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EXISTENCE AND UNIQUENESS OF SOLUTIONS OF A SEMILINEAR FUNCTIONAL-DIFFERENTIAL EVOLUTION NONLOCAL CAUCHY PROBLEM

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Two theorems about the existence and uniqueness of mild and classical solutions of a semilinear functional-differential evolution nonlocal Cauchy problem in a general Banach space are proved. Methods of semigroups and the Banach contraction theorem are applied.

Key words: Abstract Cauchy Problem, Evolution Equation, Functional-Differential Equation, Nonlocal Condition, Existence and Uniqueness of the Solutions, Mild and Classical Solutions, Banach Contraction Theorem.

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1. Introduction

In this paper we study the existence and uniqueness of mild and classical solutions of a semilinear functional-differential evolution nonlocal Cauchy problem in a general Banach space. Methods of C_0 semigroups and the Banach theorem about the fixed point are applied. The functional-differential evolution nonlocal Cauchy problem considered here is of the form

$$u'(t) + Au(t) = F_1(t, u(t), u(\sigma_1(t)), \ldots, u(\sigma_m(t)))$$

$$+ \int_{t_0}^{t} F_2(t, s, u(s), \int_{t_0}^{s} f(s, \tau, u(\tau))d\tau)ds,\ t \in (t_0, t_0 + a] \tag{1.1}$$

and

$$u(t_0) + G(u) = u_0, \tag{1.2}$$

where $t_0 \geq 0$, $a > 0$, $-A$ is the infinitesimal generator of a C_0 semigroup of operators on a Banach space, F_i $(i=1,2)$, G, f, σ_i $(i=1,\ldots,m)$ are given functions satisfying some assumptions and u_0 is an element of the Banach space.

The results obtained pertaining to the nonlocal evolution problem are generalizations of those given by Byszewski [2, 4, 5], and by Balasubramaniam and Chandrasekaran [1]. Moreover, the results obtained concerning the evolution problem (1.1)-(1.2), where $F_2=0$ and $G=0$, are generalizations of those given by Winiarska [10] and Pazy [9].

Nonlocal semilinear and nonlinear functional-differential evolution Cauchy problems in general Banach spaces have also been studied by Byszewski [3, 6, 7] and by Lin, Liu [8].

2. Notation and Definitions

Let E be a Banach space with norm $\|\cdot\|$ and let $\{T(t)\}_{t\geq 0}$ be a C_0 semigroup of operators on E.

In this paper we assume that $-A$ is the infinitesimal generator of a C_0 semigroup of operators on E, $D(A)$ is the domain of A, $t_0\geq 0$, $a>0$,

$$I:=[t_0, t_0+a], \quad \Delta:=\{(t,s): t_0\leq s\leq t\leq t_0+a\}$$

$$M:=\sup_{t\in[0,a]}\|T(t)\|_{BL(E,E)}, \tag{2.1}$$

$$X:=C(I,E)$$

and

$$F_1: I\times E^{m+1}\to E, \quad F_2:\Delta\times E^2\to E, \quad G: X\to E,$$

$$f:\Delta\times E\to E, \quad \sigma_i: I\to I \quad (i=1,\ldots,m)$$

are given functions satisfying some assumptions.

In the sequel, the operator norm $\|\cdot\|_{BL(E,E)}$ will be denoted by $\|\cdot\|$.

We will need the following two definitions of mild and classical solutions of the nonlocal Cauchy problem (1.1)-(1.2):

Definition 2.1: A function $u\in X$ satisfying the integral equation

$$\begin{aligned} u(t) &= T(t-t_0)u_0 - T(t-t_0)G(u) \\ &+ \int_{t_0}^{t} T(t-s)F_1(s,u(s),u(\sigma_1(s)),\ldots,u(\sigma_m(s)))ds \\ &+ \int_{t_0}^{t} T(t-s)\left(\int_{t_0}^{s} F_2\Big(s,\tau,u(\tau),\int_{t_0}^{\tau} f(\tau,\mu,u(\mu))d\mu\Big)d\tau\right)ds, \quad t\in I, \end{aligned} \tag{2.2}$$

is said to be a *mild solution* of the nonlocal Cauchy problem (1.1)-(1.2) on I.

Definition 2.2: A function $u: I\to E$ is said to be a *classical solution* of the nonlocal Cauchy problem (1.1)-(1.2) on I if:

(*i*) u is continuous on I and continuously differentiable on $I\backslash\{t_0\}$,

(*ii*) $u'(t)+Au(t)=F_1(t,u(t),u(\sigma_1(t)),\ldots,u(\sigma_m(t)))$

$$+\int_{t_0}^{t} F_2\left(t,s,u(s),\int_{t_0}^{s} f(s,\tau,u(\tau))d\tau\right)ds, \quad t\in I\backslash\{t_0\},$$

(*iii*) $u(t_0+G(u)=u_0$.

3. Theorem about a Mild Solution

Theorem 3.1: *Assume that*

(*i*) *for all* $z_i\in E$ $(i=0,1,\ldots,m)$, *the function* $I\ni t\rightarrow F_1(t,z_0,z_1,\ldots,z_m)\in E$ *is continuous on* I, *for all* $z_i\in E$ $(i=1,2)$ *the function* $\Delta\ni(t,s)\rightarrow F_2(t,s,z_1,z_2)\in E$ *is continuous on* Δ, *for all* $z\in E$ *the function* $\Delta\ni(t,s)\rightarrow f(t,s,z)\in E$ *is continuous on* Δ, $G: X\rightarrow E$, $\sigma_i\in C(I,I)$ $(i=1,\ldots,m)$ *and* $u_0\in E$;

(*ii*) *there are constants* $L_i>0$ $(i=1,2,3,4)$ *such that*

$$\| F_1(t,z_0,z_1,\ldots,z_m)-F_1(t,\widetilde{z}_0,\widetilde{z}_1,\ldots,\widetilde{z}_m)\| \leq L_1\sum_{i=0}^{m}\| z_i-\widetilde{z}_i\|$$

$$\text{for } t\in I,\ z_i,\widetilde{z}_i\in E \quad (i=0,1,\ldots,m); \tag{3.1}$$

$$\| F_2(t,s,z_1,z_2)-F_2(t,s,\widetilde{z}_1,\widetilde{z}_2)\| \leq L_2\sum_{i=1}^{2}\| z_i-\widetilde{z}_i\|$$

$$\text{for } (t,s)\in\Delta,\ z_i,\widetilde{z}_i\in E \quad (i=1,2); \tag{3.2}$$

$$\| f(t,s,z)-f(t,s,\widetilde{z})\| \leq L_3\| z-\widetilde{z}\|$$

$$\text{for } (t,s)\in\Delta,\ z,\widetilde{z}\in E; \tag{3.3}$$

$$\| G(w)-G(\widetilde{w})\| \leq L_4\| w-\widetilde{w}\|_X \text{ for } w,\widetilde{w}\in X; \tag{3.4}$$

(*iii*) $M\left[L_1a(m+1)+L_2a^2(1+L_3a)+L_4\right]<1$.

Then the nonlocal Cauchy problem (1.1)-(1.2) *has a unique mild solution on* I.

Proof: Introduce an operator $\mathfrak{F}$ on X by the formula

$$(\mathfrak{F}w)(t):=T(t-t_0)u_0-T(t-t_0)G(w)$$

$$+\int_{t_0}^{t} T(t-s)F_1(s,w(s),w(\sigma_1(s)),\ldots,w(\sigma_m(s)))ds$$

$$+\int_{t_0}^{t} T(t-s)\left(\int_{t_0}^{s} F_2\left(s,\tau,w(\tau),\int_{t_0}^{\tau} f(\tau,\mu,w(\mu))d\mu\right)d\tau\right)ds$$

for $w \in X$ and $t \in I$.

It is easy to see that

$$\mathcal{F}: X \to X. \tag{3.5}$$

Now, we shall show that $\mathcal{F}$ is a contraction on X. For this purpose, consider the difference

$$\begin{aligned}(\mathcal{F}w)(t)-(\mathcal{F}\widetilde{w})(t) = {} & -T(t-t_0)[G(w)-G(\widetilde{w})] \\ & +\int_{t_0}^{t} T(t-s)[F_1(s,w(s),w(\sigma_1(s)),\ldots,w(\sigma_m(s))) \\ & -F_1(s,\widetilde{w}(s),\widetilde{w}(\sigma_1(s)),\ldots,\widetilde{w}(\sigma_m(s)))]ds \\ & +\int_{t_0}^{t} T(t-s)\left(\int_{t_0}^{s}\left[F_2\left(s,\tau,w(\tau),\int_{t_0}^{\tau} f(\tau,\mu,w(\mu))d\mu\right)\right.\right. \\ & \left.\left.-F_2\left(s,\tau,\widetilde{w}(\tau),\int_{t_0}^{\tau} f(\tau,\mu,\widetilde{w}(\mu))d\mu\right)\right]d\tau\right)ds\end{aligned} \tag{3.6}$$

for $w, \widetilde{w} \in X$ and $t \in I$.

From (3.6), (2.1) and (3.1)-(3.4),

$$\begin{aligned}\|(\mathcal{F}w)(t)-(\mathcal{F}\widetilde{w})(t)\| \le {} & \|T(t-t_0)\|\,\|G(w)-G(\widetilde{w})\| \\ & +\int_{t_0}^{t}\|T(t-s)\|\,\|F_1(s,w(s),w(\sigma_1(s)),\ldots,w(\sigma_m(s))) \\ & -F_1(s,\widetilde{w}(s),\widetilde{w}(\sigma_1(s)),\ldots,\widetilde{w}(\sigma_m(s)))\|\,ds \\ & +\int_{t_0}^{t}\|T(t-s)\|\left(\int_{t_0}^{s}\|F_2\left(s,\tau,w(\tau),\int_{t_0}^{\tau} f(\tau,\mu,w(\mu))d\mu\right)\right. \\ & \left.-F_2\left(s,\tau,\widetilde{w}(t),\int_{t_0}^{\tau} f(\tau,\mu,\widetilde{w}(\mu))d\mu\right)\|\,d\tau\right)ds\end{aligned} \tag{3.7}$$

$$\le ML_4\|w-\widetilde{w}\|_X + ML_1\int_{t_0}^{t}\left(\|w(s)-w(\widetilde{s})\| + \sum_{i=1}^{m}\|w(\sigma_i(s))-\widetilde{w}(\sigma_i(s))\|\right)ds$$

$$+ML_2\int_{t_0}^{t}\left(\int_{t_0}^{s}\left[\| w(\tau)-\widetilde{w}(\tau)\| + \int_{t_0}^{\tau}\| f(\tau,\mu,w(\mu))-f(\tau,\mu,\widetilde{w}(\mu))\| \, d\mu\right]d\tau\right)ds$$

$$\leq ML_4 \| w-\widetilde{w} \|_X + ML_1 a(m+1)\| w-\widetilde{w} \|_X$$

$$+ML_2\int_{t_0}^{t}\left(\int_{t_0}^{s}\left[\| w(\tau)-\widetilde{w}(\tau)\| + L_3\int_{t_0}^{\tau}\| w(\mu)-\widetilde{w}(\mu)\| \, d\mu\right]d\tau\right)ds$$

$$= M\left[L_1 a(m+1)+L_2 a^2(1+L_3 a)+L_4\right]\| w-\widetilde{w} \|_X$$

for $w, \widetilde{w} \in X$ and $t \in I$.

Let

$$q:= M\left[L_1 a(m+1)+L_2 a^2(1+L_3 a)+L_4\right].$$

Then, by (3.7) and by assumption (*iii*),

$$\| \mathfrak{T}w - \mathfrak{T}\widetilde{w} \|_X \leq q \| w-\widetilde{w} \|_X \text{ for } w,\widetilde{w} \in X \tag{3.8}$$

with $0 < q < 1$. This shows that operator $\mathfrak{T}$ is a contraction on X.

Consequently, from (3.5) and (3.8), operator $\mathfrak{T}$ satisfies all the assumptions of the Banach contraction theorem. Therefore, in space X there is only one fixed point of $\mathfrak{T}$ and this point is the mild solution of the nonlocal Cauchy problem (1.1)-(1.2). So the proof of Theorem 3.1 is complete.

4. Theorem about a Classical Solution

Theorem 4.1: *Suppose that assumptions (i)-(iii) of Theorem* 3.1 *are satisfied. Then the nonlocal Cauchy problem* (1.1)-(1.2) *has a unique mild solution on I. Assume, additionally, that:*

(*i*) *E is a reflexive Banach space, $u_0 \in D(A)$ and $G(u) \in D(A)$, where u denotes the unique mild solution of problem* (1.1)-(1.2);

(*ii*) *there are constants $C_i > 0$ $(i = 1,2)$ such that*

$$\| F_1(t,z_0,z_1,\ldots,z_m) - F_1(\widetilde{t},z_0,z_1,\ldots,z_m) \| \leq C_1 | t-\widetilde{t} | \quad \text{for } t,\widetilde{t} \in I,\ z_i \in E\ (i=0,1,\ldots,m) \tag{4.1}$$

and

$$\| F_2(t,s,z_1,z_2) - F_2(\widetilde{t},s,z_1,z_2) \| \leq C_2 | t-\widetilde{t} | \quad \text{for } (t,s)\in\Delta,\ (\widetilde{t},s)\in\Delta,\ z_i \in E\ \ (i=1,2); \tag{4.2}$$

(iii) there is a constant $c > 0$ *such that*

$$\| u(\sigma_i(t)) - u(\sigma_i(\tilde{t})) \| \leq c \| u(t) - u(\tilde{t}) \| \quad \text{for } t, \tilde{t} \in I \quad (i = 0, 1, \ldots, m). \tag{4.3}$$

Then u *is the unique classical solution of the nonlocal Cauchy problem* (1.1)-(1.2) *on* I.

Proof: Since all the assumptions of Theorem 3.1 are satisfied, then the nonlocal Cauchy problem (1.1)-(1.2) possesses a unique mild solution which, according to assumption (i), is denoted by u.

Now, we shall show that u is the unique classical solution of problem (1.1)-(1.2) on I. To this end, introduce

$$N_1 := \max_{s \in I} \| F_1(s, u(s), u(\sigma_1(s)), \ldots, u(\sigma_m(s))) \| \tag{4.4}$$

and

$$N_2 := \max_{(\xi, \eta) \in \Delta} \| F_2(\xi, \eta, u(\eta), \int_{t_0}^{\eta} f(\eta, \mu, u(\mu)) d\mu) \|, \tag{4.5}$$

and observe that

$$\begin{aligned} u(t+h) - u(t) &= \left[T(t+h-t_0)u_0 - T(t-t_0)u_0\right] \\ &- [T(t+h-t_0)G(u) - T(t-t_0)G(u)] \\ &+ \int_{t_0}^{t_0+h} T(t+h-s)F_1(s, u(s), u(\sigma_1(s)), \ldots, u(\sigma_m(s)))ds \\ &+ \int_{t_0+h}^{t+h} T(t+h-s)F_1(s, u(s), u(\sigma_1(s)), \ldots, u(\sigma_m(s)))ds \\ &- \int_{t_0}^{t} T(t-s)F_1(s, u(s), u(\sigma_1(s)), \ldots, u(\sigma_m(s)))ds \\ &+ \int_{t_0}^{t_0+h} T(t+h-s)\left(\int_{t_0}^{s} F_2(s, \tau, u(\tau), \int_{t_0}^{\tau} f(\tau, \mu, u(\mu))d\mu)d\tau \right)ds \\ &+ \int_{t_0+h}^{t+h} T(t+h-s)\left(\int_{t_0}^{s} F_2(s, \tau, u(\tau), \int_{t_0}^{\tau} f(\tau, \mu, u(\mu))d\mu)d\tau \right)ds \end{aligned} \tag{4.6}$$

$$- \int_{t_0}^{t} T(t-s)\left(\int_{t_0}^{s} F_2(s,\tau,u(\tau),\int_{t_0}^{\tau} f(\tau,\mu,u(\mu))d\mu)d\tau\right)ds$$

$$= T(t-t_0)[T(h)-I]u_0 - T(t-t_0)[T(h)-I]G(u)$$

$$+ \int_{t_0}^{t_0+h} T(t+h-s)F_1(s,u(s),u(\sigma_1(s)),\ldots,u(\sigma_m(s)))ds$$

$$+ \int_{t_0}^{t} T(t-s)[F_1(s+h,u(s+h),u(\sigma_1(s+h)),\ldots,u(\sigma_m(s+h)))$$

$$- F_1(s,u(s),u(\sigma_1(s)),\ldots,u(\sigma_m(s)))]ds$$

$$+ \int_{t_0}^{t_0+h} T(t+h-s)\left(\int_{t_0}^{s} F_2(s,\tau,u(\tau),\int_{t_0}^{\tau} f(\tau,\mu,u(\mu))d\mu)d\tau\right)ds$$

$$+ \int_{t_0}^{t} T(t-s)\left(\int_{t_0}^{s}\left[F_2(s+h,\tau,u(\tau),\int_{t_0}^{\tau} f(\tau,\mu,u(\mu))d\mu)\right.\right.$$

$$\left.\left.- F_2(s,\tau,u(\tau),\int_{t_0}^{\tau} f(\tau,\mu,u(\mu))d\mu)\right]d\tau\right)ds$$

$$+ \int_{t_0}^{t} T(t-s)\left(\int_{s}^{s+h} F_2(s+h,\tau,u(\tau),\int_{t_0}^{\tau} f(\tau,\mu,u(\mu))d\mu)d\tau\right)ds$$

for $t \in [t_0, t_0+a)$, $h > 0$ and $t+h \in (t_0, t_0+a]$.

Consequently, by (4.6), (2.1) and (4.1)-(4.5),

$$\| u(t+h)-u(t) \| \leq hM \| Au_0 \| + hM \| AG(u) \| + hMN_1 + ahML_1$$

$$+ ML_1 \int_{t_0}^{t}\left(\| u(s+h)-u(s) \| + \sum_{i=1}^{m} \| u(\sigma_i(s+h)) - u(\sigma_i(s)) \| \right)ds \qquad (4.7)$$

$$+ a^2ML_2h + 2aMN_2h \leq Ch + ML_1(1+mc)\int_{t_0}^{t} \| u(s+h)-u(s) \| \, ds$$

for $t \in [t_0, t_0+a)$, $h > 0$ and $t+h \in (t_0, t_0+a]$, where

$$C: = M\Big[\| Au_0 \| + \| AG(u) \| + N_1 + aL_1 + a^2L_2 + 2aN_2\Big].$$

From (4.7) and Gronwall's inequality,

$$\| u(t+h) - u(t) \| \leq Ce^{aML_1(1+mc)}h$$

for $t \in [t_0, t_0 + a)$, $h > 0$ and $t + h \in (t_0, t_0 + a]$. Hence u is Lipschitz continuous on I.

The Lipschitz continuity of u on I and inequalities (4.1), (3.1), (4.2) imply that the function

$$I \ni t \to k(t): = F_1(t, u(t), u(\sigma_1(t)), \ldots, u(\sigma_m(t)))$$

$$+ \int_{t_0}^{t} F_2\left(t, s, u(s), \int_{t_0}^{s} f(s, \tau, u(\tau))d\tau\right)ds \in E$$

is Lipschitz continuous on I. This property of $t \to k(t)$ together with assumptions of Theorem 4.1 imply by Theorem 1 from [10], by Theorem 3.1 from this paper and by (2.2), that the linear Cauchy problem

$$v'(t) + Av(t) = k(t),\ t \in I \backslash \{t_0\},$$

$$v(t_0) = u_0 - G(u)$$

has a unique classical solution v such that

$$v(t) = T(t-t_0)u_0 - T(t-t_0)G(u) + \int_{t_0}^{t} T(t-s)k(s)ds$$

$$= T(t-t_0)u_0 - T(t-t_0)G(u)$$

$$+ \int_{t_0}^{t} T(t-s)F_1(s, u(s), u(\sigma_1(s)), \ldots, u(\sigma_m(s)))ds$$

$$+ \int_{t_0}^{t} T(t-s) \left(\int_{t_0}^{s} F_2\left(s, \tau, u(\tau), \int_{t_0}^{\tau} f(\tau, \mu, u(\mu))d\mu\right)d\tau \right) ds = u(t),\ t \in I.$$

Consequently, u is the unique classical solution of the nonlocal Cauchy problem (1.1)-(1.2) on I. Therefore, the proof of Theorem 4.1 is complete.

References

[1] Balasubramaniam, P. and Chandrasekaran, M., Existence of solutions of nonlinear integrodifferential equation with nonlocal boundary conditions in Banach space, *Atti Sem. Mat. Fis. Univ. Modena* **46** (1998), 1-13.

[2] Byszewski, L., Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, *J. Math. Anal. Appl.* **162**:2 (1991), 494-505.

[3] Byszewski, L., Uniqueness criterion for solution of abstract nonlocal Cauchy problem, *J. Appl. Math. Stoch. Anal.* **6** (1993), 49-54.

[4] Byszewski, L., *Differential and Functional-Differential Problems with Nonlocal Conditions*, Cracow University of Technology, Monograph **184**, Cracow 1995.

[5] Byszewski, L., Existence and uniqueness of mild and classical solutions of semilinear functional-differential evolution nonlocal Cauchy problem, *Selected Probl. in Math.*, Cracow Univ. of Tech., Anniversary Issue **6** (1995), 25-33.

[6] Byszewski, L., On weak solutions of functional-differential abstract nonlocal Cauchy problem, *Annales Polonici Math.* **65** (1997), 163-170.

[7] Byszewski, L., Application of properties of the right-hand sides of evolution equations to an investigation of nonlocal evolution problems, *Nonlinear Anal.* **33** (1998), 413-426.

[8] Lin, Y. and Liu, J.H., Semilinear integrodifferential equations with nonlocal Cauchy problem, *Nonlinear Anal.* **26** (1996), 1023-1033.

[9] Pazy, A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, Berlin, Tokyo 1983.

[10] Winiarska, T., Nonlinear evolution equation with a parameter, *Bull. Polon. Acad. Sci. Math Ser.* **37** (1989), 157-162.