Journal of Applied Mathematics and Stochastic Analysis, **12**:3 (1999), 261-263.

THE SOLUTION OF AN OPEN PROBLEM GIVEN BY H. HARUKI AND T.M. RASSIAS

BARA KIM
Korea Advanced Institute of Science and Technology (KAIST)
Department of Mathematics and Center for Applied Mathematics
373-1 Kusong-Dong, Yusong-Gu, Taejon 305-701, Korea
e-mail: bara@mathx.kaist.ac.kr

(Received July, 1998; Revised December, 1998)

Haruki and Rassias [1] generalized the Poisson kernel in two dimensions and discussed integral formulas for each case. They presented an open problem for an integral formula. In this paper, we give a solution to that problem.

Key words: Poisson Kernel, Integral Formula.
AMS subject classifications: 31A05, 31A10.

1. Introduction

Haruki and Rassias [1] introduced two types of generalizations of the Poisson kernel. One of them is defined by

$$Q(\theta; a, b) \triangleq \frac{1-ab}{(1-ae^{i\theta})(1-be^{-i\theta})},$$

where a, b are complex parameters satisfying $|a| < 1$ and $|b| < 1$.

They proved the integral formulas:

$$\frac{1}{2\pi}\int_0^{2\pi} Q(\theta; a, b)d\theta = 1, \tag{1}$$

$$\frac{1}{2\pi}\int_0^{2\pi} Q(\theta; a, b)^2 d\theta = \frac{1+ab}{1-ab}. \tag{2}$$

They set the open problem as follows:
"Let

$$I_n \triangleq \frac{1}{2\pi}\int_0^{2\pi} Q(\theta; a,b)^{n+1} d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1-ab}{(1-ae^{i\theta})(1-be^{-i\theta})}\right)^{n+1} d\theta, \quad (n = 0, 1, \ldots), \tag{3}$$

where a, b are complex parameters satisfying $|a| < 1$ and $|b| < 1$. Compute I_n for $n = 2, 3, 4, \ldots$."

In the next section, we will give the solution to the problem.

2. Solution of the Problem

Theorem 1: *I_n, defined by (3), satisfies*

$$I_n = \sum_{j=0}^{n} \frac{(2n-j)!}{j!((n-j)!)^2}\left(\frac{ab}{1-ab}\right)^{n-j},$$

for $n = 0, 1, 2, \ldots$, and complex values a, b are such that $|a| < 1$ and $|b| < 1$.

Proof: By the change of variables, with $z = e^{i\theta}$, (3) becomes

$$I_n = \frac{1}{2\pi i}\oint_{|z|=1}\left(\frac{1-ab}{(1-az)(1-bz^{-1})}\right)^{n+1} z^{-1} dz$$

$$= \frac{1}{2\pi i}\oint_{|z|=1}\left(\frac{1-ab}{1-az}\right)^{n+1} z^n (z-b)^{-n-1} dz.$$

Let

$$f(z) \triangleq \left(\frac{1-ab}{1-az}\right)^{n+1} z^n (z-b)^{-n-1}.$$

Then $f(z)$ is analytic on $\{z \in \mathbb{C}: |z| \le 1, z \ne b\}$ and has a pole at $z = b$. Therefore, by the residue theorem, I_n is the residue of $f(z)$ at $z = b$.

The Laurent series expansion of $f(z)$ at $z = b$ gives:

$$f(z) = \left(\frac{1}{1 - \frac{a}{1-ab}(z-b)}\right)^{n+1} (b + (z-b))^n (z-b)^{-n-1}$$

$$= \sum_{k=0}^{\infty}\binom{n+k}{k}\left(\frac{a}{1-ab}\right)^k (z-b)^k \sum_{j=0}^{n}\binom{n}{j} b^{n-j}(z-b)^j (z-b)^{-n-1}$$

$$= \sum_{k=0}^{\infty}\sum_{j=0}^{n}\binom{n+k}{k}\binom{n}{j}\left(\frac{a}{1-ab}\right)^k b^{n-j}(z-b)^{k+j-n-1}.$$

Therefore, the residue of $f(z)$ at b, which is I_n, is given by

$$I_n = \sum_{j=0}^{n} \binom{2n-j}{n-j} \binom{n}{j} \left(\frac{ab}{1-ab}\right)^{n-j}$$

$$= \sum_{j=0}^{n} \frac{(2n-j)!}{j!((n-j)!)^2} \left(\frac{ab}{1-ab}\right)^{n-j}. \qquad \square$$

Note that we obtain (1) and (2) by substituting $n = 0$ and $n = 1$, respectively.

References

[1] Haruki, H. and Rassias, T.M., New generalizations of the Poisson kernel, *J. Appl. Math. Stoch. Anal.* **10**:2 (1997), 191-196.