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ON CERTAIN RANDOM POLYGONS OF LARGE AREAS

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Consider the tesselation of a plane into convex random polygons determined by a unit intensity Poissonian line process. Let $M(A)$ be the ergodic intensity of random polygons with areas exceeding a value A. A two-sided asymptotic bound

$$\exp\{-2\sqrt{A/\pi}+c_0A^{1/6}\} < M(A) < \exp\{-2\sqrt{A/\pi}+c_1A^{1/6}\}$$

is established for large A, where $c_0 > 2.096$, $c_1 < 6.36$.

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1. The Problem Statement and Main Results

This paper is devoted to the discussion of a problem posed by D.G. Kendall in his foreword to the book [9]. The problem concerns the investigation of the tail of the distribution of the area of a random polygon.

Consider a unit intensity Poissonian line process in $\Re^2$. Such a process can be determined by a planar Poissonian process of points (p_i, φ_i), with a planar intensity $1/\pi$ in the band $\Re_+ \times (0, 2\pi)$, in such a way that each of the points generates a random line with polar coordinates (p_i, φ_i) for the foot of the perpendicular from the origin O to the line. Processes of this kind have been investigated thoroughly by Miles [6-8] and others. (Miles uses an equivalent definition: $p_i \in \Re$, $\varphi_i \in (0, \pi)$.) The line processes determines the tesselation of the plane into convex random polygons.

Consider the ergodic intensity $M(A)$ of random polygons with areas exceeding a value A.

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It is necessary to specify a precise definition of ergodic intensities for random polygons. The simplest way to do this is a reduction to the ergodic intensity of an appropriate point process. Evidently, all the sides of a random polygon have different lengths (almost surely). Thus every random polygon K can be associated with a well-defined point P_K, namely, its vertex such that, when running along the contour of K in the positive direction, the greatest of its sides has P_K as its endpoint.

We define the function $M(A)$ as the ergodic intensity of the planar point process of random points P_K considering only random polygons K with areas $A(K) > A$. In turn, the ergodic intensity of a planar point process is defined as the mean number of random points in a unit area. From the Korolyuk theorem [3], for a simple stationary point process, the probability that a random point falls into an area element is equivalent to the ergodic intensity of the point process multiplied by the area.

Two asymptotic bounds are established for $M(A)$, as follows.

Theorem 1: *The bound*

$$M(A) > \exp\Big\{ -2\sqrt{A/\pi} + c_0 A^{1/6}(1+o(1))\Big\} \tag{1}$$

holds for a constant $c_0 > 2.096$ *as* $A \to \infty$.

Theorem 2: *The bound*

$$M(A) < \exp\Big\{ -2\sqrt{A/\pi} + c_1 A^{1/6}(1+o(1))\Big\} \tag{2}$$

holds for a constant $c_1 < 6.36$ *as* $A \to \infty$.

2. Proof of Theorem 1

Consider an event Ω_r: {no random line crosses the circle C_r of the radius r, with the center in the origin}. Evidently, $P\{\Omega_r\} = e^{-2r}$. If the event Ω_r occurs then the circle C_r is surrounded by a random polygon K_ϖ (a so-called Crofton cell).

Let $\{(r+X(t),t), 0 \le t \le 2\pi\}$ be the graph of K_ϖ in polar coordinates. Then

$$A(K_\varpi) = \frac{1}{2}\int_0^{2\pi} (r+X(t))^2 dt. \tag{3}$$

It is convenient to consider the positive square root $a(K)$ of the area $A(K)$ of a polygon K. By the Bounjakowsky (Cauchy) inequality,

$$2\pi \int_0^{2\pi} X^2(t)dt \ge \left(\int_0^{2\pi} X(t)dt \right)^2.$$

Thus equation (3) implies the bound

$$a(K_\varpi) \ge r\sqrt{\pi} \quad \left(1 + \frac{1}{2\pi r}\int_0^{2\pi} X(t)dt\right), \tag{4}$$

provided Ω_r occurred.

From Miles' theory, the probability of a random line not crossing a convex figure equals $e^{-S/\pi}$, where S is the perimeter of the figure. The event $\Omega_r \cap \{X(t) > x\}$ means no crossing by a random line of the convex hull of C_r completed by the point $(r+x, t)$. The perimeter of the hull equals $2r(\pi + \tan\alpha - \alpha)$, where $\alpha = \arccos(r/(r+x))$. Hence,

$$P\{X(t) > x \mid \Omega_r\} = \exp\{-2r(\tan\alpha - \alpha)/\pi\}. \tag{5}$$

It is convenient to introduce a monotonic transformation $\alpha(t) = \arccos(r/(r+X(t)))$. Since $\{X(t) > x\} = \{\alpha(t) > \alpha\}$, the lefthand side of equation (5) is just $P(\alpha(t) > \alpha \mid \Omega_r\}$. Moreover, consider truncated random variables

$$\bar{\alpha}(t) = \begin{cases} \alpha(t) & \text{if } \alpha(t) < \alpha_0, \\ 0 & \text{if } \alpha(t) > \alpha_0 \end{cases}$$

and $\bar{X}(t) = r((1/\cos\bar{\alpha}(t) - 1)$. Thanks to equation (5), the conditional p.d.f. $f(\alpha)$ of the random variable $\bar{\alpha}(t)$ has the form

$$f(\alpha) = (2r/\pi)\tan^2\alpha\exp\{-2r(\tan\alpha - \alpha)/\pi\}$$

inside the interval $(0, \alpha_0)$, provided Ω_r occurred. Choose α_0 as $\alpha_0 = (\ln r/r)^{1/3}\ln\ln r$. It can be easily seen that, $\tan\alpha - \alpha \cong \alpha^3/3$, $\tan\alpha \cong \alpha$ uniformly in the interval $\{0 < \alpha < \alpha_0\}$; hence

$$E\bar{\alpha}^m(t \mid \Omega_r) \cong (2r/\pi)\int_0^{\alpha_0} \alpha^{m+2}\exp\{-2r\alpha^3/(3\pi)\}d\alpha; \quad \text{as } r \to \infty$$

for any positive m. Moreover, the corresponding integral over the interval $\{\alpha > \alpha_0\}$ can be proved to decrease more rapidly than r^{-N} for a given N; therefore,

$$E\bar{\alpha}^m \cong (2r/\pi)\int_0^{\infty} \alpha^{m+2}\exp\{-2r\alpha^3/(3\pi)\}d\alpha, \text{ as } r \to \infty.$$

In particular,

$$E\{\alpha^2(t) \mid \Omega_r\} \cong (3\pi/(2r))^{2/3}\Gamma(5/3) \text{ as } r \to \infty; \tag{6}$$

$$E\{\alpha^4(t) \mid \Omega_r\} \cong O(r^{-4/3}) \text{ as } r \to \infty. \tag{7}$$

As $(1/\cos z) > 1 + z^2/2$ for any $z: 0 < z < \pi/2$, we have the inequality

$$\bar{X}(t) > r\bar{\alpha}^2(t)/2. \tag{8}$$

From Equations (8), (6) and (4),

$$E\{a(K_{\varpi}) \mid \Omega_r\} \geq r\sqrt{\pi}(1 + \tfrac{1}{2}\left(\tfrac{3\pi}{2r}\right)^{2/3}\Gamma(5/3)(1 + o(1))) \text{ as } r \to \infty.$$

Note that $\operatorname{cov}(\bar{\alpha}(t), \bar{\alpha}(\tau)) = 0$ as soon as the angle distance between t and τ exceeds $2\alpha_0$. Applying also equation (7), one obtains the bound

$$\operatorname{Var}\left\{\int_0^{2\pi} \bar{\alpha}^2(t)dt \mid \Omega_r\right\} = O\Big(r^{-5/3}(\ln r)^{1/3}\ln\ln r\Big). \tag{9}$$

Equations (9), (8), (6) and (4) imply the bound

$$P\left\{a(K_{\varpi}) \geq r\sqrt{\pi}\left(1+\frac{1}{2}\left(\frac{3\pi}{2r}\right)^{2/3}\Gamma(5/3)(1-\delta)\right) \mid \Omega_r\right\} \to 1 \text{ as } r \to \infty \tag{10}$$

for any $\delta > 0$.

Choose r as follows:

$$r = \frac{a}{\sqrt{\pi}}\left(1 - Ca^{-2/3}(1-2\delta)\right) \tag{11}$$

where

$$C = \pi(3/2)^{2/3}\Gamma(5/3)/2. \tag{12}$$

Then equation (10) implies the bound

$$P\{a(K_{\varpi}) > a \mid \Omega_r\} \to 1 \text{ as } a \to \infty \tag{13}$$

and hence

$$P\{\Omega_r \cap \{a(K_{\varpi}) > a\}\} \cong P\{\Omega_r\} = e^{-2r}. \tag{14}$$

Inserting r, defined by equation (11), into equation (14), and changing a^2 to A leads to the following bound:

$$P\{C_r \subset K_{\varpi}; A(K_{\varpi}) > A\} > \exp\{-2\sqrt{A/\pi} + c_0 A^{1/6}(1+o(1))\}$$

with $c_0 = (3/2)^{2/3}\sqrt{\pi}\Gamma(5/3) > 2.096$.

To make a passage from probabilities to ergodic intensities, we will prove that a typical Crofton cell lies inside a certain circle of the radius $3r$, provided the event Ω_r occurred. Consider three concentric circles C_r, C_{2r}, C_{3r} of the radii $r, 2r, 3r$, respectively. The event $\Omega_r \cap \{X(t) > 2r\}$ implies the event $\Omega_r \cap \{X(\tau) > r, \tau \in \Delta_t\}$ where Δ_t is an interval of a positive length. If N is large enough then

$$\Omega_r \cap \{X(\tau) > r, \tau \in \Delta_t\} \subset \bigcup_{k=0}^{N-1} \Omega_r \cap \{X(2\pi k/N) > r\}$$

for every t, $0 \leq t \leq 2\pi$. Therefore, due to equation (5) with $\alpha = \pi/3$,

$$P\left\{\max_t X(t) > 2r \mid \Omega_r\right\} \leq N\exp\{-2r(\sqrt{3} - \pi/3)/\pi\} = o(1) \text{ as } r \to \infty.$$

We have the relation

$$\Omega_r \cap \left\{\max_t X(t) \leq 2r\right\} \subset \Omega_r \cap \left\{P_{K_{\varpi}} \in C_{3r}\right\};$$

since

$$A(C_{3r}) = 9\pi r^2,$$

we have

$$9\pi r^2 M(A) \geq P\left\{\Omega_r \cap \left\{\max_t X(t) \leq 2r\right\}\right\} \cong 9\pi r^2 P\{\Omega_r\}. \tag{15}$$

Equations (11-15) imply the relation in equation (1).

3. A Bound for Ergodic Intensities $M_n(A)$

Evidently,

$$M(A) = \sum_{n=3}^{\infty} M_n(A),$$

where $M_n(A)$ is the contribution of n-gons to $M(A)$. The following lemma presents an upper bound for $M_n(A)$; we set $A = a^2$.

Lemma 1: *A bound*

$$M_n(a^2) \leq \frac{cn}{(2n-3)!a}(2\pi)^{2n} L_{n-1} \tag{16}$$

holds true with

$$L_{n-1} = \int_{y_i > 0,\, y_1 + \ldots + y_n > 2a(1+\Delta_n)/\sqrt{\pi}} \cdots \int exp\{-(y_1 + \ldots + y_n)\} dy_1 \ldots dy_{n-1}, \tag{17}$$

where y_n is a function of the remaining y_i's;

$$y_n > (y_1 + \ldots + y_{n-1})/(n-1), \tag{18}$$

$$\Delta_n = \left(\tfrac{n}{\pi} tan \tfrac{\pi}{n}\right)^{1/2} - 1. \tag{19}$$

Proof: Let K be a random n-gon such that its vertex P_K (see Section 1 for a definition) falls into the circle C_ρ of a small radius ρ, with center at the origin. Denote vertices of K by $P_1, \ldots, P_n$ in the positive direction; for definiteness, set $P_n = P_K$ (i.e., the endpoint of the side of the greatest length). Also, denote the exterior angle of K corresponding to the vertex P_i by θ_i, and set $x_1 = |P_n P_1|$; $x_i = P_{i-1}P_i|$, $2 \leq i \leq n$. For simplicity, we will also give the line $P_{i-1}P_i$ the name X_i. An n-gon K can be coded as $(\boldsymbol{\theta}, \boldsymbol{x})$, where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_{n-2})$ and $\boldsymbol{x} = (x_1, \ldots, x_{n-1})$. Given a position of the line X_1, these parameters determine x_n and θ_{n-1} uniquely as $\rho \to 0$. For example, $x_n = |OP_{n-1}|$ in the limit. By definition, see Section 1, $x_n = \max_{1 \leq i \leq n} \{x_i\}$. It is well known that the perimeter of an n-gon K does not exceed the perimeter of a regular n-gon, given a value of the area. Thus, $x_1 + \ldots + x_n > 2a\sqrt{\pi}(1 + \Delta_n)$ as soon as $A(K) \geq a^2$. These notes explain the bounds for y_i in equations (17-19), where $y_i = x_i/\pi$. Consider the probability of the occurrence of a random polygon in elementary volumes $d\boldsymbol{\theta}$, $d\boldsymbol{x}$, and $\pi\rho^2$ for parameters, $\boldsymbol{\theta}, \boldsymbol{x}$, and P_K, respectively. This differential probability can be considered as the product of the following expressions:

- *(i)* $2\rho =$ the probability of a random line X_1 crossing C_ρ;
- *(ii)* $(2\pi)^{-n+2} d\theta_1 \ldots d\theta_{n-2} =$ the probability of the choice of the directions of random lines $X_2, \ldots, X_{n-1}$;
- *(iii)* $2^{n-2} \sin\theta_1 dx_1 \ldots \sin\theta_{n-2} dx_{n-2} =$ the probability of the crossing of X_i by X_{i+1} in the intervals dx_i, given θ_i;
- *(iv)* $2\rho \sin\theta_{n-1} dx_{n-1}/(\pi x_n) =$ the probability of the occurrence of a random line crossing both an interval dx_{n-1} of the line X_{n-1} and the circle C_ρ; and
- *(v)* $\exp\{-(x_1 + \ldots + x_n)/\pi\} =$ the probability of no random line crossing the polygon K. [The above formulation is close to that of Miles [6-8]].

Applying a usual principle

$$\int f(\boldsymbol{\theta},\boldsymbol{x})d\boldsymbol{\theta}d\boldsymbol{x} \le \int d\boldsymbol{\theta} \int \sup_{\boldsymbol{\theta}} f(\boldsymbol{\theta},\boldsymbol{x})d\boldsymbol{x}$$

we may integrate in $\boldsymbol{\theta}$ separately:

$$\int \cdots \int \sin\theta_1 \ldots \sin\theta_{n-1}(d\theta_1 \ldots d\theta_{n-2})$$

$$< \int \cdots \int \theta_1 \ldots \theta_{n-2}(2\pi - \theta_1 - \ldots - \theta_{n-2})d\theta_1 \ldots d\theta_{n-2}$$

$$= (2\pi)^{2n-3}/(2n-3)!. \tag{20}$$

Having collected expressions (i) to (v), the bound in equation (20), applying the inequality $x_n > (x_1 + \ldots + x_n)/n > 2a\sqrt{\pi}/n$, and also having divided by $\pi\rho^2$, one obtains the bound in equations (16) and (17) directly.

4. A Proof of Theorem 2

Let L^0_{n-1} denote the contribution of the domain $\{y_1 + \ldots + y_{n-1} < 2a(1+\Delta_n)/\sqrt{\pi}\}$ to the integral L_{n-1} [see equation (17)]; $L^1_{n-1} = L_{n-1} - L^0_{n-1}$. Due to equation (16), we have

$$M_n(a^2) \le M^0_n(a^2) + M^1_n(a^2)$$

where $M^j_n(a^2)$ is defined as the righthand side of equation (16) with L_{n-1} changed by L^j_{n-1} $(j = 0, 1)$. We have

$$L^0_{n-1} < \exp\{-2a(1+\Delta_n)/\sqrt{\pi}\} \quad \int_{y_i > 0,\, y_1 + \ldots + y_{n-1} < 2a(1+\Delta_n)/\sqrt{\pi}} \cdots \int dy_1 \ldots dy_{n-1}$$

$$= (2a(1+\Delta_n)/\sqrt{\pi})^{n-1}\exp\{-2a(1+\Delta_n)/\sqrt{\pi}\}/(n-1)!. \tag{21}$$

Equation (21) implies the bound $\sum_{n=3}^{N} M^0_n(a^2) = O(-2a(1+\varepsilon)/\sqrt{\pi})$ as $a \to \infty$ for a given N; thus, only the case of large n should be investigated. Applying Stirling's formula to the factorials and also applying the relation

$$\Delta_n \cong \pi^2/(6n^2) = o(1/n),$$

one obtains the bound

$$M^0_n(a^2) < c_1(n^4/a^2)(2\pi^{3/2}e^3a/n^3)^n \exp\{-2a(1+\pi^2/(6n^2))/\sqrt{\pi}n^2)\}.$$

To apply a usual asymptotic analysis, see Dingle [2], introduce a variable

$$x = n/(2\pi^{3/2}e^3a)^{1/3}$$

and search the maximum of the expression

$$\ln(\exp\{-2a\Delta_n/\sqrt{\pi}\}/x^{3n})$$
$$\approx \{-18.211x\ln x - 0.0504x^{-2}\}a^{1/3}.$$

As the result of computations, the value 6.36 is obtained as an upper bound. By a standard argument in Dingle [2] it can be shown that

$$M^0(a^2) < \exp\{-2a/\sqrt{\pi} + 6.36a^{1/3}\} \tag{22}$$

for large a.

As for $M_n^1(a^2)$, consider two cases:

(*i*) $n \geq a/\ln a$.

(*ii*) $n < a/\ln a$.

In the case (*i*) it is sufficient to note that $L_{n-1}^1 < 1$ whereas

$$\sum_n \frac{n}{(2n-3)!}(2\pi)^{2n} = O(e^{-2(1-\delta)a}) \text{ as } a \to \infty$$

for a given $\delta > 0$, and thus

$$\sum_n M_n^1(a^2) < \exp\{-2a/\sqrt{\pi}\} \tag{23}$$

for large a. In case (*ii*), we omit the factor $(1+\Delta_n)$ in equation (17) and note that

$$\int \cdots \int_{y_i > 0, y_1 + \ldots + y_n > x} e^{-(y_1 + \ldots + y_n)} dy_1 \ldots dy_n < (ex/n)^n e^{-x}$$

as soon as $n < x$. Hence,

$$M_{n+1}^1(a^2) < \frac{c(n+1)}{a(2n-1)!}\left(\frac{2ae}{\sqrt{\pi}n}\right)^n$$
$$\times (2\pi)^{2n+2}\exp\{-2a(1+1/n)/\pi^{1/2}\} = Q_n(a), \text{ say.} \tag{24}$$

From equation (24), we have a relation

$$Q_{n+1}(a)/Q_n(a) \cong 2a\pi^{3/2}n^{-3}\exp\{2a(1+o(1))/(n^2\pi^{1/2})\} \tag{25}$$

for large n. The righthand side of equation (25) is large as $n < a^{(1-\varepsilon)/2}$ and small as $n > a^{1/2}$ Hence $\arg\max Q_n(a) = a^\theta$ where $(1-\varepsilon)/2 < \theta < 1/2$. For such n, equation (24) implies the relation

$$M_{n+1}(a^2) < \exp\{-2(a + a^{1/2})/\pi^{1/2}\};$$

hence,

$$\Sigma_{n < a/\ln a} M_{n+1}^1(a^2) < \exp\{-2a/\pi^{1/2}\}. \tag{26}$$

Combining the bounds in equations (22), (23) and (26) leads to the desired equation (2).

5. Remarks

The problem considered here is closely related to a "long-standing conjecture of D.G. Kendall" concerning shapes of random polygons. In a version suggested by Miles [8], this conjecture is as follows: Let $\mu(A)dA$ be the ergodic intensity of random polygons of the type considered as above, and $\mu_\varepsilon(A)dA$ be the ergodic intensity of those contours which, moreover, are surrounded by concentric circles of radii $\sqrt{A/\pi}(1 \pm \varepsilon)$. Then:

$$\mu_\varepsilon(A)/\mu(A) \to 1 \text{ as } A \to \infty \tag{27}$$

for a given $\varepsilon > 0$. For two different proofs of equation (27), both based on an inequality of Bonnesen [1], see Kovalenko [4, 5].

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