*Journal of Applied Mathematics and Stochastic Analysis*, **12**:3 (1999), 233-252.

# NONCONVEX EVOLUTION INCLUSIONS GENERATED BY TIME-DEPENDENT SUBDIFFERENTIAL OPERATORS

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(Received June, 1998; Revised September, 1998)

We consider nonlinear nonconvex evolution inclusions driven by time-varying subdifferentials $\partial\phi(t,x)$ without assuming that $\phi(t,\cdot)$ is of compact type. We show the existence of extremal solutions and then we prove a strong relaxation theorem. Moreover,r we show that under a Lipschitz condition on the orientor field, the solution set of the nonconvex problem is path-connected in $C(T,H)$. These results are applied to nonlinear feedback control systems to derive nonlinear infinite dimensional versions of the "bang-bang principle." The abstract results are illustrated by two examples of nonlinear parabolic problems and an example of a differential variational inequality.

**Key words:** Subdifferential, Strong Solution, Strong Relaxation, Path-Connected, Feedback Control System, Parabolic Equation.

**AMS subject classifications:** 34G20, 35K55.

## 1. Introduction

Recently in a series of papers (see Papageorgiou [17-19]), the third author investigated evolution inclusions of the subdifferential type and proved existence theorems, determined the structure of the solution set and studied problems depending on a parameter. Crucial in these works was the assumption that the function $\phi(t,x)$ involved in the subdifferential is of compact type as a function of $x$. In this paper we remove this hypothesis and instead we impose a compactness-type condition on the multivalued term $F(t,x)$. We focus our attention to the nonconvex problem and in particular on the existence and density of the set of extremal trajectories. Also for the nonconvex problem, we show that the solution set is path-connected. We present applications to nonlinear feedback control systems.

 

So let $T=[0,b]$ and $H$ a separable Hilbert space. The problem under consideration is the following:

$$\left\{\begin{array}{ll} -\dot{x}(t)\in\partial\phi(t,x(t))+F(t,x(t)) & \text{a.e. on } T \\ x(0)=x_0 & \end{array}\right\}. \tag{1}$$

In conjuction with (1) we also consider the following multivalued Cauchy problem, which is important in the study of control systems in connection with the "bang-bang principle":

$$\left\{\begin{array}{ll} -\dot{x}(t)\in\partial\phi(t,x(t))+extF(t,x(t)) & \text{a.e. on } T \\ x(0)=x_0 & \end{array}\right\}. \tag{2}$$

Here $extF(t,x)$ denotes the extreme points of the orientor field $F(t,x)$. The solutions of (2) are called "extremal solutions" (or "extremal trajectories"). Their study was initiated by DeBlasi-Pianigiani [8], who considered differential inclusions with no subdifferential term present. That formulation precludes the applicability of their work to partial differential equations with multivalued terms. Their proof is based on an ingenious application of the Baire category theorem. Their method was formalized in a "nonconvex" continuous selection theorem due to Tolstonogov (see Hu-Papageorgiou [10], Theorem II.8.31, p. 260). We use this result to show the existence of extremal solutions and then to prove that the solution set of (2) is a dense $G_\delta$-set in the solution set of (1), in the space $C(T,H)$. These results are then applied to nonlinear control systems driven by evolution equations with a priori feedback. For such systems we obtain nonlinear versions of the celebrated "bang-bang principle." Finally we illustrate our abstract results with two examples of parabolic boundary value problems and an example of a "differential variational inequality" in $R^N$.

Problems related to (1) were studied by Attouch-Damlamian [1], Watanabe [24], Kenmochi [11], Yamada [25], Yotsutani [26] and Kubo [13]. All (except Attouch-Damlamian), assumed $F(t,x)$ to be independent of $x\in H$ and single-valued. In Attouch-Damlamian the subdifferential term is independent of $t\in T$ and the authors deal only with the convex problem (see Theorem 4.1 where $\phi(\cdot)$ is of compact-type and Theorem 4.6 where $\phi(\cdot)$ is not necessarily of compact-type but $F(t,x)$ is single-valued and weakly continuous in $x$).

## 2. Mathematical Preliminaries

In this section we fix our notation and recall some basic definitions and facts from multivalued analysis and convex analysis that we will need in the sequel. For details we refer to Hu-Papageorgiou [10].

Let $X$ be a separable Banach space. We will be using the following notations:

$$P_{f(c)}(X)=\{A\subseteq X\colon\ A \text{ nonempty, closed (and convex)}\}$$

$$P_{(w)k(c)}(X)=\{A\subseteq X\colon A \text{ nonempty, (weakly-) compact (and convex)}\}.$$

Let $(\Omega, \Sigma, \mu)$ be a finite measure space. A multifunction $F: \Omega \to P_f(X)$ is said to be measurable, if for all $x \in X$, the function $\omega \to d(x, F(\omega)) = \inf\{ \| x - z \| : z \in F(\omega)\}$ is measurable. Note that if $F(\cdot)$ is measurable, then $GrF \in \Sigma \times B(X)$ with $B(X)$ being the Borel $\sigma$-field of $X$ (graph measurability of $F(\cdot)$), while the converse is true if $\Sigma$ is $\mu$-complete. If $1 \leq p \leq \infty$, by $S_F^p$ we denote the set of selectors of $F(\cdot)$ that belong to the Lebesgue-Bochner space $L^p(\Omega, X)$ , i.e., $S_F^p = \{f \in L^p(\Omega, X): f(\omega) \in F(\omega) \mu$-a.e.$\}$. For a graph measurable multifunction $F: \Omega \to 2^X \backslash \{\emptyset\}$, $S_F^p$ is nonempty if and only if $\omega \to \inf\{ \| z \| : z \in F(\omega)\} \in L^p(\omega)$. Also the set $S_F^p$ is decomposable, i.e. for every triple $(f, g, A) \in S_F^p \times S_F^p \times \Sigma$, we have $\chi_{A^c} f + \chi_A g \in S_F^p$.

On $P_f(X)$ we can define a generalized metric, known in the literature as the "Hausdorff metric," by setting for $A, C \in P_f(X)$,

$$h(A, C) = \max\{\sup[d(a, C): a \in A], \sup[d(c, A): c \in C]\}.$$

A multifunction $F: X \to P_f(X)$ is said to be "h-continuous" (Hausdorff continuous), if it is continuous from $X$ into the metric space $(P_f(X), h)$. Note that $(P_f(X), h)$ is complete, while $(P_k(X), h)$ is separable.

Let $\phi: X \to \overline{R} = R \cup \{ +\infty\}$. We say that $\phi(\cdot)$ is proper if it is not identically $+\infty$. The family of all functions $\phi: X \to \overline{R}$ which are proper, convex and lower semicontinuous, is denoted by $\Gamma_o(X)$. By dom$\phi$ we denote the effective domain of $\phi(\cdot)$, i.e., dom$\phi = \{x \in X: \phi(x) < +\infty\}$. The subdifferential of $\phi(\cdot)$ at $x \in X$ is the set $\partial\phi(x) = \{x^* \in X^*: (x^*, y - x) \leq \phi(y) - \phi(x)$ for all $y \in \text{dom}\phi\}$. In this direction by $(\cdot, \cdot)$, we denote the duality brackets for the pair $(X, X^*)$. If $\phi(\cdot)$ is Gateaux differentiable at $x$, then $\partial\phi(x) = \{\phi'(x)\}$. We say that $\phi \in \Gamma_o(X)$ is of "compact-type" if for all $\lambda > 0$, the lower level set $\{x \in X: \phi(x) + \| x \|^2 \leq \lambda\}$ is compact.

For our problem $T = [0, b]$ and $H$ is a separable Hilbert space. The following hypotheses concerning $\phi(t, x)$ will be valid throughout this paper and originally were formulated by Yotsutani [26] (see also Kenmochi [11] and Yamada [25] for a little more restricted versions). In what follows for $A \subseteq R$ by $|A|$ we denote the Lebesgue measure of $A$.

**H($\phi$):** $\phi: T \times H \to \overline{R} = R \cup \{ +\infty\}$ is a function such that

(*i*) for every $t \in T \backslash N$, $|N| = 0, \phi(t, \cdot)$ is proper, convex and lower semicontinuous (i.e., $\phi(t, \cdot) \in \Gamma_o(X)$);

(*ii*) for every positive integer $r$, there exist a constant $K_r > 0$, an absolutely continuous function $g_r: T \to R$ with $\dot{g}_r \in L^\beta(T, R)$ and a function of bounded variation $h_r: T \to R$ such that if $t \in T \backslash N$, $x \in \text{dom}\phi(t, \cdot)$ with $\| x \| \leq r$ and $s \in [t, b] \backslash N$, then there exists $\widehat{x} \in \text{dom}\phi(s, \cdot)$ satisfying

$$\| \widehat{x} - x \| \leq | g_r(s) - g_r(t) | (\phi(t, x) + K_r)^\alpha$$

$$\text{and} \quad \phi(s, \widehat{x}) \leq \phi(t, x) + | h_r(s) - h_r(t) | (\phi(t, x) + K_r)^\alpha$$

where $\alpha \in [0, 1]$ and $\beta = 2$ if $\alpha \in [0, 1/2]$ or $\beta = \frac{1}{1 - \alpha}$ if $\alpha \in [\frac{1}{2}, 1]$.

**Remark:** Hypothesis H($\phi$)(*ii*) allows the effective domain dom$\phi(t, \cdot)$ of $\phi(t, \cdot)$ to vary in a regular way with respect to $t \in T$, without excluding the possibility that dom$\phi(t, \cdot) \cap \text{dom}\phi(s, \cdot) = \emptyset$ if $t \neq s$. This situation arises in the study of obstacle problems.

By a solution of problem (1) we mean a function $x \in C(T, H)$ such that $x(\cdot)$ is absolutely continuous on any closed subinterval of $(0, b)$ and satisfies:

(a) $x(t) \in \text{dom}\phi(t, \cdot)$ a.e. on $T$;
(b) there exists $f \in S^2_{F(\cdot, x(\cdot))}$ such that $-\dot{x}(t) \in \partial\phi(t, x(t)) + f(t)$ a.e. on $T$;
(c) $x(0) = x_0$.

By $S_c(x_0)$ (resp. $S_e(x_0)$) we denote the solution set of problem (1) (resp. of (2)).

## 3. Extremal Solutions

In this section we prove a theorem on the existence of extremal solutions (i.e., on the nonemptiness of $S_e(x_0)$). For this we need the following hypothesis on the multivalued term $F(t,x)$.

**H(F)$_1$**: $F: T \times H \to P_{wkc}(H)$ is a multifunction such that
(i) for every $x \in H$, $t \to F(t,x)$ is measurable;
(ii) for very $t \in T$, $x \to F(t,x)$ is h-continuous and for every $B \subseteq H$ bounded, we have $\overline{F(t,B)} \in P_k(H)$;
(iii) for almost all $t \in T$ and all $x \in H$, $|F(t,x)| = \sup\{\|v\| : v \in F(t,x)\} \leq \alpha(t) + c(t)\|x\|$ with $\alpha, c \in L^2(T)$.

In the next proof we use the following easy fact about the *weak** convergence in the dual $X^*$ of a Banach space $X$. Namely, if $x_n^* \xrightarrow{w^*} x^*$ in $X^*$, then for all $K \subseteq X$ compact we have that $(x_n^*, u) \to (x^*, u)$ uniformly in $u \in K$, i.e. $\sup\{|(x_n^* - x^*, u)| : u \in K\} \to 0$ as $n \to \infty$.

**Theorem 1:** *If hypotheses* H($\phi$), H(F)$_1$ *hold and* $x_0 \in dom\phi(0, \cdot)$, *then* $S_e(x_0)$ *is nonempty.*

**Proof:** First we derive an a priori bound for the elements in $S_c(x_0) \subseteq C(T,H)$. To this end, let $y \in C(T,H)$ be the unique strong solution of the Cauchy problem

$$\left\{\begin{array}{l} -\dot{y}(t) \in \partial\phi(t,(t)) \text{ a.e. on } T \\ y(0) = x_0 \end{array}\right\} \tag{3}$$

(see Yotsutani [26]). Let $x \in S_c(x_0)$. Then $-\dot{x}(t) \in \partial\phi(t, x(t)) + f(t)$ a.e. on $T$, $x(0) = x_0$ with $f \in S^2_{F(\cdot, x(\cdot))}$. Exploiting the monotonicity of the subdifferential operator, we have

$$(-\dot{x}(t) + \dot{y}(t), y(t) - x(t)) \leq (f(t), y(t) - x(t)) \text{ a.e. on } T$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|y(t) - x(t)\|^2 \leq (f(t), y(t) - x(t)) \text{ a.e. on } T$$

$$\Rightarrow \frac{1}{2}\|y(t) - x(t)\|^2 \leq \int_0^t (f(s), y(s) - x(s))ds$$

$$\leq \int_0^t \|f(s)\| \, \|y(s) - x(s)\| \, ds.$$

Invoking Lemma A.5, p. 157 of Brezis [5] and using hypotheses H(F)$_1$(*iii*) we have

$$\| y(t) - x(t) \| \leq \int_0^t \| f(s) \| ds \leq \int_0^t (\alpha(s) + c(s) \| x(s)) ds$$

$$\Rightarrow \| x(t) \| \leq \| y \|_\infty + \int_0^t (\alpha(s) + c(s) \| x(s) \| ) ds \text{ for all } t \in T.$$

By Gronwall's inequality, we deduce that there exists $M_1 > 0$ such that for all $x \in S_c(x_0)$ and all $t \in T$ we have $\| x(t) \| \leq M_1$. Thus without any loss of generality we may assume that for almost all $t \in T$ and all $x \in H$, $| F(t,x) | \leq \psi(t)$, with $\psi(t) = \alpha(t) + M_1 c(t) \in L^2(T)$ (otherwise we replace $F(t,x)$ by $F(t, p_{M_1}(x))$, where $p_{M_1}(\cdot)$ is the $M_1$-radical retraction on $H$; this substitution does not change the solution set $S_c(x_0)$). Let $B_{M_1} = \{z \in H: \| z \| \leq M_1\}$ and set $V(t) = \overline{F(t, B_{M_1})}$. From hypothesis $\mathrm{H(F)}_1(ii)$ we know that $V(t) \in P_k(H)$ for every $t \in T$. Also if $\{x_m\}_{m \geq 1}$ is dense in $B_{M_1}$, since by hypothesis $\mathrm{H(F)}_1(ii)$, $F(t, \cdot)$ is h-continuous, we have for the support function, $\sigma(h, V(t)) = \sup[(h,v): v \in V(t)]$, $h \in H$:

$$\sigma(h, V(t)) = \sup_{m \geq 1} \sigma(h, F(t, x_m)).$$

But for each $m \geq 1$, $t \rightarrow F(t, x_m)$ is measurable (see hypothesis $\mathrm{H(F)}_1(i)$) and so $t \rightarrow \sigma(h, F(t, x_m))$ is measurable for every $h \in H$ (see Proposition II.2.33, p. 164 of Hu-Papageorgiou [10]). Therefore $t \rightarrow \sigma(h, V(t))$ is measurable for every $h \in H$ and since $t \rightarrow V(t)$ is $P_k(H)$-valued, from Proposition II.2.39, p. 166 of Hu-Papageorgiou [10] we infer that $t \rightarrow V(t)$ is measurable from $T$ into $P_k(H)$.

Let $p: L^2(T,H) \rightarrow C(T,H)$ be the map which to each $g \in L^2(T,H)$ assigns the unique solution of the Cauchy problem

$$\left\{ \begin{array}{l} -\dot{x}(t) \in \partial\phi(t, x(t)) + g(t) \text{ a.e. on } T \\ x(0) = x_0 \end{array} \right\} \tag{4}$$

(see Yotsutani [26]). It is easy to verify that $p(\cdot)$ is nonexpansive (hence continuous). Let $K = p(S_V^2)$. We claim that $K \subseteq C(T,H)$ is compact. To this end, first we show that $K$ is equicontinuous and then using the equicontinuity, we show that every sequence in $K$ has a $C(T,H)$-convergent subsequence.

First we show that equicontinuity of $K$. To this end, let $x \in K$ and let $t < t'$. We have

$$\| x(t') - x(t) \| \leq \int_t^{t'} \| \dot{x}(s) \| ds \leq \left( \int_0^b \chi_{[t,t']}(s) ds \right)^{\frac{1}{2}} \left( \int_0^b \| \dot{x}(s) \|^2 ds \right)^{\frac{1}{2}}.$$

Since $S_V^2$ is bounded in $L^2(T,H)$, from Lemma 6.11 of Yotsutani [26] we infer that there exists $M_2 > 0$ such that for all $x \in K$, $\| \dot{x} \|_2 \leq M_2$. Hence for all $x \in K$ we have

$$\| x(t') - x(t) \| \leq M_2 (t' - t)^{\frac{1}{2}}$$

$$\Rightarrow K \text{ is equicontinuous.}$$

Next, let $\{x_n\}_{n \leq 1} \subseteq K$. Then $x_n = p(v_n)$ with $v_n \in S^2_V$, $n \geq 1$. Since $\| v_n(t) \| \leq \psi(t)$ a.e. on $T$ with $\psi \in L^2(T)$, by passing to a subsequence if necessary, we may assume that $v_n \xrightarrow{w} v$ in $L^2(T,H)$, $v \in S^2_V$. Let $x = p(V) \in K$ and $z_n(\cdot) = (x_n - x)(\cdot)$. Then for all $n \geq 1$ and all $t \in T$, $\| z_n(t) \| \leq 2 \mid K \mid = M_3$ where $\mid K \mid = \sup\{ \| x \|_\infty : x \in K\} < \infty$, since $K$ is bounded. Therefore for all $n \geq 1$ and all $t \in T$, $z_n(t) \in B_{M_3} = \{y \in H: \| Y \| \leq M_3\} \in P_{wkc}(H)$. Moreover, since $K$ is equicontinuous, $\{z_n\}_{n \geq 1}$ is equicontinuous, a fortiori then weakly equicontinuous. So invoking the Arzela-Ascoli theorem, we deduce that $\{z_n\}_{n \geq 1}$ is relatively compact in $C(T, B^w_{M_3})$ where $B^w_{M_3}$ denotes the closed ball $B_{M_3}$ furnished with the weak topology. Recall that $B^w_{M_3}$ is compact metrizable. So Theorem 8.2(3), pp. 269-270 of Dugundji [9], tells us that $\{z_n\}_{n \geq 1} \subseteq C(T, H_w)$ is relatively sequentially compact (here by $H_w$ we denote the Hilbert space $H$ equipped with the weak topology). So we may assume that $z_n \rightarrow z \in C(T, H_w)$ as $n \rightarrow \infty$. Exploiting the monotonicity of the subdifferential operator as before, we have

$$\| z_n(t) \|^2 = \| x_n(t) - x(t) \|^2 \leq \int_0^t (v_n(s) - v(s), x_n(x) - x(s))ds$$

$$= \int_0^t (v_n(s) - v(s), z_n(s))ds$$

$$= \int_0^t (v_n(s) - v(s), z_n(s) - z(s))ds + \int_0^t (v_n(s) - v(s), z(s))ds.$$

Note that $z \in C(T, H_w)$ and so $z \in L^\infty(T,H) \subseteq L^2(T,H)$. Hence $\int_0^t (v_n(s) - v(s), z(s))ds \rightarrow 0$ as $n \rightarrow \infty$. Also note that $f_n(s) - f(s) \in V(s) - V(s) = W(s) \in P_k(H)$ for all $s \in T$. So $\sup[(z_n(s) - z(s), w): w \in W(s)] \rightarrow 0$ as $n \rightarrow \infty$ and by the dominated convergence theorem, we have that $\int_0^t (v_n(s) - v(s), z_n(s) - z(s))ds \rightarrow 0$ as $n \rightarrow \infty$. Since we already know that $\{x_n\}_{n \geq 1} \subseteq C(T,H)$ is equicontinuous, from the Arzela-Ascoli theorem, we infer that $\{x_n\}_{n \geq 1}$ is relatively compact in $C(T,H)$. Because $x_n(t) \rightarrow x(t)$ in $H$ for all $t \in T$, we have $x_n \rightarrow x$ in $C(T,H)$ and $x \in K$. Thus we have proved the compactness of the set $K \subseteq C(T,H)$. Let $\widehat{K} = \overline{\text{conv}} K$. Mazur's theorem tells us that $\widehat{K} \in P_{kc}(C(T,H))$. Observe that $S_c(x_0) \subseteq \widehat{K}$.

Let $R: \widehat{K} \rightarrow P_{wkc}(L^1(T,H))$ be defined by $R(y) = S^1_{F(\cdot, y(\cdot))}$. Apply Theorem II.8.31, p. 260, of Hu-Papageorgiou [10] to produce a continuous map $r: \widehat{K} \rightarrow L^1_w(T,H)$ such that $r(y) \in \text{ext} R(y)$ for all $y \in \widehat{K}$. By $L^1_w(T,H)$ we denote the Lebesgue-Bochner space equipped with the weak norm $\| g \|_w = \sup[ \| \int_0^t g(s)ds \| : t \in T]$. Let $u = p \circ r: \widehat{K} \rightarrow \widehat{K}$. Using the lemma, p. 327 of Papageorgiou [19], we have that $u$ is continuous. So we can apply Schauder's fixed point theorem to obtain $x \in \widehat{K}$ such that $x = p(r(x))$. Since $r(x) \in \text{ext} R(x) = \text{ext} S^1_{F(\cdot, x(\cdot))} = S^1_{\text{ext} F(\cdot, x(\cdot))}$, we conclude that $x \in S_e(x_0)$. <u>*Q.E.D.*</u>

## 4. Strong Relaxation Theorem

In this section we prove a strong relaxation theorem, which says that the solution set $S_e(x_0)$ of problem (2) is dense in the solution set $S_c(x_0)$ of problem (1) for the $C(T,H)$-norm topology. In other words, convexification of the orientor field $(t,x)\to \mathrm{ext}F(t,x)$ does not add too many solutions. This result is of significance in control theory, in connection with the bang-bang principle. The interpretation of the result there is that it is possible to have essentially (i.e., within closure) the same attainable set by economizing on the set of controls. So we can have practically the same results with controls that are much simpler to build. To do this, we need the following stronger conditions on the multifunction $t(t,x)\to F(t,x)$.

**H(F)$_2$:** $F: T\times H \to P_{wkc}(H)$ is a multifunction such that

(*i*) for every $x\in H$, $t\to F(t,x)$ is measurable;

(*ii*) for almost all $t\in T$ and all $x,y\in H$, $h(F(t,x),F(t,y))\le k(t)\,\|x-y\|$ with $k(\cdot)\in L^1(T)$;

(*iii*) for all $t\in T$ and all $B\subseteq H$ bounded, we have that $\overline{F(t,B)}$ is compact;

(*iv*) for almost all $t\in T$ and all $x\in H$, $|F(t,x)| = \sup\{\|v\|: v\in F(t,x)\}\le \alpha(t)+c(t)\,\|x\|$ with $\alpha,c\in L^2(T)$.

For this, let $\{x_n^*\}_{n\ge 1}$ be a sequence of points in $B_1^*$ which is dense in the dual closed unit ball $B_1^*=\{x^*\in X^*: \|x^*\|\le 1\}$ for the Mackey topology $m(X^*,X)$. Let $\gamma_F$: $T\times H\times H\to \widehat{R} = R\cup\{+\infty\}$ be defined by

$$\gamma_F(t,x,v)=\begin{Bmatrix} \sum_{n\ge 1}\frac{(x_n^*,v)^2}{2^n} & \text{if } v\in F(t,x) \\ +\infty & \text{otherwise} \end{Bmatrix}.$$

Let $Aff(H)=\{$the set of all continuous affine functions $\alpha: H\to R\}$. Let $\widehat{\gamma}_F$: $T\times H\times H\to \widehat{R}=R\cup\{+\infty\}$ be defined by

$$\widehat{\gamma}_F(t,x,v)=\inf\{\alpha(v):\alpha\in Aff(H) \text{ and } \alpha(z)\ge \gamma_F(t,x,z) \text{ for all } z\in F(t,x)\}.$$

As always, $\inf\emptyset = -\infty$. The "Choquet function" $\xi_F: T\times H\times H\to \overline{R} = R\cup\{-\infty\}$ corresponding to $F(t,x)$ is defined by

$$\xi_F(t,x,v)=\widehat{\gamma}_F(t,x,v)-\gamma_F(t,x,v).$$

The next lemma lists the basic properties of this function and can be found in Hu-Papageorgiou [10] (Proposition II.4.1 and Theorem II.4.2, p. 190).

**Lemma 1:** *If hypothesis* H(F)$_2$ *holds, then*

(*a*) *$(t,x,v)\to \xi_F(t,x,v)$ is measurable;*

(*b*) *for every $t\in T$, $(x,v)\to\xi_F(t,x,v)$ is upper semicontinuous;*

(*c*) *for every $t\in T$, $x\in H$, $v\to \xi_F(t,x,v)$ is concave and strictly concave on $F(t,x)$;*

(*d*) *for almost all $t\in T$ and all $(x,v)\in GrF(t,\cdot)$, $0\le \xi_F(t,x,v)\le 4\alpha(t)^2+4c(t)^2\,\|x\|^2$;*

(*e*) *$\xi_F(t,x,v)=0$ if and only if $v\in extF(t,x)$.*

Now we are ready to prove the strong relaxation theorem

**Theorem 2:** *If hypotheses* H($\phi$), H(F)$_2$ *hold and $x_0\in dom\phi(0,\cdot)$, then $S_e(x_0)$ is a dense, $G_\delta$-subset of $S_c(x_0)$ in $C(T,H)$.*

**Proof:** Let $x \in S_c(x_0)$. Then by definition, $x = p(f)$ with $f \in S^2_F(\cdot, x(\cdot))$. Let $\widehat{K} \in P_{kc}(C(T,H))$ be as in the proof of Theorem 1. For $\varepsilon > 0$ and $y \in \widehat{K}$, we define

$$\Delta_\varepsilon(t) = \{h \in F(t,y(t)) \colon \| f(t) - h \| < \varepsilon + d(f(t), F(t,y(t)))\}.$$

From Proposition II.7.8, p. 229 of Hu-Papageorgiou [10], we know that $(t,x) \to F(t,x)$ is jointly measurable. Hence $t \to F(t,y(t))$ is measurable and then so is $t \to d(f(t), F(t,y(t)))$. Therefore we infer that

$$Gr\Delta_\varepsilon \in \mathcal{L}(T) \times B(H)$$

where $\mathcal{L}$ is the Lebesgue $\sigma$-field of $T$ and $B(H)$ the Borel $\sigma$-field of $H$. Thus we can apply the Yankov-von Neumann-Aumann selection theorem (see Hu-Papageorgiou [10], Theorem II.2.14, p. 158) and obtain $h: T \to H$, a measurable map such that $h(t) \in \Delta_\varepsilon(t)$ a.e. on $T$. Let $\Gamma_\varepsilon: \widehat{K} \to 2^{L^1(T,H)}$ be defined by

$$\Gamma_\varepsilon(y) = \{h \in S^1_{F(\cdot, y(\cdot))} \colon \| f(t) - h(t) \| < \varepsilon + d(f(t), F(t,y(t)) \text{ a.e. on } T\}.$$

We have just seen that $\Gamma_\varepsilon(y) \neq \emptyset$ for every $y \in \widehat{K}$. Moreover, from Lemma II.8.3, p. 239 of Hu-Papageorgiou [10], we know that $y \to \Gamma_\varepsilon(y)$ is a lower semicontinuous (lsc) multifunction with decomposable values. Hence so is $y \to \overline{\Gamma_\varepsilon(y)}$. Apply Theorem II.8.7, p. 245 of [10] to obtain $u_\varepsilon: \widehat{K} \to L^1(T,H)$, a continuous map such that $u_\varepsilon(y) \in \overline{\Gamma_\varepsilon(y)}$ for all $y \in \widehat{K}$. Hence we have

$$\| f(t) - u_\varepsilon(y)(t) \| \leq \varepsilon + d(f(t), F(t,y(t))) \leq \varepsilon k(t) \| x(t) - y(t) \| \text{ a.e. on } T.$$

Now apply Theorem II.8.31, p. 260 of [10] to obtain $v_\varepsilon: \widehat{K} \to L^1_w(T,H)$, a continuous map such that $v_\varepsilon(y) \in \text{ext} S^1_{F(\cdot, y(\cdot))} = S^1_{\text{ext} F(\cdot, y(\cdot))}$ and $\| u_\varepsilon(y) - v_\varepsilon(y) \|_w < \varepsilon$ for all $y \in R$ (recall that $L^1_w(T,H)$ denotes the Lebesgue-Bochner space $L^1(T,H)$ furnished with the weak norm $\| \cdot \|_w$ defined by $\| g \|_w = \sup[\| \int_0^1 g(s)ds \| : t \in T]$; see also the proof of Theorem 1).

Let $\varepsilon_n = \frac{1}{n}$ and $u_n = u_{\varepsilon_n}$, $v_n = v_{\varepsilon_n}$, $n \geq 1$ be as above. As in the proof of Theorem 1, via Schauder's fixed point theorem, we can find $x_n \in \widehat{K}$ such that $x_n = (p \circ v_n)(x_n)$, $n \geq 1$. Since $\widehat{K} \subseteq C(T,H)$ is compact, we may assume that $x_n \to z$ in $C(T,H)$ as $n \to \infty$. We have

$$(-\dot{x}_n(t) + x(t), x(t) - x_n(t)) \leq (v_n(x_n)(t) - f(t), x(t) - x_n(t)) \text{ a.e. on } T$$

$$\Rightarrow \frac{1}{2} \frac{d}{dt} \| x(t) - x_n(t) \|^2 \leq (v_n(x_n)(t) - f(t), x(t) - x_n(t)) \text{ a.e. on } T$$

$$\Rightarrow \frac{1}{2} \| x(t) - x_n(t) \|^2 \leq \int_0^1 (v_n(x_n)(s) - f(s), x(s) - x_n(s))ds$$

$$= \int_0^t (v_n(x_n)(s) - u_n(x_n)(s), x(s) - x_n(s))ds + \int_0^t (u_n(x_n)(s) - f(s), x(s) = x_n(s))ds.$$

From the lemma, p. 327, of Papageorgiou [19], we have that $v_n(x_n) - u_n(x_n) \xrightarrow{w} 0$ in $L^2(T,H)$ as $n \to \infty$. So we can say that

$$\int_0^t (v_n(x_n)(s) - u_n(x_n)(s), x(s) - x_n(s))ds \to 0 \text{ as } n \to \infty.$$

Also we have

$$\begin{aligned} &\int_0^t (u_n(x_n)(s) - f(s), x(s) - x_n(s))ds \\ &\leq \int_0^t \| u_n(x_n)(s) - f(s) \| \; \| x(s) - x_n(s) \| ds \\ &\leq \int_0^t (\varepsilon_n + k(s) \| x(s) - x_n(s) \| ) \| (x(s) - x_n(s) \| ds. \end{aligned}$$

Thus in the limit as $n \to \infty$, we obtain

$$\frac{1}{2} \| x(t) - z(t) \|^2 \leq \int_0^t k(s) \| x(s) - z(s) \|^2 ds$$

$$\Rightarrow x = z \text{ (by Gronwall's lemma).}$$

Hence we have $x_n \to x$ in $C(T,H)$ and clearly, $x_n \in S_e(x_0)$, $n \geq 1$. So we have proved that $S_c(x_0) \subseteq \overline{S_e(x_0)}$, the closure taken in $C(T,H)$. But $S_c(x_0)$ is closed (in fact, compact) in $C(T,H)$. Thus $S_c(x_0) = \overline{S_e(x_0)}$.

Now let $E_n = \{x \in S_c(x_0)$: $\int_0^b \xi_F(t, x(t), f(t))dt \leq \frac{1}{n}$ where $f \in S^2_{F(\cdot, x(\cdot))}$, $x = p(f)\}$. By virtue of Lemma 2 and Theorem 2.1 of Balder [2], we see that $S_e(x_0) \subseteq \cap E_n$. On the other hand, if $x \in \cap E_n$ we have $0 \leq \int_0^b \xi_F(t, x(t), f(t))dt < \frac{1}{n}$ for all $n \geq 1$, hence $\int_0^b \xi_F(t, x(t), f(t))dt = 0$. Because $\xi_F \geq 0$, we infer that $S_F(t, x(t), f(t)) = 0$ a.e. on $T$ which implies that $f(t) \in \text{ext}F(t, x(t))$ a.e. on $T$ (see Lemma 2(e)). So indeed $S_e(x_0)$ is a dense $G_\delta$-subset of $S_c(x_0)$. <u>*Q.E.D.*</u>

## 5. Topological Structure of the Solution Set

In this section we examine the topological structure of the solution set of the nonconvex problem. In previous works we investigated the structure of the solution set of the convex problem (see Papageorgiou [16, 17]). Here, for the nonconvex problem, we show that the solution set is path-connected. It will be compact in $C(T,H)$ if and only if the orientor field $F(t,x)$ is convex-valued (see Papageorgiou [17]). Note that our conclusion is stronger than that of the usual Kneser-type theorems which assert that the solution set is connected. Recall that a path-connected set is automatically connected but the converse is not necessarily true (see Dugundji [9], Example 4, p. 115). However, in order to achieve this stronger conclusion, we need to strengthen our continuity hypothesis on $F(t, \cdot)$ to h-

Lipschitzness. Recently, related results for single-valued semilinear parabolic problems were established by Ballotti [3] and Kikuchi [12].

**$\mathbf{H(F)_3}$:** $F: T \times H \rightarrow P_f(H)$ is a multifunction such that

(*i*) for every $x \in H$, $t \rightarrow F(t,x)$ is measurable;

(*ii*) for almost all $t \in T$ and all $x, y \in H$, $h(F(t,x), F(t,y)) \leq k(t) \| x - y \|$ with $k \in L^1(T)$;

(*iii*) for almost all $t \in T$ and all $x \in H$, $| F(t,x) | \leq \alpha(t) + c(t) \| x \|$ with $\alpha, c \in L^2(T)$.

Let $S(x_0) \subseteq C(T,H)$ denote the solution set of (1) when the orientor field $F(t,x)$ satisfies the above hypotheses. We shall assume that $S(x_0) \neq \emptyset$. With an extra compactness-type condition either on $\phi(t, \cdot)$ (see Papageorgiou [16, 17]) or on $F(t, \cdot)$ (see hypothesis $\mathrm{H(F)_1}(ii)$), we know that we can have that $S(x_0)$ is nonempty.

**Theorem 3:** *If hypotheses* $\mathrm{H}(\phi)$, $\mathrm{H(F)_3}$ *hold and* $x_0 \in dom\phi(0, \cdot)$ *holds, then* $S(x_0)$ *is a path-connected subset of* $C(T,H)$.

**Proof:** As in the proof of Theorem 1, through a priori estimation, we know that without any loss of generality we may assume that for almost all $t \in T$ and all $x \in H$, we have $| F(t,x) | \leq \psi(t)$, with $\psi \in L^2(T)$. Let $V = \{u \in L^1(T,H): \| u(t) \| \leq \psi(t)$ a.e. on $T\}$. Consider the multifunction $R: V \rightarrow P_f(L^1(T,H))$ defined by $R(u) = S^1_{F(\cdot, p(u)(\cdot))}$. Consider the following equivalent norm on the Lebesgue-Bochner space $L^1(T,H)$:

$$| h | = \int_0^b \exp(-L\theta(t)) \| h(t) \| dt,$$

with $L > 1$ and $\theta(t) = \int_0^t k(s)ds$, $t \in T$. In what follows, by $d_1(\cdot, \cdot)$ we denote the distance function corresponding to this new norm and by $h_1(\cdot, \cdot)$ the Hausdorff metric on $P_f(L^1(T,H))$ which is generated by $| \cdot |$. We will show that $u \rightarrow R(u)$ is an $h_1$-contraction with constant $\frac{1}{L} < 1$. To this end, let $u_1 u_2 \in V$ and let $h_1 \in R(u_1)$. We have

$$d_1(h_1, R(u_2)) = \inf\{ | h_1 - h_2 | : h_2 \in R(u_2)\}$$

$$= \inf\left\{ \int_0^b \exp(-L\theta(t)) \| h_1(t) - h_2(T) \| dt: h_2 \in R(u_2)) \right\}$$

$$= \int_0^b \exp(-L\theta(t)) \inf[ \| h_1(t) - v \| : v \in F(t, p(u_2)(t)]dt$$

(see Hu-Papageorgiou [10], Theorem II.3.24, p. 183)

$$= \int_0^b \exp(-L\theta(t)) d(h_1(t), F(t, p(u_2)(t)))dt.$$

Hence we have:

$$\sup[d_1(h_1, R(u_2)): h_1 \in R(u_1)]$$

$$= \sup\left[\int_0^b \exp(-L\theta(t)d(h_1(t), F(t, p(u_2)(t)))dt: h_1 \in R(u_1)\right]$$

$$= \int_0^b \exp(-L\theta(t))\sup[d(v, F(t, p(u_2)(t)): v \in F(t, p(u_1)(t)))]dt$$

(again Hu-Papageorgiou [10], Theorem II.3.24, p. 183)

$$= \int_0^b \exp(-L\theta(t))h^*(F(t, p(u_1)(t)), F(t, p(u_2)(t)))dt. \tag{5}$$

Interchanging the roles of $u_1, u_2$ we also have

$$\sup[d_1(h_2, R(u_1)): h_2 \in R(u_2)]$$
$$= \int_0^b \exp(-L\theta(t))h^*(F(t, p(u_2)(t)), F(t, p(u_1)(t)))dt. \tag{6}$$

From (5) and (6), we obtain that

$$h_1(R(u_1), R(u_2)) \le \int_0^b \exp(-L\theta(t))h(F(t, p(u_1)(t)), F(t, p(u_2)))dt$$

$$\le \int_0^b \exp(-L\theta(t))k(t) \| p(u_1)(t) - p(u_2)(t) \| dt$$

$$\le \int_0^b \exp(-L\theta(t))k(t) \int_0^t \| u_1(s) - u_2(s) \| dsdt \text{ (see the proof of Theorem 1)}$$

$$= -\frac{1}{L}\int_0^b \left(\int_0^t \| u_1(s) - u_2(s) \| ds\right) d(\exp(-L\theta(t)))$$

$$\le \frac{1}{L}\int_0^b \exp(-L\theta(t)) \| u_1(t) - u_2(t) \| dt \quad \text{(by integration by parts)}$$

$$= \frac{1}{L} | u_1 - u_2 | .$$

Set $\mathcal{T} = \{u \in V: u \in R(u)\}$. From Theorem V.1.11, p. 524 of Hu-Papageorgiou [10], we know that $\mathcal{T} \neq \emptyset$. Also the theorem of Bressan-Cellina-Fryszkoski [4], implies that $\mathcal{T}$ is an absolute retract of $V$. Note that $V$, being a closed, convex subset of $L^1(T, H)$ is by Dugundji's extension theorem (see Dugundji [9], Theorem 6.1., p.188 or Hu-Papageorgiou [10], Theorem I.2.88, p. 70) an absolute retract of $L^1(T, H)$. So $V$ is an absolute retract in $L^1(T, H)$ (see Kuratowski [14], Theorem 6, p. 341), which means that $V$ is path-connected in $L^1(T, H)$ (see Kuratowski [14], p.

339). Note that $S(x_0) = p(\mathfrak{T})$. Recalling that the solution map $p(\cdot)$ is continuous and that the continuous image of a path-connected set is path-connected (see Dugundji [9], p. 115), we conclude that $S(x_0)$ is path-connected in $C(T, H)$. Q.E.D.

**Remark:** If hypotheses $\mathrm{H(F)}_2$ hold, then $S(x_0)$ is compact and connected in $C(T, H)$, thus a continuum in $C(T, H)$. Moreover, arguing as in Papageorgiou-Shahzad [22], we can show that in fact, $S(x_0)$ is a $R_\delta$-set in $C(T, H)$ (i.e., a decreasing sequence of compact absolute retracts). Recall that an $R_\delta$-set is acyclic and connected, but not necessarily path-connected (consider the topologists sine curve; see Dugundji [9], p. 115).

## 6. Control Systems

In this section we consider infinite dimensional control systems with a prior feedback and we use our results on subdifferential evolution inclusions to obtain nonlinear versions of the bang-bang principle and investigate the properties of the reachable sets.

So consider the following nonlinear feedback control system:

$$\left\{ \begin{array}{l} -\dot{x}(t) \in \partial\phi(t, x(t)) + g(t, x(t)) + B(t)u(t) \text{ a.e. on } T \\ x(0) = x_0, \quad u(t) \in U(t, x(t)) \text{ a.e. on } T \end{array} \right\}. \tag{7}$$

In conjunction with (5), we also consider the system, which has an admissible controls the "bang-bang" controls. More precisely, we consider the following system:

$$\left\{ \begin{array}{l} -\dot{x}(t) \in \partial\phi(t, x(t)) + g(t, x(t)) + B(t)u(t) \text{ a.e. on } T \\ x(0) = x_0, \quad u(t) \in \mathrm{ext}U(t, x(t)) \text{ a.e. on } T \end{array} \right\}. \tag{8}$$

We start by showing that system (8) has trajectories (solutions), which we call "extremal solutions" and as before denote them by $S_e(x_0) \subseteq C(T, H)$. The set of trajectories of (7) is denoted by $S(x_0) \subseteq C(T, H)$.

In what follows, $Y$ is a separable Banach space and models the control space. Our hypotheses on the data are the following:

**H(g):** $g: T \times H \to H$ is a multifunction such that

(*i*) for every $x \in H$, $t \to g(t, x)$ is measurable;

(*ii*) for every $t \in T$, $x \to g(t, x)$ is completely continuous;

(*iii*) for almost all $t \in T$ and all $x \in H$, $\| g(t, x) \| \leq \alpha(t) + c(t) \| x \|$ with $\alpha, c \in L^2(T)$.

**H(B):** For every $t \in T$, $B(T) \in \mathcal{L}(Y, H)$, is compact, for all $u \in Y$, $t \to B(t)u$ is measurable and $\| B(t) \|_{\mathcal{L}} \leq M$ a.e. on $T$, with $\| \cdot \|_{\mathcal{L}}$ denoting the operator norm on the Banach space $\mathcal{L}(Y, H)$.

**H(U):** $U: T \times H \to P_{wkc}(H)$ is a multifunction such that

(*i*) for every $x \in H, t \to U(t, x)$ is measurable;

(*ii*) for every $t \in T$, $x \to U(t, x)$ is h-continuous;

(*iii*) for almost all $t \in T$ and all $x \in H$, $| U(t, x) | \leq \alpha_1(t) + c_1(t) \| x \|$ with $\alpha_1, c_1 \in L^2(T)$.

**Proposition 1:** *If hypotheses* H($\phi$), H(g), H(B), H(U) *hold and* $x_0 \in dom\phi(0, \cdot)$, *then* $S_e(x_0) \subseteq C(T, H)$ *is nonempty.*

**Proof:** Let $F: T \times H \rightarrow P_{wkc}(H)$ be defined by $F(t,x) = g(t,x) + B(t)U(t,x)$. It is easy to see that $t \rightarrow F(t,x)$ is measurable. Also for every $t \in T$ and all $x, y \in H$, we have

$$h(F(t,x), F(t,y) \leq \| g(t,x) - g(t,y) \| + h(B(t)U(t,x), B(t)U(t,y))$$

$$\leq \| g(t,x) - g(t,y) \| + Mh(U(t,x), U(t,u))$$

$$\Rightarrow x \rightarrow F(t,x) \text{ is h-continuous.}$$

Moreover, hypotheses H(g)(*ii*) and H(B) imply that for all $B \subseteq H$ bounded and all $t \in T$, $\overline{F(t,B)} \in P_k(H)$, while hypotheses H(g)(*iii*) and H(U)(*iii*) tell us that for almost $t \in T$ and all $x \in H$, $| F(t,x) | \leq \widehat{\alpha}(t) + \widehat{c}(t) \| x \|$ with $\widehat{\alpha}, \widehat{c} \in L^2(T)$. A straightforward application of the Yanikov-von Neumann-Aumann selection theorem shows that (7) is equivalent to the following evolution inclusion (control free or deparametrized system):

$$\left\{ \begin{array}{l} -\dot{x}(t) \in \partial\phi(t, x(t)) + F(t, x(t)) \text{ a.e. on } T \\ x(0) = x_0 \end{array} \right\}. \tag{9}$$

Finally note that

$$\mathrm{ext}F(t,x(t)) = g(t,x(t)) + \mathrm{ext}B(t)U(t,x(t)) \subseteq g(t,x(t)) + B(t)\mathrm{ext}U(t,x(t)).$$

So an application of Theorem 1 gives us the nonemptiness of $S_e(x_0)$. *Q.E.D.*

In fact, the above proposition remains valid if $Y$ is assumed to be the dual of a separable Banach space; i.e., $Y = V^*$ with $V$ being a separable Banach space. In this case, our hypotheses on the control constraining multifunction are the following:

**H(U)$_1$:** $U: H \rightarrow 2^Y \backslash \{\emptyset\}$ is an h-continuous multifunction with $w^*$-compact and convex values and for all $x \in H$, $| U(x) | \leq c \| x \|$, $c > 0$.

Then with minor changes in the proof of Proposition 1, we obtain the following result:

**Proposition 2:** *If hypotheses* H($\phi$), H(g), H(B), H(U)$_1$ *hold and* $x_0 \in dom\phi(0, \cdot)$, *then* $S_e(x_0) \subseteq C(T,H)$ *is nonempty.*

If we strengthen our hypotheses on the data, we can have nonlinear versions of the "bang-bang principle." The stronger hypotheses needed are the following:

**H(g)$_1$:** $g: T \times H \rightarrow H$ is a map such that

(*i*) for every $x \in H$, $t \rightarrow g(t,x)$ is measurable;

(*ii*) for almost all $t \in T$ and all $x, y \in H$, $\| g(t,x) - g(t,y) \| \leq k_1(t) \| x - y \|$ with $k_1 \in L^1(T)$;

(*iii*) for almost all $t \in T$ and all $x \in H$, $\| g(t,x) \| \leq \alpha(t) + c(t) \| x \|$ with $\alpha, c \in L^2(T)$.

If the control space $Y$ is a separable Banach space, then our hypotheses on $U$ have the following form:

**H(U)$_2$:** $U: T \times H \rightarrow P_{wkc}(Y)$ is a multifunction such that

(*i*) for every $x \in H, t \rightarrow U(t,x)$ is measurable;

(*ii*) for almost all $t \in T$ and all $x, y \in H$, $h(U(t,x), U(t,y)) \leq k_2(t) \| x - y \|$ with $k_2 \in L^2(T)$;

(*iii*) for almost all $t \in T$ and all $x \in H$, $| U(t,x) | \leq \alpha_1(t) + c_1(t) \| x \|$ with

$\alpha_1, c_1 \in L^2(T)$.

If the control space $Y$ is the dual of a separable Banach space $V$ (i.e., $Y = V^*$), then the hypotheses on $U$ take the following form:

**H(U)$_3$:** $U: T \to 2^Y \backslash \{\emptyset\}$ is a multifunction with $w^*$-compact and convex values, for all $x, y \in H$, $h(U(x), U(y)) \leq k_2 \| x - y \|$, $k_2 > 0$ and for all $x \in H$ $| U(x) | \leq c \| x \|$, $c > 0$.

In what follows by $R(t)$ (resp. $R_e(T)$), we denote the reachable set at time $t \in T$ of (7) (resp. (8)); i.e., $R(t) = \{x(t): x \in S(x_0)\}$ and $R_e(t) = \{x(t): x \in S_e(x_0)\}$, $t \in T$.

Using Theorem 2, we obtain the following nonlinear "bang-bang principle":

**Proposition 3:** *If hypotheses* H($\phi$), H(g)$_1$, H(B), H(U)$_2$ *hold and* $x_0 \in dom\phi(0, \cdot)$, *then* $S(x_0) = \overline{S_e(x_0)}^{C(T,H)}$ *and for all* $t \in T$, $R(t) = \overline{R_e(t)}^H \in P_k(H)$.

We can also treat the case when the feedback control constraint multifunction need not have convex values. In this case, $Y$ is a separable Banach space and the hypotheses on $U(t,x)$ are the following:

**H(U)$_4$:** $U: T \times H \to P_{wk}(Y)$ is a multifunction satisfying hypotheses H(U)$_3$($i$), ($ii$) and ($iii$).

Using Theorem 3, we obtain the following result:

**Proposition 4:** *If hypotheses* H($\phi$), H(g), H(B), H(U)$_4$ *hold and* $x_0 \in dom\phi(0, \cdot)$, *then* $S(x_0)$ *is path-connected in* $C(T,H)$ *and for every* $t \in T$, $R(t)$ *is path-connected in* $H$.

Finally, note that if $\eta: C(T,H) \to R$ is a continuous cost functional and we consider the following optimization problem:

$$\inf\{\eta(x): x \in S(x_0)\} = m \tag{10}$$

then (10) has a solution $x^*$ and given any $\varepsilon > 0$, we can find an extremal trajectory $x_\varepsilon^* \in S_e(x_0)$ which is $\varepsilon$-optimal, i.e., $m \leq \eta(x_\varepsilon^*) \leq m + \varepsilon$.

## 7. Examples

In what follows, $Z \subseteq R^N$ is a bounded domain with a $C^1$-boundary $\Gamma$.

(1) First we consider the following nonlinear control system:

$$\left\{ \begin{array}{l} \frac{\partial x}{\partial t} - \operatorname{div}(\alpha(t,z) \| Dx \|^{p-2} Dx) + \beta(x(z)) - \int_Z k(t,z,z',x(t,z'))dz' \\ + \int_Z b(t,z,z')u(t,z'dz') \text{ on } T \times Z \\ \\ x(0,z) = x_0(z) \text{ a.e. on } Z,\ x|_{T \times \Gamma} = 0,\ u(\cdot,\cdot) \text{ measurable}, \\ 0 \leq u(t,z) \leq 1 \text{ a.e. on } T \times Z,\ p \geq 2 \end{array} \right\}. \tag{11}$$

We make the following hypotheses:

**H($\alpha$):** $\alpha: T \times Z \to R$ is a function such that for all $t \in T$, $z \to \alpha(t,z)$ is measurable, for all $z \in Z$ and all $t, s \in T$, $| \alpha(t,z) - \alpha(s,z) | \leq \eta(z) | t - s |$ with $\eta \in L^\infty(Z)$ and for almost all $(t,z) \in T \times Z$, $0 < m_1 \leq \alpha(t,z) \leq m_2$.

**H($\beta$):** $\beta: R \to 2^R$ is a maximal monotone set with $0 \in \operatorname{dom}\beta$. This implies (see Brezis [5], p. 43) that $\beta(\cdot)$ is cyclically maximal monotone and so there exists $j$: $R \to$

$\bar{R} = R \cup \{+\infty\}$ proper, convex and lower semicontinuous (i.e., $j \in \Gamma_0(R)$) such that $\eta = \partial j$.

**H(k):** $k: T \times Z \times Z \times R \to R$ is a function such that

(*i*) for every $r \in R$, $(t, z, z') \to k(t, z, z', r)$ is measurable;

(*ii*) for every $(t, z, z') \in T \times Z \times Z$, $r \to k(t, z, z', r)$ is continuous;

(*iii*) for almost all $(t, z, z') \in T \times Z \times Z$ and all $r \in R$, $|k(t, z, z', r)| \leq \gamma(t, z, z')(1 + |r|)$ with $\gamma \in L^2(T \times Z \times Z)$.

**H(b):** $b: T \times Z \times Z \to R$ is a continuous map.

Let $H = L^2(Z)$ and define $\phi: T \times H \to \bar{R} = R \cup \{+\infty\}$ by

$$\phi(t,x) = \left\{ \begin{array}{ll} \frac{1}{p}\int_Z \alpha(t,z) \| Dx \|^p dz + \int_Z j(z, x(z)) dz & \text{if } w \in W_0^{1,p}(Z), j(\cdot, x(\cdot)) \in L^1(Z) \\ +\infty & \text{otherwise} \end{array} \right\}. \tag{12}$$

Note that for $t, s \in T$ and $x \in W_0^{1,p}(Z)$ we have

$$\phi(s,x) - \phi(t,x) \leq \frac{1}{p}\int_Z (\alpha(s,z) - \alpha(t,z)) \| Dx \|^p dz$$

$$\frac{1}{p}\int_Z \eta(z) | t - s | \; \| Dx \|^p dz.$$

Since $j \in \Gamma_0(R)$ (see hypothesis H($\beta$)), we can find $c_1, c_2 > 0$ such that $-c|y| - c_2 \leq j(y)$. So if $\|x\|_2 \leq r$, then $\int_Z j(x(z))dz \geq -c_3\|x\|_2 - c_4 \geq -c_3 r - c_4$ for some $c_3, c_4 > 0$. Thus if we take $K_r > c_3 r + c_4 > 0$, we see that hypothesis H($\phi$) is satisfied. Moreover, from Proposition 5.2, pp. 194-195 of Showalter [23] we have that

$$\partial\phi(t,x) = -\operatorname{div}(\| Dx \|^{p-2} Dx) + \beta(x) = L_p^\beta(t,x)$$

for all $x \in D(t) = \{y \in W_0^{1,p}(Z): L_p^\beta(t,x) \in L^2(Z)\}$. Let $g: T \times H \to H$ be defined by $g(t,x)(z) = \int_Z k(t,z,z',x(z'))dz'$. From Fubini's theorem we see that for every $v \in L^2(Z) = H$, $t \to (g(t,x), v)$ is measurable. So $t \to g(t,x)$ is weakly measurable and since $H$ is separable from the Pettis measurability theorem, we have that $t \to g(t,x)$ is measurable. Moreover, by virtue of hypothesis $H(k)$ and the Krasnosel'skii-Ladyzenskii theorem we have that $g(t,\cdot)$ is compact. Also $\| g(t,x) \| \leq \| \gamma(t,\cdot,\cdot) \|_2 (1 + \| x \|)$ a.e. on $T$. So hypothesis H(g) in Section 6 holds.

Let $Y = L^\infty(Z) = L^1(Z)^*$ and set $U = \{u \in Y: \|u\|_\infty \leq 1, u \geq 0\}$. Let $B: T \to \mathcal{L}(Y, H)$ be defined by

$$B(t)u(z) = \int_Z b(t,z,z')u(z')dz'.$$

Hypothesis H(b) implies that $B(t)(\cdot)$ is compact. Thus we have satisfied H(B).

Now rewrite (11) in the following equivalent evolution inclusion form:

$$\left\{ \begin{array}{l} -\dot{x}(t) \in \partial\phi(t, x(t)) + g(t, x(t)) + B(t)u(t) \text{ a.e. on } T \\ x(0) = x_0, u(t) \in U \text{ a.e. on } T,\ u(\cdot) \text{ measurable} \end{array} \right\}. \tag{13}$$

Recall that ext$U = \{\chi_A: A \subseteq Z$ Borel$\}$. Using Proposition 3 to obtain:

**Proposition 5:** *If hypothesis* H($\alpha$), H($\beta$), H(k), H(b) *hold,* $x_0 \in W^{2,p}(Z) \cap W_0^{1,p}(Z)$ *and* $x(\cdot,\cdot) \in C(T, L^2(Z))$ *is a trajectory (weak solution) of* (11), *then given* $\varepsilon > 0$, *we can find* $A \subseteq Z$ *Borel such that if* $y \in C(T, L^2(Z))$ *is the trajectory of* (11) *generated by the control function* $u(\cdot) = \chi_E(\cdot)$ *(time-invariant), we have* $\sup_{t \in T} \| x(t,\cdot) - y(t,\cdot) \|_2 < \varepsilon$.

(2) We consider the following multivalued porous-medium equation:

$$\left\{ \begin{array}{l} \frac{\partial x}{\partial t} - \Delta\beta(t, x(t,z)) \in f(t, x(t))(z) \text{ on } T \times Z \\ \\ x(0,z) = x_0(z) \text{ a.e. on } Z,\ \beta(t, x(t,z)) = 0 \text{ on } T \times \Gamma \text{ measurable}, \\ 0 \le u(t,z) \le 1 \text{ a.e. on } T \times z,\ p \ge 2. \end{array} \right\}. \tag{14}$$

In conjunction with (14), we consider the following problem:

$$\left\{ \begin{array}{l} \frac{\partial x}{\partial t} - \Delta\beta(t, x(t,z)) \in \operatorname{ext} f(t, x(t))(z) \text{ on } T \times Z \\ \\ x(0,z) = x_0(z) \text{ a.e. on } Z,\ \beta(t, x(t,z)) = 0 \text{ on } T \times \Gamma \text{ measurable}, \\ 0 \le u(t,z) \le 1 \text{ a.e. on } T \times z,\ p \ge 2. \end{array} \right\}. \tag{15}$$

We need the following hypotheses:

**H($\beta$)$_1$:** $\beta: T \times Z \to 2^R \backslash \{\emptyset\}$ is a measurable maximal monotone map and $j(t,z)$ is a normal integrand such that $\partial j(t,z) = \beta(t,z)$ (see Hu-Papageorgiou [10]). Assume $t \to j(t,z)$ is nonincreasing.

Let $f: T \times H^{-1}(Z) \to P_{fc}(H^{-1}(Z))$ be defined by

$$f(t,x) = \overline{\{h \in L^2(Z): |h(z)| \le r(t, z, \|x\|_{H^{-1}(Z)}) \text{ a.e. on } Z\}}^{\|\cdot\|_{H^{-1}(z)}}$$

with $r(t,z,v)$ satisfying:

**H(r):** $r: T \times Z \times R_+ \to R_+$ is a function such that

(*i*) for all $v \in R_+$, $(t,z) \to r(t,z,v)$ is measurable;

(*ii*) for almost all $(t,z) \in T \times Z$ and all $v, v' \in R_+$, $|r(t,z,v) - r(t,z,v')| \le k(t,z)|v - v'|$ with $k \in L^1(T \times Z)$;

(*iii*) for almost all $(t,z) \in T \times Z$ and all $v \in R_+$, $r(t,z,v) \le \alpha(t,z) + c(t,z)v$ with $\alpha, c \in L^2(T \times Z)$.

Let $H = H^{-1}(Z)$ and define $\phi: T \times H \to \overline{R}$ by

$$\phi(t,x) = \left\{ \begin{array}{ll} \int_Z j(t, x(z))dz & \text{if } x \in L^1(Z), j(t, x(\cdot)) \in L^1(Z) \\ +\infty & \text{otherwise} \end{array} \right\}. \tag{16}$$

Hypothesis H($\beta$)$_1$ guarantees that $H(\phi)$ holds and from Brezis [6], we know that $\partial\phi(t,x) = A(t,x)$ where $A: T \times H^{-1}(Z) \to 2^{H^{-1}(Z)}$ is defined by

$$A(t,x) = \{-\Delta u: u \in H_0^1(Z) \text{ an } du(z) \in \beta(t, x(z)) \text{ a.e. on } Z\}$$

for all $x \in D(t) = \{x \in H^{-1}(Z) \cap L^1(Z)$: there is some $u \in H_0^1(Z)$ such that $u(z) \in \beta(t, x(z))$ a.e. on $Z\}$. Recall that $-\Delta$ is the canonical isomorphism from $H_0^1(Z)$ onto $H^{-1}(Z)$.

Since $L^2(Z)$ is embedded compactly in $H^{-1}(Z)$, we verify that $H(f)_2$ holds for the multifunction $f(t,x)$. As before, by $S_c(x_0)$ (resp. $S_e(x_0)$) we denote the set of weak solutions of (14) (resp. (15)). Theorem 3 gives us:

**Proposition 6:** *If hypotheses* H$(\beta)_1$, H(f) *hold and* $x_0 \in L^1(Z)$, $j(t, x_0(\cdot)) \in L^1(Z)$ *for almost all* $t \in T$, *then* $S_c(x_0) = \overline{S_e(x_0)}^{C(T, H^{-1}(Z))}$. *Moreover, for every* $x \in S_c(x_0)$, $\frac{\partial x}{\partial t} \in L^2(T, H^{-1}(Z))$, $j(\cdot, x(\cdot, \cdot)) \in L^\infty(T, L^1(Z))$ *and* $\beta(\cdot, x(\cdot, \cdot)) \in L^2(t, H_0^1(Z))$.

(3) Now let $\phi(t,x) = \partial \delta_{K(t)}(x)$ with $\delta_{K(t)}(x) = \left\{ \begin{array}{cc} 0 & \text{if } x \in K(t) \\ +\infty & \text{otherwise} \end{array} \right\}$. It is well-known that $\partial \delta_{K(t)}(x) = N_{K(t)}(x)$ ( = the normal cone to $K(t)$ at $x$). Then problems (1) and (2) become

$$\left\{ \begin{array}{l} -\dot{x}(t) \in N_{K(t)}(x(t)) + F(t, x(t)) \text{ a.e. on } T \\ x(0) = x_0 \in K(0) \end{array} \right\} \tag{17}$$

and

$$\left\{ \begin{array}{l} -\dot{x}(t) \in N_{K(t)}(x(t)) + \operatorname{ext} F(t, x(t)) \text{ a.e. on } T \\ x(0) = x_0 \in K(0) \end{array} \right\}. \tag{18}$$

Problems of this form are known as "differential variational inequalities." They arise in theoretical mechanics in the study of elastoplastic (see Moreau [15]) and in mathematical economics in the study of planning processes (see Cornet [7] and the references therein). In particular, as it was shown in Cornet [7], if $H = R^N$ and $K(t) = K \in P_{fc}(R^N)$ for all $t \in T$, then problem (12) is equivalent to the "projected differential inclusion" $-\dot{x}(t) \in \operatorname{proj}(F(t, x(t)); T_K(x(t)))$ a.e. on $T$, $x(0) = x_0$ with $T_K(x(t))$ being the tangent cone to $K$ at $x(t)$. In many applications, when dealing with systems having constraints, in describing the effect of the constraint on the dynamics of the system, it can be assumed that the velocity $\dot{x}$ is projected at each time instant on the set of allowed directions toward $K$ at $x(t)$. This is true for electrical networks with diode nonlinearities.

We make the following hypothesis concerning the multifunction $t \rightarrow K(t)$.

**H(K):** $K: T \rightarrow P_{fc}(H)$ is a multifunction such that $h(K(t'), K(t)) \leq \int_t^{t'} v(s)ds$ with $v \in L^2(T)$.

If $\phi(t,x) = \delta_{K(t)}(x)$, then it is easy to check that $H(\phi)$ holds with $g_r(t) = V(T) = \int_0^t v(s)ds$, $\dot{g}_r(t) = v(t), \beta = 2$, $\alpha = 0$ and $K_r = 1$. Thus using Theorems 1 and 3 we obtain the following results which complement the work of Papageorgiou [20].

**Proposition 7:** *If hypotheses* H(K) *and* H(F)$_1$ *hold, then problem* (18) *has a solution* $x \in W^{1,2}(T, H)$.

Let $S_c(x_0)$ (resp. $S_e(x_0)$) be the solution set of (17) (resp. of (18)).

**Proposition 8:** *If hypotheses* H(K) *and* H(F)$_2$ *hold, then* $S_e(x_0)$ *is a dense* $G_\delta$-*subset of* $S_c(x_0)$ *in* $C(T, H)$.

Moreover, using Theorem 3, we can have the following result which complements and extends the structural result of Papageorgiou [21].

**Proposition 9:** *If hypotheses* H(K) *and* $\text{H(F)}_3$ *hold, then the solution set of problem* (17) *is path-connected in* $C(T,H)$.

Let us consider a specific such problem in $R^N$. So $H = R^N$, $R_+^N$ is the positive cone and we write $x \leq y$ if and only if $y - x \in R_+^N$, while $x < y$ if and only if $y - x \in \text{int} R_+^N$. The problem that we will examine is the following:

$$\left\{ \begin{array}{l} x(t) \geq u(t) \text{ for all } t \in T \\ -\dot{x}(t) i F(t,x(t)) \text{ a.e. on } T_1 = \{s \in T: x(s) > u(s)\} \\ -\dot{x}(t) \in F(t,x(t)) + R^N \text{ a.e. on } T_2 = \{s \in T: x(s) = u(s))\} \\ x(0) = x_0 \leq u(0) \end{array} \right\}. \tag{19}$$

Here $u \in W^{1,2}(T,R^N)$ is the "obstacle function."

Let $K: T \rightarrow P_{fc}(R^N)$ be defined by $K(t) = \{v \in R^N: v \underline{\gg} u(t)\}$. Note that for all $0 \leq t \leq t' \leq b$ we have $h(K(t'),K(t)) \leq \| u(t') - u(t) \| \leq \int_t \| u(s) \| ds$, hence $H(K)$ holds. Also for $v \in K(t)$ we have

$$T_{K(t)}(v) = R_+^N \text{ if } v = u(t)$$

and

$$T_{K(t)}(v) = R^N \text{ if } v > u(t).$$

Thus it follows that

$$N_{K(t)}(v) = T_{K(t)}(v)^* = R_-^N = -R_+^N \text{ if } v = u(t)$$

and

$$N_{K(t)}(v) = T_{K(t)}(v)^* = \{0\} \text{ if } v > u(t).$$

Therefore we can equivalently rewrite (19) as follows:

$$\left\{ \begin{array}{l} -\dot{x}(t) \in N_{K(t)}(x(t)) + F(t,x(t)) \text{ a.e. on } T \\ x(0) = x_0 \end{array} \right\}. \tag{20}$$

This means that the results of this paper apply to problem (19). In particular, it has a nonempty solution set which is compact in $C(T,R^N)$.

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