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# EXISTENCE AND UNIQUENESS OF A CLASSICAL SOLUTION TO A FUNCTIONAL-DIFFERENTIAL ABSTRACT NONLOCAL CAUCHY PROBLEM

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The aim of this paper is to investigate the existence and uniqueness of a classical solution to a functional-differential abstract nonlocal Cauchy problem in a general Banach space. For this purpose, a special kind of a mild solution is introduced and the Banach contraction theorem and a modified Picard method are applied.

**Key words:** Abstract Cauchy Problem, Ordinary Functional-Differential Equation, Nonlocal Condition, Existence and Uniqueness of a Classical Solution, Mild Solution, Banach Contraction Theorem, Picard Method.

**AMS subject classifications:** 34G20, 34K30, 34A12, 34A34, 47H10, 34A45, 34G99.

## 1. Introduction

We present four theorems (Theorems 2.1-2.4) on the existence and uniqueness of a classical solution to a functional-differential abstract nonlocal Cauchy problem in an arbitrary Banach space and give an approximation of the solution to the nonlocal problem. In the proofs of the theorems, we introduce a special kind of a mild solution and apply the Banach contraction theorem and a modified Picard method of successive approximations.

The functional-differential nonlocal problem, studied in this paper is of the form:

$$u'(t) = f(t, u(t), u(a(t))), \quad t \in I, \tag{1.1}$$

$$u(t_0) + \sum_{k=1}^{p} c_k u(t_k) = x_0, \tag{1.2}$$

where $I: = [t_0, t_0 + T]$, $t_0 < t_1 < \ldots < t_p \leq t_0 + T$, $T > 0$; $f: I \times E^2 \to E$ and $a: I \to I$ are given functions satisfying some assumptions; $E$ is a Banach space with norm $\| \cdot \|$, $x_0 \in E$, $c_k \neq 0$ $(k = 1, \ldots, p)$ and $p \in \mathbb{N}$.

 

The results obtained are generalizations and continuations of those, reported previously in [1-4], with the nonlocal condition of type (1.2). Moreover, the results of the paper include, among other things, a special kind of a mild solution to nonlocal problem (1.1)-(1.2). Therefore, throughout the proofs of the theorems, we apply properties of function $f$ in a greater measure than in [1-3]. Consequently, in contrast with [1-3], now, even if $T$ is an arbitrary positive constant, then $c_k (k = 1,\ldots, p)$ from the nonlocal condition (1.2) can satisfy the inequalities $|c_k| > 1$ $(k = 1,\ldots, p)$. The special kind of a mild solution in this paper is a modification of a mild solution introduced by the author (in [5]), for nonlocal evolution problems. In the case when $c_k = 0$ $(k = 1,\ldots, p)$ and the right-hand side of the functional-differential equation does not depend on the functional argument, some results of Theorem 2.4 are reduced to those (given in [6]) on the existence and uniqueness of a classical solution to the abstract Cauchy problem with the standard initial condition.

If $c_k \neq 0$ $(k = 1,\ldots, p)$ then the results of the paper can be applied in kinematics to determine the evolution $t \to u(t)$ of the location of a physical object for which we do not know the positions $u(t_0), u(t_1),\ldots, u(t_p)$, but we know that the nonlocal condition (1.2) holds. Consequently, to describe some physical phenomena, the nonlocal condition can be more useful than the standard initial condition $u(t_0) = x_0$.

## 2. Theorems About the Existence and Uniqueness of a Classical Solution

By $X$, we denote the Banach space $C(I, E)$ with the standard norm $\|\cdot\|_X$. So,

$$\| w \|_X := \sup_{t \in I} \| w(t) \|, \quad w \in X.$$

Assume that $\sum_{k=1}^{p} c_k \neq -1$. A function $u \in X$, satisfying the integral equation

$$u(t) = \left( x_0 - \sum_{k=1}^{p} c_k \int_{t_0}^{t_k} f(\tau, u(\tau), u(a(\tau)))d\tau \right) \Big/ \left( 1 + \sum_{k=1}^{p} c_k \right) \tag{2.1}$$

$$+ \int_{t_0}^{t} f(\tau, u(\tau), u(a(\tau)))d\tau, \quad t \in I,$$

is said to be a *mild solution* of the nonlocal problem (1.1)-(1.2).

A function $u: I \to E$ is said to be a *classical solution* of the nonlocal problem (1.1)-(1.2) if

(*i*) $u$ is continuous on $I$ and continuously differentiable on $I$,

(*ii*) $u'(t) = f(t, u(t), u(a(t)))$ for $t \in I$

and

(*iii*) $u(t_0) + \sum_{k=1}^{p} c_k u(t_k) = x_0$.

**Theorem 2.1:** *Suppose that* $f: I \times E^2 \to E$, $a: I \to I$ *and* $\sum_{k=1}^{p} c_k \neq -1$. *If* $u$ *is a classical solution of the nonlocal problem* (1.1)-(1.2), *then* $u$ *is a mild solution of this problem.*

**Proof:** Let $u$ be a classical solution of the nonlocal problem (1.1)-(1.2). Then $u$ satisfies equation (1.1) and, consequently,

$$u(t) = u(t_0) + \int_{t_0}^{t} f(\tau, u(\tau), u(a(\tau)))d\tau, \quad t \in I. \tag{2.2}$$

From (2.2),

$$u(t_k) = u(t_0) + \int_{t_0}^{t_k} f(\tau, u(\tau), u(a(\tau)))d\tau \quad (k = 1, \ldots, p). \tag{2.3}$$

By (1.2) and (2.3),

$$u(t_0) + \sum_{k=1}^{p} c_k \left[ u(t_0) + \int_{t_0}^{t_k} f(\tau, u(\tau), u(a(\tau)))d\tau \right] = x_0. \tag{2.4}$$

Since $\sum_{k=1}^{p} c_k \neq -1$, then (2.4) implies

$$u(t_0) = \left( x_0 - \sum_{k=1}^{p} c_k \int_{t_0}^{t_k} f(\tau, u(\tau), u(a(\tau)))d\tau \right) \Big/ \left( 1 + \sum_{k=1}^{p} c_k \right). \tag{2.5}$$

From (2.2) and (2.5), we obtain that $u$ is a mild solution of the nonlocal problem (1.1)-(1.2). The proof of Theorem 2.1 is complete.

**Theorem 2.2:** *Suppose that $f \in C(I \times E^2, E)$, $a: I \rightarrow I$ and $\sum_{k=1}^{p} c_k \neq -1$. If $u$ is a mild solution of the nonlocal problem* (1.1)-(1.2) *then $u$ is a classical solution of this problem.*

**Proof:** Let $u$ be a mild solution of the nonlocal problem (1.1)-(1.2). Then $u$ satisfies equation (1.1) and, from the continuity of $f, u \in C^1(I, E)$. Now, we will show that $u$ satisfies the nonlocal condition (1.2). For this purpose, observe that, by (2.1),

$$u(t_0) = \left( x_0 - \sum_{k=1}^{p} c_k \int_{t_0}^{t_k} f(\tau, u(\tau), u(a(\tau)))d\tau \right) \Big/ \left( 1 + \sum_{k=1}^{p} c_k \right) \tag{2.6}$$

and

$$u(t_i) = \left( x_0 - \sum_{k=1}^{p} c_k \int_{t_0}^{t_k} f(\tau, u(\tau), u(a(\tau)))d\tau \right) \Big/ \left( 1 + \sum_{k=1}^{p} c_k \right) \tag{2.7}$$

$$+ \int_{t_0}^{t_i} f(\tau, u(\tau), u(a(\tau)))d\tau \quad (i = 1, \ldots, p).$$

From (2.6) and (2.7), and from some computations,

$$u(t_0)+\sum_{i=1}^{p}c_iu(t_i)=\left(x_0-\sum_{k=1}^{p}c_k\int_{t_0}^{t_k}f(\tau,u(\tau),u(a(\tau)))d\tau\right)$$

$$+\sum_{i=1}^{p}c_i\int_{t_0}^{t_i}f(\tau,u(\tau),u(a(\tau)))d\tau=x_0.$$

Therefore, the proof of Theorem 2.2 is complete.

As a consequence of Theorems 2.1 and 2.2, we obtain:

**Theorem 2.3:** *Suppose that* $f\in C(I\times E^2,E)$, $a:I\to I$ *and* $\sum_{k=1}^{p}c_k\neq -1$. *Then* $u$ *is the unique classical solution of the nonlocal problem* (1.1)-(1.2) *if and only if* $u$ *is the unique mild solution of this problem.*

Now, we will prove the main theorem of the paper.

**Theorem 2.4:** *Assume that:*

(*i*) $a\in C(I,I)$, $f:I\times E^2\to E$ *is continuous with respect to the first variable on* $I$ *and there is* $L>0$ *such that*

$$\|f(s,z_1,z_2)-f(s,\tilde{z}_1,\tilde{z}_2)\|\leq L\sum_{i=1}^{2}\|z_i-\tilde{z}_i\| \tag{2.8}$$

$$\text{for } s\in I,\ z_i,\tilde{z}_i\in E\ \ (i=1,2),$$

(*ii*) $\sum_{k=1}^{p}c_k\neq -1$

*and*

(*iii*)
$$2LT\left(1+\left|\left(\sum_{k=1}^{p}c_k\right)\Big/\left(1+\sum_{k=1}^{p}c_k\right)\right|\right)<1.$$

*Then the nonlocal Cauchy problem* (1.1)-(1.2) *has a unique classical solution* $u$. *Moreover, the successive approximations* $u_n$ $(n=0,1,2,\ldots)$, *defined by the formulas*

$$u_0(t):=x_0 \text{ for } t\in I \tag{2.9}$$

*and*

$$u_{n+1}(t):=\left(x_0-\sum_{k=1}^{p}c_k\int_{t_0}^{t_k}f(\tau,u_n(\tau),u_n(a(\tau)))d\tau\right)\Big/\left(1+\sum_{k=1}^{p}c_k\right)$$
$$+\int_{t_0}^{t}f(\tau,u_n(\tau),u_n(a(\tau)))\text{ for } t\in I\ \ (n=0,1,2,\ldots), \tag{2.10}$$

*converge uniformly on* $I$ *to the unique classical solution* $u$.

**Proof:** Introduce an operator $A$ by the formula

$$(Aw)(t):=\left(x_0-\sum_{k=1}^{p}c_k\int_{t_0}^{t_k}f(\tau,w(\tau),w(a(\tau)))d\tau\right)\Big/\left(1+\sum_{k=1}^{p}c_k\right) \tag{2.11}$$

$$+\int_{t_0}^{t} f(\tau, w(\tau), w(a(\tau)))d\tau, \quad w \in X,\ t \in I.$$

It is easy to see that

$$A: X \to X. \tag{2.12}$$

Now, we will show that $A$ is a contraction on $X$. For this purpose observe that

$$\begin{aligned}(Aw)(t) - (A\widetilde{w})(t)& \\ = \left(-\sum_{k=1}^{p} c_k \int_{t_0}^{t_k} \left[f(\tau, w(\tau), w(a(\tau))) - f(\tau, \widetilde{w}(\tau), \widetilde{w}(a(\tau)))\right]d\tau\right) &\Big/ \left(1 + \sum_{k=1}^{p} c_k\right) \\ + \int_{t_0}^{t} \left[f(\tau, w(\tau), w(a(\tau))) - f(\tau, \widetilde{w}(\tau), \widetilde{w}(a(\tau)))\right]d\tau,& \quad w, \widetilde{w} \in X,\ t \in I.\end{aligned} \tag{2.13}$$

From (2.13) and (2.8),

$$\begin{aligned}&\| (Aw)(t) - (A\widetilde{w})(t) \| \\ &\le 2LT\left(1 + \left|\left(\sum_{k=1}^{p} c_k\right) \Big/ \left(1 + \sum_{k=1}^{p} c_k\right)\right|\right) \| w - \widetilde{w} \|_X, \quad w, \widetilde{w} \in X,\ t \in I.\end{aligned} \tag{2.14}$$

Let

$$q := 2LT\left(1 + \left|\left(\sum_{k=1}^{p} c_k\right) \Big/ \left(1 + \sum_{k=1}^{p} c_k\right)\right|\right). \tag{2.15}$$

Then, by (2.14), (2.15) and assumption (*iii*),

$$\| Aw - A\widetilde{w} \|_X \le q \| w - \widetilde{w} \|_X \text{ for } w, \widetilde{w} \in X \tag{2.16}$$

with $0 < q < 1$.

Consequently, by (2.12) and (2.16), operator $A$ satisfies all the assumptions of the Banach contraction theorem. Therefore, in space $X$ there is only one fixed point $u$ of $A$ and this point is the mild solution of the nonlocal problem (1.1)-(1.2). Consequently, from Theorem 2.3, $u$ is the unique classical solution of the nonlocal problem (1.1)-(1.2).

Now, we will prove the second part of the thesis of Theorem 2.4. To this end, observe that by (2.10) and (2.9),

$$\begin{aligned}\| u_1 - u_0 \|_X &= \sup_{t \in I} \| u_1(t) - u_0(t) \| \\ \le &\left\| \left(-\sum_{k=1}^{p} c_k \int_{t_0}^{t_k} f(\tau, u_0(\tau), u_0(a(\tau)))d\tau\right) \Big/ \left(1 + \sum_{k=1}^{p} c_k\right) \right\|\end{aligned} \tag{2.17}$$

$$+\sup_{t\in I}\left\| \int_{t_0}^{t} f(\tau, u_0(\tau), u_0(a(\tau)))d\tau \right\|$$

$$\leq MT\left(1+\left|\left(\sum_{k=1}^{p} c_k\right)\Big/\left(1+\sum_{k=1}^{p} c_k\right)\right|\right),$$

where

$$M: = \sup\{ \| f(\tau, w(\tau), w(a(\tau))) \| : w \in X, \tau \in I\}.$$

Next, assume that

$$\| u_n - u_{n-1} \|_X \leq MT\left(1+\left|\left(\sum_{k=1}^{p} c_k\right)\Big/\left(1+\sum_{k=1}^{p} c_k\right)\right|\right) \tag{2.18}$$

$$\cdot\left[2LT\left(1+\left|\left(\sum_{k=1}^{p} c_k\right)\Big/\left(1+\sum_{k=1}^{p} c_k\right)\right|\right)\right]^{n-1}$$

for some natural $n \geq 2$.

Then, by (2.10), (2.9), (2.8) and (2.18),

$$\| u_{n+1} - u_n \|_X = \sup_{t\in I} \| u_{n+1}(t) - u_n(t) \| \tag{2.19}$$

$$\leq \left\|\left(-\sum_{k=1}^{p} c_k \int_{t_0}^{t_k} \left[f(\tau, u_n(\tau), u_n(a(\tau))) - f(\tau, u_{n-1}(\tau), u_{n-1}(a(\tau)))\right]d\tau\right)\Big/\left(1+\sum_{k=1}^{p} c_k\right)\right\|$$

$$+\sup_{t\in I}\left\| \int_{t_0}^{t} \left[f(\tau, u_n(\tau), u_n(a(\tau))) - f(\tau, u_{n-1}(\tau), u_{n-1}(a(\tau)))\right]d\tau \right\|$$

$$\leq 2LT\left(1+\left|\left(\sum_{k=1}^{p} c_k\right)\Big/\left(1+\sum_{k=1}^{p} c_k\right)\right|\right)\| u_n - u_{n-1} \|_X$$

$$\leq MT\left(1+\left|\left(\sum_{k=1}^{p} c_k\right)\Big/\left(1+\sum_{k=1}^{p} c_k\right)\right|\right)\cdot\left[2LT\left(1+\left|\left(\sum_{k=1}^{p} c_k\right)\Big/\left(1+\sum_{k=1}^{p} c_k\right)\right|\right)\right]^{n}.$$

Therefore, from (2.17), (2.18), (2.19), and from mathematical induction,

$$\| u_n - u_{n-1} \|_X \leq MT\left(1+\left|\left(\sum_{k=1}^{p} c_k\right)\Big/\left(1+\sum_{k=1}^{p} c_k\right)\right|\right) \tag{2.20}$$

$$\cdot\left[2LT\left(1+\left|\left(\sum_{k=1}^{p} c_k\right)\Big/\left(1+\sum_{k=1}^{p} c_k\right)\right|\right)\right]^{n-1}$$

for all $n = 1, 2, \ldots$.

Inequalities (2.20) and assumption (*iii*) imply, by the Weierstrass theorem, the uniform convergence of the series

$$u_1 + \sum_{n=1}^{\infty}(u_{n+1} - u_n)$$

on the interval $I$ and, consequently, the uniform convergence of the sequence $u_n$ on $I$. Let

$$u_*(t) := \lim_{n \to \infty} u_n(t) \text{ for } t \in I.$$

Since $u_n$ tends uniformly to $u_*$ on $I$ then, by (2.9), (2.10) and (2.8), $u_*$ is a classical solution of the nonlocal problem (1.1)-(1.2) on $I$. But, from the first part of the thesis of Theorem 2.4, we know that there exists only one classical solution $u$ of the nonlocal problem (1.1)-(1.2) on $I$. So, $u_* = u$ on $I$.

The proof of Theorem 2.4 is complete.

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