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A CLASSICAL APPROACH TO EIGENVALUE PROBLEMS ASSOCIATED WITH A PAIR OF MIXED REGULAR STURM-LIOUVILLE EQUATIONS I

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In the studies of acoustic waveguides in ocean, buckling of columns with variable cross sections in applied elasticity, transverse vibrations in nonhomogeneous strings, etc., we encounter a new class of problems of the type $L_1y_1 = -\frac{d^2y_1}{dx^2} + q_1(x)y_1 = \lambda y_1$ defined on an interval $[d_1, d_2]$ and $L_2y_2 = -\frac{d^2y_2}{dx^2} + q_2(x)y_2 = \lambda y_2$ on the adjacent interval $[d_2, d_3]$ satisfying certain matching conditions at the interface point $x = d_2$.

Here in Part I, we constructed a fundamental system for (L_1, L_2) and derive certain estimates for the same. Later, in Part II, we shall consider four types of boundary value problems associated with (L_1, L_2) and study the corresponding spectra.

Key words: Sturm-Liouville Equations, Interface Boundary Conditions, Initial Value Problems, Matching Conditions, Fundamental System, Eigenvalue Problems, Estimates, Inequalities.

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1. Introduction

In studies of acoustic waveguides in ocean [1], buckling of columns with variable cross sections in applied elasticity [9], transverse vibrations in nonhomogeneous strings [2], etc., we encounter a new class of problems of the type

$$L_1y_1 = -\frac{d^2y_1}{dx^2} + q_1(x)y_1 = \lambda y_1$$

defined on an interval $[d_1, d_2]$ and

$$L_2y_2 = -\frac{d^2y_2}{dx^2} + q_2(x)y_2 = \lambda y_2,$$

defined on the adjacent interval $[d_2, d_3]$, where λ is an unknown constant (eigenvalue) and the functions y_1, y_2 are required to satisfy certain matching conditions at the interface $x = d_2$. In most of the cases, the complete set of physical conditions give rise to (selfadjoint) eigenvalue problems associated with the pair (L_1, L_2). The spectral analysis of these boundary value problems (BVPs) can be carried out to some extent by recasting them as operator equations in an appropriate abstract space [3, 8]. But some of the nice and useful properties of the original BVPs cannot be captured so easily in the abstract space settings. In the literature, there do not seem to exist may results in this area. However, O.H. Hald [5] discusses the inverse theory of some problems of this type which arise in torsional modes of the Earth, and B.J. Harris [6] obtains series solutions for certain Riccatti equations with applications to Sturm-Liouville problems

Hence here and in the sequel, we adopt the classical approach for the study of eigenvalue problems (EVPs) associated with the pair (L_1, L_2) and prove a few spectral analysis results for the new class of BVPs.

Before proceeding to the work, we shall introduce a few notations and definitions. Let R denote the real line, and C denote the complex plane with their usual topologies. For a complex number λ, Reλ and Imλ denote the real and imaginary parts of λ, respectively. For any two nonempty sets A and B, $A \backslash B$ denotes the collection of elements in A which are not in B. Again, for any two nonempty sets V_1 and V_2, $V_1 \times V_2$ denotes the Cartesian product (space equipped with the product topology) of V_1 and V_2, taken in that order. For a compact interval $[a, b]$, of R, $L^2_C[a, b]$ $(L^2_R[a, b])$ denotes the complex (real) Hilbert space of all complex (real) valued Lebesgue square integrable functions defined on $[a, b]$. The inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$ in $L^2_C[a, b]$ $(L^2_R[a, b])$ are given by

$$(f, g) = \int_a^b f\overline{g}\, dx \text{ and } \| f \| = (f, f)^{1/2}$$

where $\overline{g}$ denotes the complex conjugate of g. For a function y, y' and y'' denote the first and second order derivatives of y, respectively, if they exist. Let $AC^2[a, b]$ denote the space of all twice continuously differentiable complex valued functions y defined on $[a, b]$ such that y' is absolutely continuous. Let $H^2_C[a, b]$ denote those functions $y \in AC^2[a, b]$ such that $y'' \in L^2_C[a, b]$. Let $0 < h < 1$ and let $(q_1, q_2) \in L^2_C[0, h] \times L^2_C[h, 1]$. Let w_1 and w_2 be nonzero constants.

We consider the pair of Sturm-Liouville equations

$$L_1 y_1 \equiv -y_1'' + q_1(x) y_1 = \lambda y_1, \quad 0 \le x \le h, \tag{1}$$

$$L_2 y_2 \equiv -y_2'' + q_2(x) y_2 = \lambda y_2, \quad h \le x \le 1, \tag{2}$$

together with the matching conditions at the interface $x = h$ given by

$$y_1(h) = y_2(h), \quad w_1 y_1'(h) = w_2 y_2'(h) \tag{3}$$

where λ is a complex constant.

Definition 1: By a solution of the problem (1)-(3), we mean a pair of functions $\{y_1, y_2\}$ satisfying the following conditions:

(*i*) $y_1 \in AC^2[0,h]$ and satisfies Equation (1) for almost all $x \in [0,h]$,
(*ii*) $y_2 \in AC^2[h,1]$ and satisfies Equation (2) for almost all $x \in [h,1]$,
(*iii*) y_1, y_2 satisfy the matching conditions (3).

Definition 2: We say that the nontrivial pairs $\{y_{11}, y_{12}\}, \{y_{21}, y_{22}\}$ where y_{11}, y_{21} are defined on $[0,h]$ and y_{12}, y_{22} are defined on $[h,1]$ are linearly independent if for any two scalars α and β, the equations

$$\alpha y_{11}(x) + \beta y_{21}(x) = 0 \text{ for all } x \in [0,h] \text{ and}$$

$$\alpha y_{12}(x) + \beta y_{22}(x) = 0 \text{ for all } x \in [h,1]$$

imply $\alpha = \beta = 0$.

Definition 3: By a fundamental system (FS) for the problem (1)-(3), we mean a set of two linearly independent solutions of (1)-(3), which span the solution space of (1)-(3).

In Part I, we construct a FS for the problem (1)-(3) and establish certain estimates for the components of FS. In Part II, we present results concerning the location of the spectra of various associated BVPs.

2. A Fundamental System for (1)-(3): Construction and Estimates

For the sake of simplicity, we denote $C_\lambda(x) = (\cos\sqrt{\lambda}x)$ and $S_\lambda(x) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}$. Before proving the main theorem, we state the following easily verified lemmas.

Lemma 1: *Let* $(g_1, g_2) \in L^2_C[0,h] \times L^2_C[h,1]$. *Then for* $x \in [h,1]$,

$$\int_0^h |g_1(t)|\,dt + \int_h^x |g_2(t)|\,dt \leq (\|g_1\|^2 + \|g_2\|^2)^{1/2}\sqrt{x}.$$

Lemma 2: (*A*) *The problem* (1)-(3) *along the initial conditions*

$$y_1(0) = 1, \quad y_1'(0) = 0 \tag{4}$$

is equivalent to the Liouville integral equation

$$y_1(x) = C_\lambda(x) + \int_0^x S_\lambda(x-t)q_1(t)y_1(t)dt, \quad 0 \leq x \leq h, \tag{5}$$

$$y_2(x) = y_1(h)C_\lambda(x-h) + (w_1/w_2)y_1'(h)S_\lambda(x-h)$$
$$+ \int_h^x S_\lambda(x-t)q_2(t)y_2(t)dt, \quad h \leq x \leq 1. \tag{6}$$

(*B*) *The problem* (1)-(3) *along with the initial conditions*

$$y_1(0) = 0, \quad y_1'(0) = 1 \tag{7}$$

is equivalent to the Liouville integral equation

$$y_1(x) = S_\lambda(x) + \int_0^x S_\lambda(x-t)q_1(t)y_1(t)dt, \quad 0 \le x \le h, \tag{8}$$

$$y_2(x) = y_1(h)C_\lambda(x-h) + (w_1/w_2)y_1'(h)S_\lambda(x-h)$$

$$+ \int_h^x S_\lambda(x-t)q_2(t)y_2(t)dt, \quad h \le x \le 1. \tag{9}$$

Theorem 1: (Construction of a FS for (1)-(3)) (A) *The unique solution of initial value problem* (1)-(4) *is given by the pair* (y_{11}, y_{12}) *where*

$$y_{11}(x) = C_\lambda(x) + \sum_{n=1}^{\infty} \int_{0 \le t_1 \le \ldots \le t_n \le t_{n+1} = x} C_\lambda(t_1) \prod_{i=1}^{n}$$

$$\times S_\lambda(t_{i+1} - t_i)q_1(t_i)dt_1 \ldots dt_n, \quad 0 \le x \le h, \tag{10}$$

$$y_{12}(x) = C_0(x,\lambda) + \sum_{n=1}^{\infty} \int_{h \le t_1 \le \ldots \le t_n \le t_{n+1} = x} C_0(t_1,\lambda) \prod_{i=1}^{n}$$

$$\times S_\lambda(t_{i+1} - t_i)q_2(t_i)dt_1 \ldots dt_n, \quad h \le x \le 1, \tag{11}$$

where

$$C_0(x,\lambda) = y_{11}(h)C_\lambda(x-h) + \frac{w_1}{w_2}(y_{11}'(h)S_\lambda(x-h)), \quad h \le x \le 1. \tag{12}$$

(B) *The unique solution of initial value problem* (1)-(3) *and* (7) *is given by the pair* (y_{21}, y_{22}) *where*

$$y_{21}(x) = S_\lambda(x) + \sum_{n=1}^{\infty} \int_{0 \le t_1 \le \ldots \le t_n \le t_{n+1} = x} S_\lambda(t_1) \prod_{i=1}^{n}$$

$$\times S_\lambda(t_{i+1} - t_i)q_1(t_i)dt_1 \ldots dt_n, \quad 0 \le x \le h, \tag{13}$$

$$y_{22}(x) = S_0(x,\lambda) + \sum_{n=1}^{\infty} \int_{h \le t_1 \le \ldots \le t_n \le t_{n+1} = x} S_0(t_1,\lambda) \prod_{i=1}^{n}$$

$$\times S_\lambda(t_{i+1} - t_i)q_2(t_i)dt_1 \ldots dt_n, \quad h \le x \le 1, \tag{14}$$

where

$$S_0(x,\lambda) = y_{21}(h)C_\lambda(x-h) + \frac{w_1}{w_2}(y_{21}'(h)S_\lambda(x-h)), \quad h \le x \le 1. \tag{15}$$

Here

$$\int\limits_{0 \le t_1 \le \ldots \le t_n \le t_{n+1} = x} dt_1 \ldots dt_n = \int_0^x \int_0^{t_n} \cdots \int_0^{t_2} \ldots dt_1 \ldots dt_n$$

and

$$\int\limits_{h \le t_1 \le \ldots \le t_n \le t_{n+1} = x} dt_1 \ldots dt_n = \int_h^x \int_h^{t_n} \cdots \int_h^{t_2} \ldots dt_1 \ldots dt_n.$$

Proof: We prove part (A). The proof of part (B) follows similarly. Expression (10) follows from Theorem 1 (on p. 7 [7]). One can also refer to Equation (8) on page 9 [4], for the power series representations in (10) and (13). Below we derive Expression (11).

We assume that y_{12} is a power series in q_2, that is

$$y_{12}(x) = C_0(x, \lambda) + \sum_{n=1}^{\infty} C_n(x, \lambda, q_2), \quad h \le x \le 1 \tag{16}$$

where $C_n(x, \lambda, q_2) = C_n(x, \lambda, q_{12}, \ldots, q_{n2})\big|_{q_{12} = q_{22} = \ldots = q_{n2} = q_2}$, and $C_n(x, \lambda, q_{12}, \ldots, q_{n2})$, for each x and λ, is a bounded, multilinear symmetric form on $L^2_C[h, 1] \times \ldots$ (n times) $\ldots \times L^2_C[h, 1]$.

Formally differentiating the power series for y_{12} twice with respect to x and substituting into Equation (2) then equating the terms which are homogeneous of the same degree in q_2, we obtain

$$-C_0'' = \lambda C_0 \tag{17}$$

$$-C_n'' = \lambda C_n - q_2 C_{n-1}, n \ge 1, h \le x \le 1. \tag{18}$$

In view of the matching conditions (3) to be satisfied by the pair (y_{11}, y_{12}) at the interface $x = h$, we impose the following initial conditions on the C_n's:

$$C_0(h, \lambda) = y_{11}(h), C_0'(h, \lambda) = \frac{w_1}{w_2} y_{11}'(h) \tag{19}$$

$$C_n(h) = C_n'(h) = 0, \quad n \ge 1. \tag{20}$$

Clearly, the solution of (17) satisfying (19) is given by

$$C_0(x, \lambda) = y_{11}(h) C_\lambda(x - h) + \frac{w_1}{w_2}(y_{11}'(h) S_\lambda(x - h)).$$

Also the solution of (18) satisfying (20) is given by

$$C_n(x, \lambda, q_2) = \int_h^x S_\lambda(x - t) q_2(t) C_{n-1}(t, \lambda, q_2) dt, \quad n \ge 1.$$

Proceeding by induction, we get

$$C_n(x, \lambda, q_2) = \int\limits_{h \le t_1 \le \ldots < t_n < t_{n+1} = x} C_0(t_1, \lambda) \prod_{i=1}^{n} S_\lambda(t_{i+1} - t_i) q_2(t_i) dt_1 \ldots dt_n.$$

Substituting the expressions for C_0, C_n in (16), we obtain (11).

Note 1: We note that

$$y_{j1}(x) = y_{j1}(x, \lambda, q_1)$$

and

$$y_{j2}(x) = y_{j2}(x, \lambda, w_1, w_2, q_1, q_2), j = 1, 2.$$

Theorem 2: (A) *The formal power series for* y_{j1}, $j = 1, 2$ *converge uniformly on bounded subsets of* $[0, h] \times C \times L^2_C[0, h]$ *and*

$$|\, y_{j1}(x) \,| \leq exp(\,|\, Im\sqrt{\lambda} \,|\, x + \|\, q_1 \,\| \sqrt{x}), \quad 0 \leq x \leq h.$$

(B) *The formal power series for* y_{j2}, $j = 1, 2$ *converge uniformly on bounded subset of* $[h, 1] \times C \times (C \backslash \{0\}) \times (C \backslash B_r(0)) \times L^2_C[0, h] \times L^2_C[h, 1]$, *for any* $r > 0$ *and*

$$\left| y_{j2}(x) \right| \leq \left(1 + \left| 1 - \frac{w_1}{w_2} \right| \right) exp\left(\left| Im\sqrt{\lambda} \right| x + \left(\|\, q_1 \,\|^2 + \|\, q_2 \,\|^2 \right)^{1/2} \sqrt{x} \right), \quad h \leq x \leq 1.$$

Proof: The proof of part (A) follows from Theorem 1 (on p. 7 [7]). Below we shall derive the estimate for $|\, y_{12}(x) \,|$. The estimate for $|\, y_{22}(x) \,|$ can be derived along similar lines.

We note that

$$|\, C_\lambda(x) \,| \leq \exp\left(\left| \mathrm{Im}\sqrt{\lambda} \right| x \right),$$

and for $0 \leq x \leq 1$,

$$|\, S_\lambda(x) \,| \leq \exp\left(\left| \mathrm{Im}\sqrt{\lambda} \right| x \right), \quad \text{(see p. 8 [7])}.$$

Substituting the series (10) and its derived series for $y_{11}(h)$ and $y'_{11}(h)$, respectively, into Expression (12), regrouping the terms, taking modulus and using the triangle inequality, we obtain

$$\left| C_0(x, \lambda) \right| \leq \left| C_\lambda(x) + \left(1 - \frac{w_1}{w_2} \right) \mathrm{Sin}\left(\sqrt{\lambda}(x - h) \right) \mathrm{Sin}\left(\sqrt{\lambda} h \right) \right| + \sum_{n=1}^{\infty}$$

$$\Bigg| \int\limits_{0 \leq t_1 < \ldots < t_n < t_{n+1} = h} C_\lambda(t_1) q_1(t_n) \prod_{i=1}^{n-1} S_\lambda(t_{i+1} - t_i) q_1(t_i)$$

$$\times \left[S_\lambda(x - t_n) + \left(\frac{w_1}{w_2} - 1 \right) S_\lambda(x - h) C_\lambda(h - t_n) \right] dt_1 \ldots dt_n \Bigg|$$

$$\leq \left(1 + \left| 1 - \frac{w_1}{w_2} \right| \right) \exp\left(\left| \mathrm{Im}\sqrt{\lambda} \right| x \right) \times \left[1 + \sum_{n=1}^{\infty} \int\limits_{0 \leq t_1 \leq \ldots < t_n \leq t_{n+1} = h} \prod_{i=1}^{n} |\, q_1(t_i) \,|\, dt_1 \ldots dt_n \right]$$

$$= \left(1 + \left| 1 - \frac{w_1}{w_2} \right| \right) \exp\left(\left| \mathrm{Im}\sqrt{\lambda} \right| x \right) \times \left[1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\int_0^h |\, q_1(t) \,|\, dt \right)^n \right] \quad \text{(see p. 8 [7])}$$

$$|C_0(x,\lambda)| \leq \left(1+\left|1-\frac{w_1}{w_2}\right|\right)\exp\left[\left|\mathrm{Im}\sqrt{\lambda}\right|x+\int_0^h |q_1(t)|\,dt\right], \quad h \leq x \leq 1. \tag{21}$$

Finally from Equation (11), we get

$$|y_{12}(x)| \leq |C_0(x,\lambda)| + \sum_{n=1}^{\infty} \int_{h \leq t_1 \leq \ldots \leq t_n \leq t_{n+1} = x}$$

$$\times |C_0(t_1,\lambda)| \prod_{i=1}^{n} S_\lambda(t_{i-1}-t_i)\,|q_2(t_i)|\,dt_1\ldots dt_n$$

$$\leq \left(1+\left|1-\frac{w_1}{w_2}\right|\right)\exp\left[\left|\mathrm{Im}\sqrt{\lambda}\right|x+\int_0^h |q_1(t)|\,dt\right]$$

$$\times\left[1+\sum_{n=i}^{\infty} \int_{h \leq t_1 \leq \ldots \leq t_{n+1} = x} \prod_{i=1}^{n} |q_2(t_i)|\,dt_1\ldots dt_n\right]$$

(using (21) and simplifying)

$$= \left(1+\left|1-\frac{w_1}{w_2}\right|\right)\exp\left[\left|\mathrm{Im}\sqrt{\lambda}\right|x+\int_0^h |q_1(t)|\,dt+\int_h^x |q_2(t)|\,dt\right] \text{(as before)}$$

$$\leq \left(1+\left|1-\frac{w_1}{w_2}\right|\right)\exp\left[\left|\mathrm{Im}\sqrt{\lambda}\right|x+\left(\|q_1\|^2+\|q_2\|^2\right)^{1/2}\sqrt{x}\right], \ h \leq x \leq 1 \tag{22}$$

(by Lemma 1).

The above estimate readily implies the uniform convergence of the power series for y_{12}.

Note 2: Theorem 2 and Lemma 2 readily imply the uniqueness of the solution stat in Theorem 1. Moreover, every solution (y_1, y_2) of the problem (1)-(3) is uniquely expressed in the form

$$y_1(x) = y_1(0)y_{11}(x) + y_1'(0)y_{21}(x), \quad 0 \leq x \leq h,$$

$$y_2(x) = y_1(0)y_{12}(x) + y_1'(0)y_{22}(x), \quad h \leq x \leq 1.$$

Lastly, we prove the following theorem on the asymptotic estimates for the components of the FS.

Theorem 3: *(A) On* $[0,h] \times C \times L_C^2[0,h]$,

(*i*) $\quad |y_{11}(x) - cos\sqrt{\lambda}x| = \frac{1}{\sqrt{\lambda}}\, exp\left(\left|Im\sqrt{\lambda}\right|x + \|q_1\|\sqrt{x}\right),$

$$(ii) \quad | y_{21}(x) - \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} | \leq \frac{1}{|\lambda|} exp(| Im\sqrt{\lambda} | x + \| q_1 \| \sqrt{x}),$$

$$(iii) \quad | y'_{11}(x) + \sqrt{\lambda} sin\sqrt{\lambda}x | \leq \| q_1 \| exp(| Im\sqrt{\lambda} | x + \| q_1 \| \sqrt{x}),$$

$$(iv) \quad | y'_{21}(x) - cos\sqrt{\lambda}x | \leq \frac{\| q_1 \|}{| \sqrt{\lambda} |} exp(| Im\sqrt{\lambda} | x + \| q_1 \| \sqrt{x}).$$

(B) *On* $[h,1] \times C \times (C\backslash\{0\}) \times (C\backslash B_r(0)) \times L^2_C[0,h] \times L^2_C[h,1]$,

$$(v) \quad | y_{12}(x) - cos\lambda x | \leq \left|1 - \frac{w_1}{w_2}\right| exp(| Im\sqrt{\lambda} | x)$$
$$+ \left(\frac{1 + \left|1 - \frac{w_1}{w_2}\right|}{\left|\sqrt{\lambda}\right|} \right) exp\left(\left| Im\sqrt{\lambda} \right| x + \left(\| q_1 \|^2 + \| q_2 \|^2 \right)^{1/2} \sqrt{x} \right),$$

$$(vi) \quad | y_{22}(x) - \frac{\sin\sqrt{x}}{\sqrt{\lambda}} | \leq \frac{1}{| \sqrt{\lambda} |} | 1 - \frac{w_1}{w_2} | exp(| Im\sqrt{\lambda} | x)$$
$$+ \left(\frac{1 + \left|1 - \frac{w_1}{w_2}\right|}{| \lambda |} \right) exp\left(\left| Im\sqrt{\lambda} \right| x + \left(\| q_1 \|^2 + \| q_2 \|^2 \right)^{1/2} \sqrt{x} \right),$$

$$(vii) \quad | y'_{12}(x) + \sqrt{\lambda} sin\sqrt{x} | \leq | \sqrt{\lambda} | \, | 1 - \frac{w_1}{w_2} | exp(| Im\sqrt{\lambda} | x)$$
$$+ \left(1 + \left| 1 - \frac{w_1}{w_2} \right| \right) \left(\| q_1 \|^2 + \| q_2 \|^2 \right)^{1/2} exp\left(\left| Im\sqrt{\lambda} \right| x + \left(\| q_1 \|^2 + \| q_2 \|^2 \right)^{1/2} \sqrt{x} \right)$$

$$(viii) \quad | y'_{22}(x) - cos\sqrt{\lambda}x | \leq | 1 - \frac{w_1}{w_2} | exp(| Im\sqrt{\lambda} | x)$$
$$+ \frac{1}{| \sqrt{\lambda} |} \left(1 + \left| 1 - \frac{w_1}{w_2} \right| \right) \left(\| q_1 \|^2 + \| q_2 \|^2 \right)^{1/2}$$
$$\times exp\left(\left| Im\sqrt{\lambda} \right| x + \left(\| q_1 \|^2 + \| q_2 \|^2 \right)^{1/2} \sqrt{x} \right).$$

Proof: The proof of part (A) follows from Theorem 3 (p. 13 [7]). We establish inequalities (v) and (vii) of (B). Inequalities (vi) and $(viii)$ can be established similarly.

(v) From

$$C_0(x, \lambda) = C_\lambda(x) + \left(1 - \frac{w_1}{w_2} \right) \text{Sin}(\sqrt{\lambda}(x - h)) \text{Sin}(\sqrt{\lambda} h)$$
$$+ \sum_{n=1}^{\infty} \int_{0 \leq t_1 \leq \ldots \leq t_n \leq t_{n+1} = h} C_\lambda(t_1) q_1(t_n) \prod_{i=1}^{n-1} s_\lambda(t_{i+1} - t_i) q_1(t_i)$$
$$\times \left[S_\lambda(x - t_n) + \left(\frac{w_1}{w_2} - 1 \right) S_\lambda(x - h) C_\lambda(h - t_n) \right] dt_1 \ldots dt_n$$

it follows by using the same type of estimating as in (21) that

$$| C_0(x,\lambda) - \cos\sqrt{\lambda}x |$$

$$\leq \left|1 - \frac{w_1}{w_2}\right| \exp\left(\left|\operatorname{Im}\sqrt{\lambda}\right|x\right) + \frac{1}{\left|\sqrt{\lambda}\right|}\left(1 + \left|1 - \frac{w_1}{w_2}\right|\right) \exp\left(\left|\operatorname{Im}\sqrt{\lambda}\right| + \int_0^h | q_1(t) | \, dt\right). \tag{23}$$

By (11)

$$| y_{12}(x) - \cos(\sqrt{\lambda}x)$$

$$\leq | C_0(x,\lambda) - C_\lambda(x) | + \sum_{n=1}^{\infty}\left| \int_{h \leq t_1 \ldots < t_{n+1} = x} C_0(t_1,\lambda) \prod_{i=1}^{n} S_\lambda(t_{i+1} - t_i) q_2(t_i) dt_1 \ldots dt_n \right|$$

$$\leq \left|1 - \frac{w_1}{w_2}\right| \exp\left(\left|\operatorname{Im}\sqrt{\lambda}\right|\right)$$

$$\frac{1}{\left|\sqrt{\lambda}\right|}\left(1 + \left|1 - \frac{w_1}{w_2}\right|\right)\exp\left(\left|\operatorname{Im}\sqrt{\lambda}\right|x + \int_0^h | q_1(t) | \, dt\right)\left(1 + \sum_{n=1}^{\infty}\frac{1}{n!}\left(\int_h^x | q_2(t) | \, dt\right)^n\right)$$

(by using (21), (23) and the second sum estimation in (22))

$$\leq \left|1 - \frac{w_1}{w_2}\right| \exp\left(\left|\operatorname{Im}\sqrt{\lambda}\right|x\right) + \frac{1}{|\sqrt{\lambda}|}\left(1 + \left|1 - \frac{w_1}{w_2}\right|\right)\exp\left(\left|\operatorname{Im}\sqrt{\lambda}\right|x + \left(\| q_1 \|^2 + \| q_2 \|^2\right)^{1/2}\sqrt{x}\right)$$

(by Lemma 1).

(*vii*) Differentiating integral equation (6) for y_{12} with respect to x, inserting $y_{11}(h)$, $y'_{11}(h)$ from integral equation (5), and simplifying we obtain

$$y'_{12}(x) = -\sqrt{\lambda}\left[\cos(\sqrt{\lambda}h)\sin(\sqrt{\lambda}(x-h)) + \frac{w_1}{w_2}\sin(\sqrt{\lambda}h)\cos(\sqrt{\lambda}(x-h))\right]$$

$$+ \int_0^h q_1(t) y_{11}(t)\left[-\sin(\sqrt{\lambda}(h-t))\sin(\sqrt{\lambda}(x-h)) + \frac{w_1}{w_2}\cos(\sqrt{\lambda}(h-t))\cos(\sqrt{\lambda}(x-h))\right]dt$$

$$+ \int_h^x C_\lambda(x-t) q_2(t) y_{12}(t) dt.$$

Hence

$$\left|y'_{12}(x) + \sqrt{\lambda}\sin(\sqrt{\lambda}x)\right|$$

$$\leq \left|\sqrt{\lambda}\right|\left|1 - \frac{w_1}{w_2}\right| \exp\left(\left|\operatorname{Im}\sqrt{\lambda}\right|x\right)$$

$$+ \int_0^h \exp\left(\left|\operatorname{Im}\sqrt{\lambda}\right|t + \| q_1 \|\right)\left(1 + \left|1 - \frac{w_1}{w_2}\right|\exp\left(\left|\operatorname{Im}\sqrt{\lambda}\right|(x-t)\right)\right) | q_1(t) | \, dt$$

$$+\int_h^x \exp\left(\left|\operatorname{Im}\sqrt{\lambda}\right|(x-t)\right)\left(1+\left|1-\frac{w_1}{w_2}\right|\right)\exp\left(\left|\operatorname{Im}\sqrt{\lambda}\right|t+\left(\|q_1\|^2+\|q_2\|^2\right)^{1/2}\sqrt{t}\right)$$

$$\times \mid q_2(t) \mid dt \text{ (by Theorem 2)}$$

$$\leq \mid \sqrt{\lambda} \mid \left|1-\frac{w_1}{w_2}\right| \exp\left(\left|\operatorname{Im}\sqrt{x}\right|x\right)$$

$$+\left(1+\left|1-\frac{w_1}{w_2}\right|\right)\left(\|q_1\|^2+\|q_2\|^2\right)^{1/2}\exp\left(\left|\operatorname{Im}\sqrt{\lambda}\right|x+\left(\|q_1\|^2+\|q_2\|^2\right)^{1/2}\sqrt{x}\right)$$

(by Lemma 1).

Note 3: It follows from standard results for initial value problems that y_{ij}, $i, j = 1, 2$ and their derivatives are analytic functions of λ.

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