$  such that

$$x_1 = P_k(x_1 - tf(u_1 - v_1)). \tag{3.7}$$

This implies that  $x_1$  is in  $K$ , and by Lemma 3.1, we find

$$\langle x_1 - (x_1 - tf(u_1 - v_1)), y - x_1 \rangle \geq 0 \text{ for all } y \text{ in } K. \tag{3.8}$$

Since  $t > 0$ , it follows that

$$[u_1 - v_1, y - x_1] \geq 0 \text{ for all } y \text{ in } K. \tag{3.9}$$

Hence  $x_1, u_1$  and  $v_1$  form a solution of the GNVI problem (2.2).

**Lemma 3.3:** ([2]) *Let  $(X, d)$  be a complete, metrically convex metric space and let  $F: X \rightarrow P(X)$  be a contraction mapping. Then  $F$  has a fixed point; and for any  $x_0$  in  $X$ , the sequence  $\{x_n\}$ , defined so that  $x_n$  is in  $F(x_{n-1})$  for  $n \geq 1$ , converges to a fixed point of  $F$  in  $X$ .*

**Theorem 3.4:** *Let  $H$  be a real Hilbert space and  $K$  be a nonempty, closed, convex subset of  $H$ . Let  $T: H \rightarrow P(H^*)$  be strongly monotone and Lipschitz continuous with respective constants  $r > 0$  and  $s > 0$ . Let  $U: H \rightarrow P(H^*)$  be strongly Lipschitz and Lipschitz continuous with respective constants  $k \geq 0$  and  $m \geq 1$ . Then the GNVI problem (2.2) has a solution for an arbitrary constant  $t$  such that  $0 < t < 2(r + k)/(s + m)^2$ .*

**Proof:** If we define an operator  $F: K \rightarrow P(K)$  by

$$F(x) = \bigcup_{u \in T(x)} \bigcup_{v \in U(x)} P_K(x - tf(u - v)) \text{ for all } x \text{ in } K, \tag{3.10}$$

then (by Lemma 3.2) it would suffice to show that  $F$  has a fixed point.  $P_K$  is non-expanding,  $T$  is strongly monotone and Lipschitz continuous, and  $U$  is strongly Lipschitz and Lipschitz continuous. Therefore, we find that, for all  $x, y$  in  $K$ ,  $u_1$  in  $T(x)$ ,  $u_2$  in  $T(y)$ ,  $v_1$  in  $U(x)$  and  $v_2$  in  $U(y)$ ,

$$\begin{aligned} & \| P_K[x - tf(u_1 - v_1)] - P_K[y - tf(u_2 - v_2)] \| \\ & \leq \| x - y - (tf(u_1 - u_2) - tf(v_1 - v_2)) \|, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} & \| x - y - (tf(u_1 - u_2) - tf(v_1 - v_2)) \|^2 \\ = & \| x - y \|^2 - 2t \langle x - y, f(u_1 - u_2) - f(v_1 - v_2) \rangle + t^2 \| f(u_1 - u_2) - f(v_1 - v_2) \|^2 \\ = & \| x - y \|^2 - 2t \langle x - y, f(u_1 - u_2) \rangle + 2t \langle x - y, f(v_1 - v_2) \rangle \\ & + t^2 \| f(u_1 - u_2) - f(v_1 - v_2) \|^2 \end{aligned}$$

$$\begin{aligned}
 &= \|x - y\|^2 - 2t[u_1 - u_2, x - y] + 2t[v_1 - v_2, x - y] \\
 &\quad + t^2 \|f(u_1 - u_2) - f(v_1 - v_2)\|^2 \\
 &\leq \|x - y\|^2 - 2tr \|x - y\|^2 - 2tk \|x - y\|^2 + t^2[\|f(u_1 - u_2)\| + \|f(v_1 - v_2)\|]^2 \\
 &\leq \|x - y\|^2 - 2t(r + k) \|x - y\|^2 + t^2[\partial(Tx, Ty) + \partial(Ux, Uy)]^2 \text{ by [6]} \\
 &\leq \|x - y\|^2 - 2t(r + k) \|x - y\|^2 + t^2[s \|x - y\| + m \|x - y\|]^2 \\
 &= \{1 - 2t(r + k) + t^2(s + m)^2\} \|x - y\|^2. \tag{3.12}
 \end{aligned}$$

From (3.11) and (3.12), it follows that

$$\partial(Fx, Fy) \leq L \|x - y\| \text{ for all } x, y \text{ in } K, \tag{3.13}$$

where  $L = (1 - 2t(r + k) + t^2(s + m)^2)^{\frac{1}{2}}$ . Now, under the assumptions,  $0 < L < 1$  for all  $t$  such that  $0 < t < 2(r + k)/(s + m)^2$ . Since each Hilbert space is a metrically convex metric space, it follows from Lemma 3.3 that  $F$  has a fixed point  $x_1$  in  $K$ , and hence  $x_1, u_1$  and  $v_1$  form a solution to the GNVI problem (2.2).

**Theorem 3.5:** *Let  $K$  be a nonempty, closed, convex subset of a real Hilbert space  $h$ . Let  $T: H \rightarrow P(H^*)$  be strongly monotone and Lipschitz continuous with the strong monotonicity constant  $r > 0$  and Lipschitz continuity constant  $s > 0$ . Also, let  $U: H \rightarrow P(H^*)$  be strongly Lipschitz and Lipschitz continuous with strong Lipschitzity constant  $k \geq 0$  and Lipschitz continuity constant  $m \geq 1$ . Consider the sequences  $\{x_n\}$ ,  $\{u_n\}$  and  $\{v_n\}$  as generated by the iterative algorithm defined by*

$$x_{n+1} = (1 - a_n)x_n + a_n P_k(x_n - tf(u_n - v_n)) \text{ for any } x_0 \text{ in } K \tag{3.14}$$

and for all  $t$  such that  $0 < t < 2(r + k)/(s + m)^2$ , where  $u_n$  is in  $T(x_n)$ ,  $v_n$  is in  $U(x_n)$ ,  $0 \leq a_n < 1$ , and the series  $a_0 + a_1 + a_2 + \dots$  is divergent. Then  $\{x_n\}$ ,  $\{u_n\}$  and  $\{v_n\}$  converge to  $\bar{x}$  in  $K$ ,  $\bar{u}$  in  $H^*$  and  $\bar{v}$  in  $H^*$ , respectively, and  $\bar{x}$ ,  $\bar{u}$  and  $\bar{v}$  form a solution of the GNVI problem (2.2).

When  $f, T: H \rightarrow H$  are the identities,  $U: H \rightarrow H$  is single-valued and  $a_n = a > 0$ , then Theorem 3.5 reduces to [13, Theorem 3.6].

**Corollary 3.6:** *Let  $U: H \rightarrow H$  be strongly Lipschitz and Lipschitz continuous with respective constants  $k \geq 0$  and  $m \geq 1$ . Let the sequence  $\{x_n\}$  be generated by an iterative scheme:*

$$x_{n+1} = P_K((1 - a)x_n + aU(x_n)) \text{ for any } x_0 \text{ in } K \tag{3.15}$$

and  $0 < a < 2(1 + k)/(1 + 2k + m^2)$ . Then  $\{x_n\}$  converges to the unique fixed point of  $U$ .

**Proof of Theorem 3.5:** Under the assumptions, it follows from Theorem 3.4 that  $\bar{x}$  in  $K$ ,  $\bar{u}$  in  $T(\bar{x})$  and  $\bar{v}$  in  $U(\bar{x})$  form a solution of the GNVI problem (2.2). Since  $P_k$  is nonexpansive, we have

$$\|x_{n+1} - \bar{x}\| \leq (1 - a_n) \|x_n - \bar{x}\| + a_n \|x_n - \bar{x} - tf(u_n - \bar{u}) + ft(v_n - \bar{v})\|. \tag{3.16}$$

Using the strong monotonicity and Lipschitz continuity of  $T$  and applying the strong Lipschitzity and Lipschitz continuity of  $U$ , we find that

$$\begin{aligned}
 & \| x_n - \bar{x} - tf(u_n - \bar{u}) + tf(v_n - \bar{v}) \|^2 \\
 &= \| x_n - \bar{x} \|^2 - 2t\langle x_n - \bar{x}, f(u_n - \bar{u}) \rangle + 2t\langle x_n - \bar{x}, f(v_n - \bar{v}) \rangle \\
 &\quad + t^2 \| f(u_n - \bar{u}) - f(v_n - \bar{v}) \|^2 \\
 &= \| x_n - \bar{x} \|^2 - 2t[u_n - \bar{u}, x_n - \bar{x}] + 2t[v_n - \bar{v}, x_n - \bar{x}] \\
 &\quad + t^2 \| f(u_n - \bar{u}) - f(v_n - \bar{v}) \|^2 \\
 &\leq \| x_n - \bar{x} \|^2 - 2t(r+k) \| x_n - \bar{x} \|^2 + t^2(\partial(Tx_n, T\bar{x}) + \partial(Ux_n, U\bar{x}))^2 \\
 &\leq (1 - 2t(r+k)) \| x_n - \bar{x} \|^2 + t^2(s+m)^2 \| x_n - \bar{x} \|^2 \\
 &= (1 - 2t(r+k) + t^2(s+m)^2) \| x_n - \bar{x} \|^2. \tag{3.17}
 \end{aligned}$$

Now applying (3.17) to (3.16), we have that

$$\begin{aligned}
 & \| x_{n+1} - \bar{x} \| \\
 &\leq (1 - a_n) \| x_n - \bar{x} \| + a_n[1 - 2t(r+k) + t^2(s+m)^2]^{\frac{1}{2}} \| x_n - \bar{x} \| \\
 &= (1 - (1 - M)a_n) \| x_n - \bar{x} \| \\
 &\leq \prod_{j=0}^n [1 - (1 - M)a_j] \| x_0 - \bar{x} \|, \tag{3.18}
 \end{aligned}$$

where  $0 < M = (1 - 2t(r+k) + t^2(s+m)^2)^{\frac{1}{2}} < 1$  for all  $t$  such that  $0 < t < 2(r+k)/(s+m)^2$ . Since the series  $a_0 + a_1 + a_2 + \dots$  diverges and  $M < 1$ , this implies that  $\lim_{n \rightarrow \infty} \prod_{j=0}^n (1 - (1 - M)a_j) = 0$  and, consequently,  $\{x_n\}$  converges strongly to  $\bar{x}$ . The Lipschitz continuity of  $T$  and  $U$  implies that  $\{u_n\}$  and  $\{v_n\}$  converge  $\bar{u}$  to  $\bar{v}$  respectively. This completes the proof.

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