Journal of Applied Mathematics and Stochastic Analysis, **12**:2 (1999), 121-131.

PERIODIC IN DISTRIBUTION SOLUTION FOR A TELEGRAPH EQUATION

A.YA. DOROGOVTSEV
Kiev University
Mechanics and Mathematics Department
Vladimirskay 64, 252033 Kiev-33 Ukraine

(Received July, 1998; Revised November, 1998)

In this paper we study an abstract stochastic equation of second order and stochastic boundary problem for the telegraph equation in a strip. We prove the existence of solutions, which are d-periodic (periodic in distribution) random processes.

Key words: Stochastic Evolution Equation, Stochastic Boundary Problems, d-Periodic (in Distribution) Solution.

AMS subject classifications: 34G10, 60G20, 60H15, 60H99.

1. Introduction

Let H be a separable Hilbert space, $A \in C(\boldsymbol{R}, \mathcal{L}(H))$ be a periodic function, and $a \in \boldsymbol{R}$. Let $w = \{w(t): t \in \boldsymbol{R}\}$ be an H-valued Wiener process. Consider the following abstract stochastic boundary problem for the telegraph equation

$$\begin{cases} u''_{tt}(t,x) + au'_t(t,x) - u''_{xx}(t,x) = A(t)u(t,x) + g(x)w'(t), \quad (t,x) \in Q; \\ u(t,0) = u(t,\pi) = \overline{0}\,, \end{cases} \tag{1}$$

where $\overline{0}$ denotes the zero element in H and $Q: = \boldsymbol{R} \times [0,\pi], g: [0,\pi] \to \boldsymbol{C}$. We are interested in d-periodic time variable t and a w-adapted solution u for problem (1) in the sense defined below.

The existence of periodic solutions for deterministic partial differential equations are intensively studied, see for example, the well-known book [19]. The problem of the existence of stationary and d-periodic solutions to stochastic ordinary differential equations is also well-known, see the books [7, 13], and the survey [8] for more references. During the past years, it has become apparent that it is natural and more adequate in many applications to consider an input source for partial differential equations as random source or a random disturbance. Thus the problem of investigating stochastic partial differential equations is of importance, see [3, 9, 18], where the problems of this kind were studied. We prove the existence of d-periodic solutions for stochastic boundary problem (1).

2. Periodic Solution for Ordinary Stochastic Equation

First, we recall some standard notations and terminology. Let $(H, (\cdot, \cdot), \|\cdot\|)$ be a complex separable Hilbert space, $\bar{0}$ the zero element in H, and $\mathfrak{L}(H)$ the Banach space of bounded linear operators on H, with the operator norm denoted also by the symbol $\|\cdot\|$. The adjoint of $D \in \mathfrak{L}(H)$ will be denoted by D^*. For an H-valued or $\mathfrak{L}(H)$-valued function, the continuity and differentiability means correspondingly the continuity and differentiability in the norm. Let I be the identity operator.

In what follows, we shall consider all random elements on the same complete probability space $(\Omega, \mathfrak{F}, P)$. The uniqueness of a random process, satisfying some equation, means its uniqueness up to stochastic equivalence. We consider only H-valued random functions which are continuous with probability one. All equalities for random elements are assumed to hold with probability one.

Definition 1: Let $\tau > 0$ be fixed. A random H-valued process $\{x(t): t \in \boldsymbol{R}\}$ is called a d-periodic (in distribution) with period τ if

$$\forall n \in \boldsymbol{N}, \quad \forall \{t_1, t_2, \ldots, t_n\} \subseteq \boldsymbol{R}, \text{ and } \forall \{B_1, B_2, \ldots, B_n\} \subseteq \mathfrak{B}(H),$$
$$P\left(\bigcap_{k=1}^{n} \{\omega: x(\omega, t_k + \tau) \in B_k\}\right) = P\left(\bigcap_{k=1}^{n} \{\omega: x(\omega, t_k) \in B_k\}\right),$$

where $\mathfrak{B}(H)$ is the Borel σ-algebra in H.

Let $w := \{w(t): t \in \boldsymbol{R}\}$ be an H-valued Wiener process with $P\{w(0) = \bar{0}\} = 1$. Note that for any $\{z_1, z_2\} \subset H$ and $0 < s < t$,

$$\boldsymbol{E}((w(t), z_1)\overline{(w(s), z_2)}) = s(z_2, Wz_1) \tag{2}$$

with a nuclear operator W and

$$\boldsymbol{E} \| w(t) - w(s) \|^2 = |t - s| \operatorname{tr} W, \quad \{s, t\} \subseteq \boldsymbol{R}.$$

Let $\mathfrak{F}_t := \sigma(w(v) - w(u): u \leq v \leq t)$, $t \in \boldsymbol{R}$.

To prove the existence of periodic solutions for problem (1), we first consider the following stochastic differential equation

$$x'(t) = A(t)x(t) + w'(t), \quad t \in \boldsymbol{R}, \tag{3}$$

where $A \in C(\boldsymbol{R}, \mathfrak{L}(H))$, and

$$\forall t \in \boldsymbol{R}, \ A(t + \tau) = A(t).$$

Definition 2: An H-valued continuous $\mathfrak{F}_t$-adapted random process $\{x(t): t \in \boldsymbol{R}\}$ is called a *nonanticipating solution of equation* (3) if for every $t \in \boldsymbol{R}$, the random element $x(t)$ is $\mathfrak{F}_t$-measurable, $\boldsymbol{E}\| x(t) \|^2 < +\infty$, and for every $s < t$ with probability 1,

$$x(t) - x(s) = \int_s^t A(u)x(u)du + w(t) - w(s).$$

The last integral is a Riemann integral of H-valued continuous function with probability one.

Let $U: \boldsymbol{R} \rightarrow \mathcal{L}(H)$ be the unique solution of the following problem:

$$\begin{cases} U'(t) = A(t)U(t), & t \in \boldsymbol{R}; \\ U(0) = I. \end{cases}$$

It is well-known that for every $t \in \boldsymbol{R}$, the operator $U(t)$ is invertible and

$$U(t) = U(t - n\tau)U(\tau)^n, \quad t \in \boldsymbol{R}, \quad n \in \boldsymbol{Z},$$

$$\| U(t)U(t_0)^{-1} \| \leq \exp\left(\int_{t_0}^{t} \| A(s) \| \, ds \right), \quad t_0 < t.$$

Let

$$G(t,s) := U(t)U^{-1}(s), \quad \{s,t\} \subset \boldsymbol{R}.$$

Now we are prepared to prove the existence of d-periodic solution of equation (3).

Theorem 1: *Stochastic equation* (3) *has a unique d-periodic with period* τ *nonanticipating solution* $\{x(t): t \in \boldsymbol{R}\}$, *with* $sup_{0 \leq t \leq \tau} \| x(t) \|^2 < +\infty$, *for every Wiener process w if and only if the following inequality*

$$\sup_{j \geq 1} \sum_{k=0}^{\infty} \int_0^{\tau} \| U^k(\tau)G(\tau,u)e_j \|^2 du < +\infty \tag{4}$$

is satisfied for every orthonormal basis $\{e_j: j \geq 1\}$ *in* H.

In order to prove Theorem 1, the following lemmas will be needed.

Lemma 1: *If the random process x is a nonanticipating solution of* (3), *then for every* $s < t$ *with probability* 1,

$$x(t) = G(t,x)x(s) + \int_s^t G(t,u)dw(u), \tag{5}$$

where the integral is a stochastic integral with respect to w.

Proof: Note that equality (5) is equivalent to the following:

$$U^{-1}(t)x(t) - U^{-1}(s)x(s) \int_s^t U^{-1}(u)dw(u), \quad s < t.$$

For every subdivision of $[s,t]$,

$$\lambda := \{s = t_0, t_1, \ldots, t_n = t\}, \quad \Delta t_k := t_{k+1} - t_k, \quad 0 \leq k \leq n-1,$$

we have

$$\begin{aligned} &U^{-1}(t)x(t) - U^{-1}(s)x(s) \\ &\quad = \sum_{k=0}^{n-1} (U^{-1}(t_{k+1})x(t_{k+1}) - U^{-1}(t_k))x(t_k) \\ &\quad = \sum_{k=0}^{n-1} (U^{-1}(t_{k+1}) - U^{-1}(t_k))x(t_{k+1}) + \sum_{k=0}^{n-1} U^{-1}(t_k)(x(t_{k+1}) - x(t_k)) \end{aligned}$$

$$= \sum_{k=0}^{n-1} U^{-1}(t_{k+1})(I - G(t_{k+1}, t_k))x(t_{k+1})$$
$$+ \sum_{k=0}^{n-1} U^{-1}(t_k)\left(\int_{t_k}^{t_{k+1}} A(u)x(u)du + w(t_{k+1}) - w(t_k)\right).$$

Hence, for every s, t, the right-hand side of (6) converges in probability to

$$-\int_s^t U^{-1}(u)A(u)x(u)du + \int_s^t U^{-1}(u)A(u)x(u)du + \int_s^t U^{-1}(u)dw(u)$$

as $\max(\Delta t_k: 0 \le k \le n-1) \to 0$.

Lemma 2: *If the random process x is a continuous nonanticipating solution of stochastic equation (5), then x is a nonanticipating solution of (3).*

Proof: The conclusion follows from a computation similar to the proof of Lemma 1.

Lemma 3: *If the random process x is a d-periodic nonanticipating solution of (3), then the stationary process $\{x(n\tau): n \in \boldsymbol{N}\}$ in H is a stationary adapted to $\{\epsilon(n): n \in \boldsymbol{Z}\}$ solution of the following difference equation in H:*

$$x((n+1)\tau) = U(\tau)x(n\tau) + \epsilon(n), \quad n \in \boldsymbol{Z}, \tag{7}$$

where

$$\epsilon(n) := \int_{n\tau}^{(n+1)\tau} G((n+1)\tau, u)dw(u) = \int_0^\tau U(\tau)U^{-1}(u)dw(n\tau + u), \quad n \in \boldsymbol{Z},$$

is a sequence of Gaussian independent identically distributed random elements in H.

Proof: The proof follows from Lemma 2. It follows by a direct computation that the covariance operator S_ϵ of the element $\epsilon(0)$ is given by

$$S_\epsilon = \int_0^\tau U(\tau)U^{-1}(u)WU^{-1}(u)^*U(\tau)^*du.$$

In addition, it can be proved as in [3, n. 4.1] that for the covariance operator S_x of $x(0)$ we have

$$S_x = \sum_{k=0}^{\infty} U^k(\tau)\int_0^\tau G(\tau, u)WG(\tau, u)^*duU^k(\tau)^*.$$

Lemma 4: *If equation (7) has stationary and adapted to $\{\epsilon(n): n \in \boldsymbol{Z}\}$ solution with $\boldsymbol{E}\| x(0)|^2 < +\infty$ for every Wiener process w, then for every orthonormal basis $\{e_j: j \ge 1\}$ in H, the inequality holds*

$$\sup_{j \ge 1} \sum_{k=0}^{\infty} \int_0^\tau \| U^k(\tau)G(\tau, u)e_j \|^2 du < +\infty.$$

Proof: This is a part of Theorem 1 [7, n. 4.1, p. 93] or Theorem 4.1 [5].

Lemma 5: *Let $\{x(n\tau): n \in \boldsymbol{Z}\}$ be a stationary and adapted to $\{\epsilon(n): n \in \boldsymbol{Z}\}$ solution of equation (7) such that $\boldsymbol{E}\| x(0)\|^2 < +\infty$ and $\{x(t): t \in \boldsymbol{R}\}$ be the H-valued process defined by*

$$x(n\tau+s) := U(s)x(n\tau) + \int_{n\tau}^{n\tau+s} G(n\tau+s,u)dw(u), \quad s\in[0,\tau],\ n\in \boldsymbol{Z}. \tag{8}$$

Then the process x is d-periodic with period τ and an $\mathfrak{F}_t$-adapted solution of equation (3). *In addition, $\boldsymbol{E}\| x(t)\|^2 < +\infty$, $t\in \boldsymbol{R}$.*

Proof: The process x is continuous with probability 1, see [1] or [7, n.8.4.2] and [2, 3, 11, 17] for general results. The last property follows from computations.

Proof of Theorem 1: (1) Suppose that equation (3) has a unique d-periodic with period τ and anticipating solution x such that $\sup_{0\le t\le\tau}\| x(t)\|^2 < +\infty$ for every Wiener process w. It follows from Lemma 3 that equation (7) has an adapted to ϵ stationary solution $\{x(n\tau): n\in \boldsymbol{Z}\}$ with $\boldsymbol{E}\| x(0)\|^2 < +\infty$. This solution of (7) is unique. Indeed, if the equation (7) had two different stationary and ϵ-adapted solutions, then by Lemma 5, equation (3) has two different d-periodic solutions. By virtue of Lemma 4, we have (4).

(2) Assume that condition (4) holds for every basis $\{e_j: j\ge 1\}$ in H. Then by Theorem 1 [3, n. 4.1, p. 93], equation (7) has a unique and ϵ-adapted stationary solution $\{x(n\tau): n\in \boldsymbol{Z}\}$ with $\boldsymbol{E}\| x(0)\|^2 < +\infty$. Hence, by Lemma 4, we have a d-periodic and nonanticipating solution for equation (2). By Lemmas 1-3, this solution is unique. This completes the proof of Theorem 1.

The following studies will be concerned with the following stochastic equation:

$$x''(t) + ax'(t) = A(t)x(t) + w'(t), \quad t\in \boldsymbol{R}. \tag{9}$$

We first give the definition of a solution for equation (9).

Definition 3: An H-valued random process $\{x(t): t\in \boldsymbol{R}\}$ is called a *solution of equation* (9) if for every $t\in \boldsymbol{R}$, the random element $x(t)$ is $\mathfrak{F}_t$-measurable, the processes x, x' are continuous with probability 1, and for every $s<t$, the equality

$$x'(t) - x'(s) + a(x(t)-x(s)) = \int_s^t A(u)x(u)du + w(t) - w(s) \tag{10}$$

holds with probability 1.

The main result of this section is a theorem which establishes a criterion of the existence of d-periodic solutions for equation (9).

Let $H^2 := H\times H$ and for x_1, x_2, y_1, y_2 from H,

$$((x_1,y_1),(x_2,y_2))_2 := (x_1,y_1) + (x_2,y_2).$$

Then $(H^2, (\cdot,\cdot)_2)$ is a Hilbert space. For $t\in \boldsymbol{R}$, let

$$\mathbb{A}(t) := \begin{pmatrix} \Theta & I \\ A(t) & -aI \end{pmatrix},$$

where Θ is the zero operator in H and

$$\mathbf{w} := \begin{pmatrix} \bar{0} \\ w \end{pmatrix}.$$

The following lemma is an immediate consequence of the above definitions.

Lemma 6: *Let x be a nonanticipating solution of equation* (9). *Then the H^2-*

valued random process

$$\mathbf{y}: = \begin{pmatrix} x \\ x' \end{pmatrix}$$

is a nonanticipating solution of the following equation in H^2:

$$\mathbf{y}'(t) = \mathbb{A}(t)\mathbf{y}(t) + \mathbf{w}', \quad t \in \boldsymbol{R}. \tag{11}$$

Lemma 7: *Let*

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

be a nonanticipating solution of (11). *Then* $y_2 = y_1'$ *with probability* 1, *and* y_1 *is a nonanticipating solution of* (9).

Proof: The proof is obvious.

Let us consider the function $\mathbb{U}$, which is a unique solution of the following problem

$$\begin{cases} \mathbb{U}'(t) = \mathbb{A}(t)\mathbb{U}(t), & t \in \boldsymbol{R}; \\ \mathbb{U}(0) = \mathbb{I}, \end{cases}$$

where $\mathbb{I}$ is the identity operator in H^2. By a direct computation, we obtain the following lemma.

Lemma 8: *We have*

$$\mathbb{U} = \begin{pmatrix} V_1 & V_2 \\ V_1' & V_2' \end{pmatrix},$$

where the functions V_1, V_2 *are the unique solutions of the equation*

$$V_j''(t) = A(t)V_j(t) - aV_j(t), \quad t \in \boldsymbol{R}; \quad j = 1, 2$$

in H, *with the boundary conditions*

$$V_1(0) = I,\ V_1'(0) = \Theta, \quad V_2(0) = \Theta,\ V_2'(0) = I.$$

Let

$$\mathbb{G}(t,s): = \mathbb{U}(t)\mathbb{U}^{-1}(s), \quad \{s,t\} \subseteq \boldsymbol{R},$$

and let

$$\mathbf{e}_j: = \begin{pmatrix} \bar{0} \\ e_j \end{pmatrix}, \quad j \geq 1,$$

for e_j, $j \geq 1$ from H.

Now, we are prepared to prove the existence of d-periodic solutions of equation (9).

Theorem 2: *The stochastic equation* (9) *has a unique nonanticipating and* d-

periodic solution x with period τ such that

$$\sup_{0 \leq t \leq \tau} \| x(t) \|^2 < +\infty \text{ and } \sup_{0 \leq t \leq \tau} \| x'(t) \|^2 < +\infty$$

for every Wiener process w if and only if the following inequality

$$\sup_{j \geq 1} \sum_{k=0}^{\infty} \int_0^{\tau} \| \mathbb{U}^k(\tau)\mathbb{G}(\tau, u)\mathbf{e}_j \|^2 du < +\infty \tag{12}$$

is satisfied for every orthonormal basis $\{e_j: j \geq 1\}$ in H.

Proof: Note that the covariance operator of $\mathbf{w}(1)$ is given by

$$\mathbb{W} = \begin{pmatrix} \Theta & \Theta \\ \Theta & W \end{pmatrix}.$$

Now, the result of Theorem 2 is a consequence of Theorem 1 and Lemmas 6-8.

Let us note that we have by (8)

$$\mathbf{y}(n\tau + s) = \sum_{l=0}^{\infty} \mathbb{U}(s)\mathbb{U}^l(\tau)\epsilon(n-l-1) + \int_0^s \mathbb{G}(s,u)d\mathbf{w}(n\tau + u), \tag{13}$$

for $n \in \boldsymbol{Z}$, $s \in [0, \tau]$ with

$$\epsilon(n) := \int_{n\tau}^{(n+1)\tau} \mathbb{G}((n+1)\tau, u)d\mathbf{w}(n\tau + u), n \in \boldsymbol{Z}.$$

3. Stochastic Boundary Problem for a Telegraph Equation

We now consider the boundary problem (1). Let us define the classes

$$C_0^1 := \{g: [0,\pi] \to \boldsymbol{C} \mid g(0) = g(\pi) = 0\} \cap C^1([0,\pi]),$$

$$C_0^3 := \{g: [0,\pi] \to \boldsymbol{C} \mid g^{(k)}(0) = g^{(k)}(\pi) = 0,\ k = 0,1,2\} \cap C^3([0,\pi]).$$

Definition 4: Let $\tau > 0$ be fixed. A random H-valued function $\{u(t,x): (t,x) \in Q\}$ is called a *d-periodic function of period τ with respect to the time t* if

$$\forall n \in \boldsymbol{N} \ \ \forall \{(t_1, x_1), (t_2, x_2), \ldots, (t_n, x_n)\} \subseteq Q \text{ and } \forall \{B_1, B_2, \ldots, B_n\} \subset \mathfrak{B}(H),$$

$$P\left(\bigcap_{k=1}^{n} \{\omega: x(\omega; t_k + \tau, x_k) \in B_k\}\right) = P\left(\bigcap_{k=1}^{n} \{\omega: x(\omega; t_k, x_k) \in B_k\}\right).$$

Definition 5: An H-valued random function u is a nonanticipating solution to the boundary problem (1) with $g \in C_0^1$, if for every $(t,x) \in Q$, the element $u(t,x)$ is $\mathfrak{F}_t$-measurable, the functions u, u_t', u_{xx}'' are continuous with probability 1, and, for every $s < t$, $x \in [0, \pi]$, we have

$$u'_t(t,x) - u'_s(s,x) + a(u(t,x) - u(t,x)) - \int_s^t u''_{xx}(r,x)dr$$

$$= \int_s^t Au(r,x)dr + g(x)(w(t) - w(s)),$$

$$u(t,0) = u(t,\pi) = \overline{0}$$

with probability 1.

Let $k \in \boldsymbol{N}$ and let V_{1k}, V_{2k} be unique solutions of the following equation in H:

$$V''_{jk}(t) = (A(t) - k^2 I)V_{jk}(t) - aV_{jk}(t), t \in \boldsymbol{R}; \quad j \in 1,2,$$

the boundary conditions

$$V_{1k}(0) = I, \ V'_{1k}(0) = \Theta, \ V_{2k}(0) = \Theta, \ V'_{2k}(0) = I.$$

Define

$$\mathbb{G}_k(t,s) := \mathbb{U}_k(t)\mathbb{U}_k^{-1}(s), \ \ \{s,t\} \subseteq \boldsymbol{R},$$

where

$$\mathbb{U}_k = \begin{pmatrix} V_{1k} V_{2k} \\ V'_{1k} V'_{2k} \end{pmatrix}.$$

Now we prove the main result of this paper.

Theorem 3: *The following two statements* (i) *and* (ii), *are equivalent:*

(i) *For any Wiener process* w *and a function* $g \in C_0^3$, *the boundary problem* (1) *has a unique nonanticipating d-periodic of period* τ *with respect to* t *solution* $u(t,x) : (t,x) \in Q\}$ *such that*

$$\sup_Q \boldsymbol{E} \parallel u(t,x) \parallel^2 < +\infty, \ \ \sup_Q \boldsymbol{E} \parallel u'_t(t,x) \parallel^2 < +\infty;$$

(ii) *For every orthonormal basis* $\{e_j : j \geq 1\}$ *in* H, *the following inequality holds:*

$$\sup_{\substack{j \geq 1, k \geq 1 \\ s \in [0,\tau]}} \left(\sum_{l=0}^{\infty} \int_0^{\tau} \parallel \mathbb{U}_k(s)\mathbb{U}_k^l(\tau)\mathbb{G}_k(\tau,u)\mathbf{e}_j \parallel^2 du + \int_0^s \parallel \mathbb{G}(s,u)\mathbf{e}_j \parallel^2 du \right) < +\infty.$$

Proof: Let us show that (ii) implies (i). Note that for each $g \in C_0^3$, we have

$$g(x) = \sum_{k=1}^{\infty} g_k \sin kx, \quad x \in [0,\pi],$$

for some $\{g_k : k \geq 1\}$. The last series converges uniformly over $[0,\pi]$.

It follows from (ii) and Theorem 2 that for every $k \geq 1$, the following stochastic equation

$$v'_k(t) - v'_k(s) + a(v_k(t) - v_k(s))$$

$$= \int_s^t (A(r) - k^2 I)v_k(r)dr + \tfrac{\pi}{2}(w(t) - w(s), t \in \boldsymbol{R} \qquad (14)$$

has a unique nonanticipating d-periodic of period τ solution $\{v_k(t): t \in \boldsymbol{R}\}$ with

$$\sup_{0 \le t \le \tau} \| v_k(t) \|^2 < +\infty, \quad \sup_{0 \le t \le \tau} \| v_k'(t) \|^2 < +\infty.$$

By (13) we know that

$$\mathbf{v}_k(n\tau + s) := \begin{pmatrix} v_k(n\tau + s) \\ v_k'(n\tau + s) \end{pmatrix}$$

$$= \sum_{l=1}^{\infty} \mathbb{U}_k(s)\mathbb{U}_k^l(\tau)\epsilon(n-l-1) + \int_0^s \mathbb{G}_k(s,u)d\mathbf{w}(n\tau + u), \tag{15}$$

for $n \in \boldsymbol{Z}$, $s \in [0,\tau]$. Consider the random function

$$u(t,x) := \sum_{k=1}^{\infty} v_k(t) g_k \sin kx, \quad (t,x) \in Q.$$

By (ii), the series for u converges on Q with probability 1. To establish the continuity of u, we have to show that the series

$$\sum_{k=1}^{\infty} \boldsymbol{E}\left(\sup_{t \in [b,c]} \| v_k(t) g_k \| \right)$$

for $b < c$, $c - b \le \tau$ is convergent. The convergence of this series follows from the well-known submartingale type inequality for stochastic integrals, see [1, 10, 14] and from the existence of the moments for Gaussian elements [15, 16]. The continuity of the random functions u_t', u_{xx}'' can be established by using similar arguments. Differentiating u with respect to t and x and using (14) we can verify that u is a solution of problem (1). The proof of the uniqueness is the same as that in the proof of theorem 2.

Let us show now that (i) implies (ii). Let $k \in \boldsymbol{N}$ and the Wiener process w be given. Let u be a unique nonanticipating d-periodic solution of (1) of period τ with respect to t such that

$$\sup_Q \boldsymbol{E} \| u(t,x) \|^2 < +\infty, \quad \sup_Q \boldsymbol{E} \| u_t'(t,x) \|^2 < +\infty$$

for $g(x) = \sin kx$, $x \in [0,\pi]$. Define

$$v_k(t) := \int_0^{\pi} u(t,x) \sin kx \, dx, \quad t \in \boldsymbol{R}.$$

It can be easily verified that v_k is a continuous nonanticipating d-periodic of period τ, H-valued process such that

$$\sup_{0 \le t \le \pi} \boldsymbol{E} \| v_k(t) \|^2 < +\infty, \quad \sup_{0 \le t \le \pi} \boldsymbol{E} \| v_k'(t) \|^2 < +\infty.$$

By virtue of Parseval's identity, see for example [7, p. 146], it follows that

$$\sup_{0 \le t \le \pi} \sum_{k=1}^{\infty} \boldsymbol{E} \| v_k(t) \|^2 < +\infty \quad \text{and} \quad \sup_{0 \le t \le \pi} \sum_{k=1}^{\infty} \boldsymbol{E} \| v'_k(t) \|^2 < +\infty.$$

From Definition 5, we also have

$$v'_k(t) - v'_k(s) + a(v_k(t) - v_k(s))$$

$$= \int_s^t (A(r) - k^2 I) v_k(r) dr + \tfrac{\pi}{2}(w(t) - w(s), \quad t \in \boldsymbol{R}, \tag{16}$$

with probability 1. The process v_k is a unique solution of (14), because if the equation (14) would have two different nonanticipating d-periodic of period τ solutions, then one could construct following the method described in part $(ii) \Rightarrow (i)$ of the proof, two different nonanticipating d-periodic solutions of problem (1). Hence, Theorem 2 applies. This leads to the following conclusion. The inequality

$$\sup_{\substack{j \ge 1 k \ge 1 \\ s \in [0, \tau]}} \left(\sum_{l=0}^{\infty} \int_0^{\tau} \| \mathbb{U}_k(s) \mathbb{U}_k^l(\tau) \mathbb{G}_k(\tau, u) \mathbf{e}_j \|^2 du + \int_0^s \| \mathbb{G}(s, u) \mathbf{e}_j \|^2 du \right) < +\infty$$

is satisfied for every orthonormal basis $\{e_j : j \ge 1\}$ in H. Thus, we have (ii). This completes the proof of Theorem 3.

References

[1] Curtain, R.F. and Falb, P.L., Stochastic differential equation in Hilbert space, *J. Differ. Eqns.* **10** (1971), 412-430.

[2] DaPrato, G., Regularity properties of a stochastic convolution integral, *Atti. Acad. Naz. Lincei Rend. Sc. Fis. Mat. e Natur.* **LXXII** (1982), 217-219.

[3] DaPrato, D. and Zabczyk, J., *Ergodicity for Infinite Dimensional Systems*, Cambridge University Press 1996.

[4] Dorogovtsev, A.Ya., Periodic solutions of an evolutional differential equation perturbed by random processes, *Ukrainskii Matem. Zhurnal* **41** (1989), 1642-1648 (in Russian).

[5] Dorogovtsev, A.Ya., Necessary and sufficient conditions for existence of stationary and periodic solutions of a stochastic difference equation in Hilbert space, *Comp. Math. Appl.* **19** (1990), 31-37.

[6] Dorogovtsev, A.Ya., Stationary and periodic solutions of stochastic difference and differential equations in Banach space, *New Trends in Prob. and Stats.* **V1**, In: *Proc. of the Bakuriani Colloquium in Honor of Yu.V. Prohorov* (ed. by V.V. Sasonov and T. Shervashidze), MOKSLAS, Vilnius (1991), 375-390.

[7] Dorogovtsev, A.Ya., *Periodic and Stationary Regimes of Infinite-Dimensional Deterministic and Stochastic Dynamically Systems*, Vissha Shkola, Kiev 1992 (in Russian).

[8] Dorogovtsev, A.Ya., Periodic processes: A survey of results, *Theory of Stochastic Processes*, TBiMC Kiev **2**:18 (1996), no. 3-4, 36-53

[9] Flandoli, F., *Regularity Theory and Stochastic Flows for Parabolic SPDEs*,

Gordon and Breach 1995.

[10] Gikhman, I.I. and Skorokhod, A.V., *Stochastic Differential Equations and Their Applications*, Naukova Dumka, Kiev 1982 (in Russian).

[11] Goldys, B., On some regularity properties of solutions to stochastic evolution equations, *Colloquium Math.* **LVIII** (1990), 327-338.

[12] Gyöngy, I. and Nualart, D., Implicit scheme for quasilinear parabolic differential equations by space-time white noise, *Stoch. Proc. and their Appl.* **58** (1995), 57-72.

[13] Khasminskii, R.Z., *Stochastic Stability of Differential Equations*, Sijthoff and Noordhoff, Alpen aan den Rijn 1980 (Russian ed. 1969).

[14] Kotelenez, P., A submartingale type inequality with applications to stochastic evolution equations, *Stochastics* **8** (1982), 139-151.

[15] Kuo, H.-H., *Gaussian Measures in Banach Spaces*, Springer Verlag, Berlin 1975.

[16] Skorokhod, A.V., A note on Gaussian measures in a Banach space, *Theory of Prob. and their Appl.* **15** (1970), 508 (in Russian).

[17] Tubaro, L., Regularity results of the process $X(t) = \int_0^t U(t,s)g(s)dW(s)$, *Rendiconti del. Sem. Mat.* **39** (1982), 241-248.

[18] Uboe, J. and Zhang, T., A stability property of the stochastic heat equation, *Stoch. Proc. and their Appl.* **60** (1995), 247-260.

[19] Vejvoda, O., et al., *Partial Differential Equations: Time-Periodic Solutions*, Noordhoff 1981.