

PALM THEORY FOR RANDOM TIME CHANGES

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Palm distributions are basic tools when studying stationarity in the context of point processes, queueing systems, fluid queues or random measures. The framework varies with the random phenomenon of interest, but usually a one-dimensional group of measure-preserving shifts is the starting point. In the present paper, by alternatively using a framework involving random time changes (RTCs) and a two-dimensional family of shifts, we are able to characterize all of the above systems in a single framework. Moreover, this leads to what we call the *detailed Palm distribution* (DPD) which is stationary with respect to a certain group of shifts. The DPD has a very natural interpretation as the distribution seen at a randomly chosen position on the extended graph of the RTC, and satisfies a general duality criterion: the DPD of the DPD gives the underlying probability P in return.

To illustrate the generality of our approach, we show that classical Palm theory for random measures is included in our RTC framework. We also consider the important special case of marked point processes with batches. We illustrate how our approach naturally allows one to distinguish between the marks within a batch while retaining nice stationarity properties.

Key words: Random Time Change, Random Measure, Point Process, Stationary Distribution, Palm Distribution, Detailed Palm Distribution, Duality.

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1. Introduction

Palm theory is especially known for its applicability to stationary queueing systems in which there is an underlying point process of arriving customers over time; see, e.g., Franken et al. [3], Brandt et al. [2], Baccelli and Brémaud [1], and Sigman [11]. The theory considers the relationship between two distributions: a time-stationary distribution and a *Palm distribution* (PD). Both describe the stochastic behavior of the system, but whereas the first does so as seen from a randomly chosen time point, the second does so from a randomly chosen arrival epoch. On the one hand, point processes can be viewed as integer-valued measures (counting the number of arrivals in subsets of the time line), and it is this view that is widely used and accepted in the literature (see in particular [3] which is a classic text in this regard, and Mecke [6]). An advantage of this “counting measure approach” is that it naturally extends to real-valued measures thus leading to a Palm theory for modern fluid queues and random measures; see, e.g., Schmidt and Serfozo [10], and Miyazawa [7]. Since a measure $g^*(\cdot)$ on the real line can be identified with a non-decreasing and right continuous functional $g(t) = g^*((0, t]), g(0) = 0$, one can also equivalently express this measure approach in a functional framework (see Geman and Horowitz [4]).

On the other hand, as presented in [2] and [11], one can alternatively view a one-dimensional point process as a sequence of non-decreasing arrival times. When the point process is simple then (meaning that only one arrival is allowed to occur at a time; no *batches* allowed), the two approaches are equivalent, but when batches are allowed they are not equivalent (see Section 1.4 and Appendix D in [11] for such details). As a result, different Palm type distributions are obtained depending on the approach taken, and they have different interpretations and different stationarity properties (see also König and Schmidt [5], page 87). It is the sequence approach which leads to the interpretation of the distribution as seen from the point of view of a “randomly selected arrival” (not arrival epoch) and is thus more appealing in applications. (The point here is that each customer within a batch has the same arrival epoch, and this sequence approach distinguishes between them, whereas the measure approach does not.)

Motivated by this “randomly selected arrival” point of view, we proceed in the present paper to make sense of it and generalize it to random measures. By generalizing the functional framework introduced in [4] to that of a *random time change* (RTC), and by introducing a two-dimensional family of shifts along an extended graph, we define a *detailed Palm distribution*. This DPD not only has the desired stationarity property but also a new and fundamental duality property: The DPD of the DPD yields the original probability P back again. As we show, all well-known distributions of Palm type follow immediately from the DPD in a natural and very intuitive way. Classical Palm theory for random measures, for example, is contained in our RTC framework. In a modified form a DPD was first mentioned in [7], on a smaller σ -field and from a more applied point of view.

In Section 2 we first introduce the framework and give the definition of a random time change Λ . In Section 3 we start with a stationary probability measure P and then introduce the DPD denoted by P_Λ . We also consider the more standard type of Palm distribution P^0 (as found in most of the literature) and relate it to P_Λ . Things are then generalized further by letting the random time change Λ be accompanied by a stochastic process S defined on its extended graph. The pair (Λ, S) is called a marked time change and its stationarity properties are revealed. Section 4 is about

duality. Using the generalized inverse of the RTC, it is proved that the DPD of the DPD is well defined and yields back P . This duality principle can be used to derive results for P_Λ from similar results for P (and vice versa). The duality between P and P_Λ and the simple relationship between P_Λ and P^0 are used to obtain a general inversion formula to express P in terms of P^0 .

In Section 5 we show that Palm theory for random measures is included in our approach. Section 6 then illustrates our approach in the context of (marked) point processes with batches. In the appendix, proofs are given for some technical results.

2. Framework

Let \tilde{G} denote the set of functions $g: \mathbb{R} \rightarrow \mathbb{R}$ such that g is non-decreasing, continuous from the right, and $\lim_{t \rightarrow \pm\infty} g(t) = \pm\infty$. Set $G := \{g \in \tilde{G} : g(0-) \leq 0 \leq g(0)\}$. Endow \tilde{G} with the smallest σ -field making all the projection mappings $t \rightarrow g(t)$, $g \in \tilde{G}$, measurable; denote this by $\tilde{\mathfrak{G}}$ and set $\mathfrak{G} := \tilde{\mathfrak{G}} \cap G$. We view \mathbb{R} as the time line, and call $g \in G$ a *time change*. For $g \in \tilde{G}$, the set

$$\Gamma(g) := \{(t, x) \in \mathbb{R}^2 : g(t-) \leq x \leq g(t)\}$$

is called the *extended graph* of g , and the function g' with

$$g'(x) := \sup\{s \in \mathbb{R} : g(s) \leq x\}, \quad x \in \mathbb{R},$$

the (*generalized*) *inverse* of g . By identifying $g \in \tilde{G}$ with its extended graph $\Gamma(g)$, we obtain measurable spaces $(\Gamma(\tilde{G}), \Gamma(\tilde{\mathfrak{G}}))$ and $(\Gamma(G), \Gamma(\mathfrak{G}))$. For a proof of the following lemma, we refer to the appendix.

Lemma 2.1: *For all $g \in G$ we have:*

- (a) $g' \in G$,
- (b) $(g')' = g$,
- (c) $(t, x) \in \Gamma(g)$ iff $(x, t) \in \Gamma(g')$,
- (d) $(g'(x), x) \in \Gamma(g)$ for all $x \in \mathbb{R}$.

Let (Ω, \mathfrak{F}) be a measurable space. A *random time change* (RTC) Λ is a measurable mapping $\Omega \rightarrow G$. For $\omega \in \Omega$ we will write $\Lambda(\cdot, \omega)$ for the corresponding function in G and $\Lambda(t, \omega)$ for its value in $t \in \mathbb{R}$. The generalized inverse of $\Lambda(\cdot, \omega)$ is denoted by $\Lambda'(\cdot, \omega)$. So Λ' is another random time change. The extended graphs of $\Lambda(\cdot, \omega)$ and $\Lambda'(\cdot, \omega)$ are denoted by $\Gamma(\omega)$ and $\Gamma'(\omega)$, respectively. In this context we will usually use s and t to denote elements of the horizontal axis of $\Gamma(\omega)$, and x and y for elements of the vertical axis.

Let $(\tilde{\Omega}, \tilde{\mathfrak{F}})$ be a measurable space such that $\tilde{\Omega} \supset \Omega$ and $\tilde{\mathfrak{F}} \cap \Omega = \mathfrak{F}$. We call $(\tilde{\Omega}, \tilde{\mathfrak{F}})$ an *extension* of (Ω, \mathfrak{F}) . Let $\Theta = \{\Theta_{(t,x)} : (t,x) \in \mathbb{R}^2\}$ be a family of transformations on $\tilde{\Omega}$, not necessarily a group. I.e., $\Theta_{(t,x)}(\omega)$ is a measurable mapping from $(\mathbb{R}^2 \times \tilde{\Omega}, \mathfrak{B}(\mathbb{R}^2) \times \tilde{\mathfrak{F}})$ to $(\tilde{\Omega}, \tilde{\mathfrak{F}})$. The assumption below expresses that the (random) extended graph Γ of Λ is *consistent* with Θ , and that the family Θ behaves itself on Γ as a group. Assume:

- (i) For all $\omega \in \Omega$, $(t, x) \in \Gamma(\omega)$ and $(s, y) \in \Gamma(\Theta_{(t,x)}\omega)$ we have:
 - (a) $\Lambda(\cdot, \Theta_{(t,x)}\omega) = \Lambda(t + \cdot, \omega) - x$,
 - (b) $\Theta_{(s,y)}(\Theta_{(t,x)}\omega) = \Theta_{(s+t, x+y)}\omega$.

(Note that with $(t, x) \in \Gamma(\omega)$ and $(s, y) \in \Gamma(\Theta_{(t,x)}\omega)$, indeed $(s+t, x+y) \in \Gamma(\omega)$.) Assumption (i) is motivated by canonical settings (useful in applications) as in the following example.

Example 2.1: In the canonical case, we take $(\tilde{\Omega}, \tilde{F}) = (\tilde{G}, \tilde{\mathfrak{G}})$ and $(\Omega, \mathfrak{F}) = (G, \mathfrak{G})$. The RTC Λ is the identity mapping on G . In this case, a natural family Θ is defined by $\Theta_{(t,x)}g: = g(t + \cdot) - x$, $(t, x) \in \mathbb{R}^2$ and $g \in \tilde{G}$. Assumption (i) is trivially satisfied.

A more general canonical case (see also the marked time change in Section 3) arises as follows. Let $\tilde{\Omega}$ be the set of pairs (g, ρ) with $g \in \tilde{G}$ and ρ a measurable function on $\Gamma(g)$. Let Ω be the restriction of $\tilde{\Omega}$ to $\tilde{g} \in G$. σ -fields $\tilde{\mathfrak{F}}$ and \mathfrak{F} are constituted by the sets $\{(g, \rho) \in \tilde{\Omega}: g \in B\}$ with $B \in \tilde{\mathfrak{G}}$ and $B \in \mathfrak{G}$, respectively. A natural family Θ is defined by

$$\Theta_{(t,x)}(g, \rho): = (g(t + \cdot) - x, \rho(t + \cdot, x + \cdot)), \quad (t, x) \in \mathbb{R}^2 \text{ and } (g, \rho) \in \tilde{\Omega},$$

and an RTC Λ by

$$\Lambda(\cdot, (g, \rho)): = g(\cdot), \quad (g, \rho) \in \Omega.$$

It is an easy exercise to prove that the consistency in (a) and the group-structure in (b) are indeed satisfied. \square

Define, for $\omega \in \Omega$, $t \in \mathbb{R}$, and $x \in \mathbb{R}$,

$$\theta_t \omega: = \Theta_{(t, \Lambda(t, \omega))} \omega \quad \text{and} \quad \eta_x \omega: = \Theta_{(\Lambda'(x, \omega), x)} \omega, \quad (2.1)$$

and put $\theta: = \{\theta_t: t \in \mathbb{R}\}$ and $\eta: = \{\eta_x: x \in \mathbb{R}\}$ for the corresponding families of transformations (*shifts*) on Ω . The results in the following lemma can be proved easily.

Lemma 2.2: *Under Assumption (i), θ and η are groups. For all $s, t, x, y \in \mathbb{R}$ and $\omega \in \Omega$ we have:*

$$\begin{aligned} \Lambda(t, \theta_s \omega) &= \Lambda(t + s, \omega) - \Lambda(s, \omega), \\ \Lambda'(y, \eta_x \omega) &= \Lambda'(y + x, \omega) - \Lambda'(x, \omega), \\ \Lambda'(x, \theta_t \omega) &= \Lambda'(x + \Lambda(t, \omega), \omega) - t, \\ \Lambda(t, \eta_x \omega) &= \Lambda(t + \Lambda'(x, \omega), \omega) - x, \\ \eta_x(\theta_t \omega) &= \eta_{x + \Lambda(t, \omega)} \omega \quad \text{and} \quad \theta_t(\eta_x \omega) = \theta_{t + \Lambda'(x, \omega)} \omega. \end{aligned}$$

Note that $\theta_0 \omega$ and $\eta_0 \omega$ are not necessarily equal to ω . In the canonical setting of Example 2.1, θ_t is the shift operator which moves the origin to the position (on the graph) belonging to t on the horizontal axis, while η_x moves the origin to the position (on the *extended* graph) which belongs to x on the vertical axis. Note also that $\Lambda(0, \theta_s \omega)$ is always zero, while $\Lambda(0, \eta_x \omega)$ need not.

We next introduce shift-invariant sets. Define

$$\begin{aligned} \mathfrak{J}^{(\theta)}: &= \{A \in \mathfrak{F}: \theta_t^{-1} A = A \text{ for all } t \in \mathbb{R}\}, \\ \mathfrak{J}^{(\eta)}: &= \{A \in \mathfrak{F}: \eta_x^{-1} A = A \text{ for all } x \in \mathbb{R}\}. \end{aligned}$$

The next lemma is an extension of Lemma 2 of Nieuwenhuis [8]. See the appendix for a proof.

Lemma 2.3: *Under Assumption (i), the above invariant σ -fields coincide.*

In view of this lemma, we denote $\mathfrak{J}^{(\theta)}$ and $\mathfrak{J}^{(\eta)}$ by a single notation \mathfrak{J} . Note that, as an immediate consequence of the lemma,

$$f \circ \theta_t = f \text{ and } f \circ \eta_x = f \tag{2.2}$$

for all \mathfrak{J} -measurable functions $f: \Omega \rightarrow \mathbb{R}$ and all $t, x \in \mathbb{R}$.

In the next sections we will occasionally use the left-continuous inverse g^{-1} of $g \in \tilde{G}$, defined by $g^{-1}(x) = \inf\{s \in \mathbb{R}: g(s) \geq x\}$, $x \in \mathbb{R}$. Let g^* be the measure generated by g , i.e.,

$$g^*((s, t]): = g(t) - g(s), \quad s < t. \tag{2.3}$$

The following lemma enables us to transform integrals with respect to g^* , on the horizontal axis, into Lebesgue-integrals on the vertical axis. It will be proved in the appendix.

Lemma 2.4: *Let $g \in \tilde{G}$ and let $f: \mathbb{R} \rightarrow \mathbb{R}$ be g^* -integrable. Then we have, for all $a, b \in \mathbb{R}$ with $a < b$,*

$$\int_{g(a)}^{g(b)} f(g'(x))dx = \int_{g(a)}^{g(b)} f(g^{-1}(x))dx = \int_{(a, b]} f(s)g^*(ds).$$

3. Detailed Palm Distribution

In this section we presume a stationary setting in which the RTC Λ has stationary increments, and then define the *detailed* Palm distribution. It has the nice property that the group η is stationary with respect to it. Intuitively it can be derived by choosing at random an x on the positive vertical axis and shifting the origin to the corresponding position $(\Lambda'(x), x)$ on the extended graph of Λ . Next, the *ordinary* Palm distribution - the one that is analogous to the well-known PD for random measures - is also defined and the relationships between the two are considered. Finally, a generalization is given to *marked* time changes: RTCs accompanied by a stochastic process on their extended graphs.

The Stationary Framework

In addition to Assumption (i) we now assume a probability measure P on (Ω, \mathfrak{F}) under which the family θ is stationary, i.e.,

$$(ii) \quad P(\theta_t^{-1}A) = P(A) \text{ for all } t \in \mathbb{R} \text{ and } A \in \mathfrak{F},$$

and assume further that the (possibly non-degenerate) limit $\bar{\Lambda} := \lim_{t \rightarrow \infty} \Lambda(t)/t = E(\Lambda(1) | \mathfrak{J})$ satisfies

$$(iii) \quad P(0 < \bar{\Lambda} < \infty) = 1.$$

Assumptions (i) and (ii) imply that the RTC Λ has stationary increments.

Detailed Palm Distribution

Definition 3.1: Under Assumptions (i)-(iii), the probability measure P_Λ on (Ω, \mathfrak{F}) , the *detailed Palm distribution* (DPD) of P with respect to Λ , is defined by

$$P_\Lambda(A) := E \left(\frac{1}{\Lambda} \int_0^{\Lambda(1)} 1_A \circ \eta_x dx \right), \quad A \in \mathcal{F}. \quad (3.1)$$

In [7], a slightly modified version of (3.1) is presented. It is defined from a more applied point of view, on a smaller σ -field.

Theorem 3.1: *Assume (i)-(iii). Then $P = P_\Lambda$ on \mathfrak{J} , and the group η of transformations on Ω is stationary with respect to P_Λ :*

$$P_\Lambda(\eta_y^{-1}A) = P_\Lambda(A) \text{ for all } y \in \mathbb{R} \text{ and } A \in \mathcal{F}. \quad (3.2)$$

Proof: By (2.2) it is obvious that $P|_{\mathfrak{J}} = P_\Lambda|_{\mathfrak{J}}$. Let $y \in \mathbb{R}$ and $A \in \mathcal{F}$. Then

$$\begin{aligned} P_\Lambda(\eta_y^{-1}A) &= E \left(\frac{1}{\Lambda} \int_y^{\Lambda(1)+y} 1_A \circ \eta_x dx \right) \\ &= P_\Lambda(A) + E \left(\frac{1}{\Lambda} \int_{\Lambda(1)}^{\Lambda(1)+y} 1_A \circ \eta_x dx \right) - E \left(\frac{1}{\Lambda} \int_0^y 1_A \circ \eta_x dx \right), \end{aligned}$$

which equals $P_\Lambda(A)$ by Lemma 2.2 and stationarity of θ . \square

Expectations under P_Λ are denoted by E_Λ . With $\Lambda \in G$, we also have $\Lambda' \in G$. As an immediate consequence of Theorem 3.1 it follows:

$$\overline{\Lambda'} = \lim_{x \rightarrow \infty} \frac{1}{x} \Lambda'(x) = E_\Lambda(\Lambda'(1) | \mathfrak{J}) \quad P_\Lambda\text{- and } P\text{-a.s.} \quad (3.3)$$

(Note that $\Lambda'(0) = 0$ P_Λ -a.s.) By part (d) of Lemma 2.1 we obtain that for all $\omega \in \Omega$ and $\epsilon > 0$,

$$\frac{\Lambda'(x)}{\Lambda(\Lambda'(x))} \leq \frac{\Lambda'(x)}{x} \leq \frac{\Lambda'(x)}{\Lambda(\Lambda'(x) - \epsilon)}$$

if x is sufficiently large. Hence, by Assumption (iii) and the first part of Theorem 3.1, we have:

$$\overline{\Lambda'} = \frac{1}{\Lambda} \quad P_\Lambda\text{- and } P\text{-a.s.} \quad (3.4)$$

The following theorem gives (at least in the canonical case) the intuitive meaning for P_Λ via ‘‘choosing at random’’ an x on the positive half-line of the vertical axis and shifting the origin to the corresponding position $(\Lambda'(x), x)$ on the extended graph of Λ . ‘‘Choosing at random’’ is made precise by taking long-run averages.

Theorem 3.2: *Assume (i)-(iii). Then, for $A \in \mathcal{F}$,*

$$\lim_{t \rightarrow \infty} \frac{1}{\Lambda(t)} \int_0^{\Lambda(t)} 1_A \circ \eta_x dx = P_\Lambda(A | \mathfrak{J})$$

$$= \frac{1}{\Lambda} E \left(\int_0^{\Lambda(1)} 1_A \circ \eta_x dx \middle| \mathfrak{F} \right), P\text{- and } P_\Lambda\text{-a.s.},$$

$$\lim_{y \rightarrow \infty} \frac{1}{y} \int_0^y P(\eta_x^{-1} A) dx = P_\Lambda(A).$$

Proof: Set $\psi(t) := \int_0^{\Lambda(t)} 1_A \circ \eta_y dy$, $t \in \mathbb{R}$. By Lemma 2.2, it follows that $\psi(t) \circ \theta_s = \psi(t+s) - \psi(s)$ for all $s, t \in \mathbb{R}$. Note that the limits

$$\lim_{y \rightarrow \infty} \frac{1}{y} \int_0^y 1_A \circ \eta_x dx \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{1}{\Lambda(t)} \int_0^{\Lambda(t)} 1_A \circ \eta_x dx \tag{3.5}$$

exist and are equal (for all $\omega \in \Omega$). Under P_Λ , the left-hand limit equals $P_\Lambda(A | \mathfrak{F})$ a.s., while under P the right-hand limit equals

$$\lim_{n \rightarrow \infty} \frac{n}{\Lambda(n)} \frac{1}{n} \sum_{i=1}^n \psi(1) \circ \theta_{i-1} = \frac{1}{\Lambda} E \left(\int_0^{\Lambda(1)} 1_A \circ \eta_x dx \middle| \mathfrak{F} \right) \text{ a.s.}$$

Since $P = P_\Lambda$ on \mathfrak{F} , the first part of the theorem follows immediately. The second part follows by taking P -expectation in the left-hand part of (3.5) and by noting that $E(P_\Lambda(A | \mathfrak{F})) = P_\Lambda(A)$. \square

On many occasions, the horizontal axis represents time. The meaning of the vertical axis depends on the system studied. If the vertical axis represents the level of a fluid coming into a reservoir, then it follows from Theorem 3.2 that the DPD describes the stochastic behavior of this system as seen from an arbitrarily chosen level onwards. (Note that this level could be located within a jump of Λ , if the system allows this.) If $\Lambda(t)$ measures the cumulative time that a service system is busy (i.e., not idle), then the time-stationary distribution considers the system from an arbitrarily chosen time point while its DPD does so from an arbitrarily chosen “busy” time point. In this case, the vertical axis represents time when the system is busy. If $\Lambda(t)$ is the cumulative traded volume of a certain share on a stock exchange, then the DPD considers the behavior of this share as seen from an arbitrarily chosen transaction of size one; see also Section 6.

Set $\Lambda\{t\} := \Lambda(t) - \Lambda(t-)$, $t \in \mathbb{R}$. A proof of the following corollary is given in the appendix.

Corollary 3.1: *Assume (i)-(iii). Then, under P_Λ , the conditional distribution of $\Lambda(0)$ given $\Lambda\{0\}$ is the uniform $[0, \Lambda\{0\}]$ distribution.*

In Section 5 we will include Palm theory for random measures as part of Palm theory for RTCs. In advance, note that two RTCs Λ_1 and Λ_2 on $(\Omega, \mathfrak{F}, P)$ which generate the same random measure Λ^* , i.e.,

$$\Lambda^*((s, t]) = \Lambda_1(t) - \Lambda_1(s) = \Lambda_2(t) - \Lambda_2(s) \tag{3.6}$$

for all $s \leq t$, have the same DPD provided that the respective families $\eta^{(1)}$ and $\eta^{(2)}$ coincide P -a.s. This is because $\Lambda_2(0) = \Lambda_1(0) = 0$, P -a.s.

Ordinary Palm Distribution

Whereas the DPD is derived intuitively by randomly moving along an extended graph in a way that keeps track of where within a jump (if any) one is, the traditional Palm distribution does not. In the present time change setting, we will refer to this traditional case as the *ordinary* Palm distribution (OPD); if Λ is continuous, the group of transformations $\{\theta_{\Lambda'(x)}\}$ is stationary under it. In the canonical setting this OPD is intuitively obtained (recall (2.1) and see Remark 3.1 below) by randomly choosing an x on the positive vertical axis and shifting the origin to $(\Lambda'(x), \Lambda \circ \Lambda'(x))$ along the graph (not extended graph) of Λ . The point here is that whenever a jump occurs for Λ , the OPD measures the magnitude of the jump size and then looks ahead after the jump (see Remark 3.2), while the DPD continues measuring continuously along the jump (vertical axis of the extended graph).

A random time change Λ generates a random measure Λ^* (recall (2.3) and (3.6)). By Assumptions (i) and (ii), Λ^* is stationary under P . In accordance with Palm theory for random measures we define the OPD of P with respect to Λ as the well-known PD of P w.r.t. Λ^* , and call it P^0 :

$$P^0(A) := E \left(\frac{1}{\Lambda} \int_{(0,1]} 1_A \circ \theta_t \Lambda^*(dt) \right), \quad A \in \mathfrak{F}. \quad (3.7)$$

This definition corresponds to (2) in [10], modified along the lines of [11] and Nieuwenhuis [9] so as to encompass the non-ergodic case. (As discussed in the last two references, it is more natural to use the random intensity $\bar{\Lambda}$ (instead of its P -expectation) in the definition of non-ergodic OPD.) As presented in [4], the family of shifts $\{\theta_{\Lambda'(x)}\}$ is a group being stationary under P^0 provided that Λ is continuous.

In order to relate OPD and DPD, we first express the OPD in terms of an integral on the other (vertical) axis. The following result is an immediate consequence of Lemma 2.4:

$$P^0(A) = E \left(\frac{1}{\Lambda} \int_0^{\Lambda(1)} 1_A \circ \theta_{\Lambda'(x)} dx \right), \quad A \in \mathfrak{F}. \quad (3.8)$$

Relationship Between DPD and OPD

We will write E^0 for expectations under P^0 . In the next theorem, the relationship between OPD and DPD is studied. It is proven in the appendix.

Theorem 3.3: *Let Λ be an RTC on $(\Omega, \mathfrak{F}, P)$ which satisfies (i)-(iii). Then the relationship between P^0 and P_Λ is as follows:*

- (a) $P^0 = P_\Lambda \theta_0^{-1}$
- (b) $P_\Lambda(A) = E^0 \left(\frac{1}{\Lambda\{0\}} \int_{-\Lambda\{0\}}^0 1_A \circ \eta_x dx \right), \quad A \in \mathfrak{F}.$

The averaged integral in (b) is interpreted as $1_A(\omega)$ if $\Lambda(\{0\}, \omega) = 0$.

Part (a) expresses the fact that the OPD looks ahead from the top of a jump (if any); the shift θ_0 does the required re-positioning. Part (b) expresses the fact that DPD looks ahead from a position uniformly within a jump.

Remark 3.1: Note that OPD and the DPD coincide in the case that Λ is continuous. Analogous to Theorem 3.2, there is an analogue for P^0 using the family of shifts $\{\theta_{\Lambda'(x)}\}$ (see also Nieuwenhuis [8] for the point process case). Since $\theta_{\Lambda'(x)} =$

$\theta_0 \circ \eta_x$, we obtain by Theorem 3.2 that

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{1}{\Lambda(t)} \int_0^{\Lambda(t)} 1_A \circ \theta_{\Lambda'(x)} dx = P_\Lambda(\theta_0^{-1} A | \mathfrak{J}) = P^0(A | \mathfrak{J}) \\ & = \frac{1}{\Lambda} E \left(\int_0^{\Lambda(1)} 1_A \circ \theta_{\Lambda'(x)} dx \middle| \mathfrak{J} \right), P\text{-}, P^0\text{-}, P_\Lambda\text{-a.s.} \\ & \lim_{y \rightarrow \infty} \frac{1}{y} \int_0^y P(\theta_{\Lambda'(x)}^{-1} A) dx = P^0(A), \quad A \in \mathfrak{F}. \end{aligned} \tag{3.9}$$

Remark 3.2: Observe that if Λ is a pure jump process with jump-times T_i and jump-sizes X_i (under the convention that $\dots < T_{-1} < T_0 \leq 0 < T_1 < \dots$), relation (3.9) becomes

$$\frac{1}{\bar{n}} \sum_{i=1}^n E \left(\frac{X_i}{\bar{X}} 1_A \circ \theta_{T_i} \right) \rightarrow P^0(A), \quad A \in \mathfrak{F}; \tag{3.10}$$

here \bar{X} is the long-run average of $\{X_1, X_2, \dots\}$. Note also that the sequences $\{T_i - T_{i-1}\}$ and $\{X_i\}$ are usually not stationary under P^0 , since they are not necessarily stationary under P_Λ (because of length-biased sampling) and their distributions do not change by shifting the origin up by applying θ_0 . But these sequences will be stationary under the distribution Q^0 with

$$Q^0(A) := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n P(\theta_{T_i}^{-1} A) = E^0(I_A \bar{X} / X_0), \quad A \in \mathfrak{F},$$

that arises from P by shifting the origin to an arbitrarily chosen jump-time. In the simple point process case, P^0 and Q^0 coincide.

Marked Time Change

For completeness, we include here the more general situation in which the RTC is accompanied by a stochastic process on its extended graph. A *marked time change* is a pair (Λ, S) consisting of a random time change Λ and a stochastic process S , on a common probability space $(\Omega, \mathfrak{F}, P)$, such that $S((\cdot, \cdot), \omega)$ is a measurable function on $\Gamma(\omega)$ for all $\omega \in \Omega$. It is assumed that a family Θ of transformations exists such that Λ satisfies Assumptions (i)-(iii). Furthermore, we assume that for all $\omega \in \Omega$ and $(t, x) \in \Gamma(\omega)$,

(iv) $S((s, y), \Theta_{(t, x)} \omega) = S((s + t, y + x), \omega)$ for all $(s, y) \in \Gamma(\Theta_{(t, x)} \omega)$. (See Example 2.1 for a canonical version.) Set $S_1(t) := S((t, \Lambda(t)))$ and $S_2(x) := S((\Lambda'(x), x))$; $t, x \in \mathbb{R}$. It is an easy exercise to prove that the stochastic processes S_1 and S_2 satisfy

$$S_1(t) \circ \theta_s = S_1(t + s) \text{ and } S_2(x) \circ \eta_y = S_2(x + y), \tag{3.11}$$

$$S_1(t) \circ \eta_y = S_1(t + \Lambda'(y)) \text{ and } S_2(x) \circ \theta_t = S_2(x + \Lambda(t)),$$

for all $s, t, x, y \in \mathbb{R}$. Consequently, S_1 is stationary under P w.r.t. θ , while S_2 is stationary under P_Λ w.r.t. η .

4. Inversion by Duality

Starting with Assumptions (i)-(iii) for the pair (Λ, P) we defined P_Λ , the DPD w.r.t. Λ . A similar approach for the pair (Λ', P_Λ) leads to a duality criterion. This criterion is used to derive an inversion formula for the OPD.

Assume (i)-(iii). We next consider Λ' instead of Λ ; we will give corresponding quantities a prime. Define the family Θ' of transformations $\Theta'_{(x,t)}$ by $\Theta'_{(x,t)}\omega := \Theta_{(t,x)}\omega$, $\omega \in \Omega$ and $(x, t) \in \mathbb{R}^2$. By Lemma 2.1 it is an easy exercise to prove that Θ' satisfies Assumption (i)' which arises from (i) by replacing Λ by Λ' and $\Gamma(\omega)$ by $\Gamma'(\omega)$. From Θ' we define θ'_x and η'_t ; $t, x \in \mathbb{R}$. Part (b) of Lemma 2.1 ensures that $\theta' = \eta$ and $\eta' = \theta$. So, we have

(ii)' θ' is stationary w.r.t. P_Λ ,

(iii)' $P_\Lambda(0 < \bar{\Lambda}' < \infty) = 1$.

(The last assertion is a consequence of (3.4).) Consequently, the DPD of P_Λ with respect to Λ' , notation $(P_\Lambda)_{\Lambda'}$, is well-defined:

$$(P_\Lambda)_{\Lambda'}(A) = E_\Lambda \left(\frac{1}{\Lambda'} \int_0^{\Lambda'(1)} 1_A \circ \theta_s ds \right).$$

Theorem 4.1: *The detailed Palm distribution of P_Λ with respect to Λ' is equal to P . Especially, for $A \in \mathfrak{F}$,*

$$P(A) = E_\Lambda \left(\frac{1}{\Lambda'} \int_0^{\Lambda'(1)} 1_A \circ \theta_s ds \right),$$

$$\frac{1}{t} \int_0^t P_\Lambda(\theta_s^{-1} A) ds \rightarrow P(A).$$

Proof: Since (i)'-(iii)' are satisfied, we can apply Theorem 3.2 replacing Λ by Λ' , P by P_Λ , and P_Λ by $(P_\Lambda)_{\Lambda'}$. This yields, for an equivalent version of the first part of Theorem 3.2,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t 1_A \circ \theta_s ds = \frac{1}{\Lambda'} E_\Lambda \left(\int_0^{\Lambda'(1)} 1_A \circ \theta_s ds \middle| \mathfrak{J} \right) P_\Lambda\text{-a.s.}$$

Since $P = P_\Lambda$ on \mathfrak{J} , we obtain:

$$P(A) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t P(\theta_s^{-1} A) ds = (P_\Lambda)_{\Lambda'}(A),$$

which gives the first assertion of the present theorem. The second is an immediate consequence. \square

Remark 4.1: By the above approach it follows that duality holds between P and its DPD w.r.t. Λ , a property which in general does not hold for classical PDs. See also [4]. Properties for P can immediately be translated into dual properties for P_Λ , and vice versa. For instance, from Theorem 3.2 and Corollary 3.1 we immediately obtain the following dual assertions:

$$P(A | \mathfrak{g}) = \frac{1}{\Lambda'} E_{\Lambda} \left(\int_0^{\Lambda'(1)} 1_A \circ \theta_s ds \mid \mathfrak{g} \right) \quad P_{\Lambda^-} \text{ and } P\text{-a.s.},$$

and under P , the conditional distribution of $\Lambda'(0)$ given $\Lambda'\{0\}$ is the uniform $[0, \Lambda'\{0\}]$ distribution.

The last result is well known in the case that Λ characterizes a simple point process (see Section 6), and obviously holds more generally, for instance for a pure jump process: with $\Lambda'(0) = :T_1$ the first jump-time on $(0, \infty)$ and $\Lambda'(0-) = :T_0$ the last jump-time on $(-\infty, 0]$, the conditional distribution of T_1 given $T_1 - T_0$ is the uniform $[0, T_1 - T_0]$ distribution. Note also that the convergence result of Theorem 4.1 means that intuitively P arises from P_{Λ} by choosing at random an s on the positive half-line of the horizontal axis and shifting the origin to the corresponding position $(s, \Lambda(s))$ on the graph of Λ . \square

Relations (3.7) and (3.8) express how P can be transformed into P^0 . An expression which works the other way round, is historically called an *inversion formula*. See also [10], Corollary 1 in Section 2. We use inversion of P_{Λ} to P , managed in Theorem 4.1 by using the duality approach, to accomplish inversion of P^0 to P . The proof of the following theorem is included in the appendix. Recall that $\bar{\Lambda}' = E_{\Lambda}(\Lambda'(1) | \mathfrak{g})$.

Theorem 4.2: *Let P^0 be the OPD of P with respect to Λ . Then*

$$P(A) = E^0 \left(\frac{1}{\bar{\Lambda}'} \int_0^{\Lambda'(1)} \left(1 \wedge \frac{1 - \Lambda(t)}{\Lambda\{0\}} \right) 1_A \circ \theta_t dt \right), \quad A \in \mathfrak{F}.$$

Here the minimum in the integrand is interpreted as 1 if $\Lambda\{0, \omega\} = 0$.

5. Stationary Random Measures and PDs

In this section, we include Palm theory for random measures in Palm theory for RTCs. Starting with a random measure and the well known PD in a common stationary setting, we construct an RTC which generates the random measure and which satisfies (i)-(iii). No additional assumptions are needed. In a sense, the OPD of this RTC is equal to the PD of the random measure we started with. The DPD of the random measure is defined as the DPD of this RTC.

Let M be the set of all measures μ on $\mathfrak{B}(\mathbb{R})$ for which $\mu(B) < \infty$ for all bounded $B \in \mathfrak{B}(\mathbb{R})$. M is endowed with the σ -field \mathcal{M} generated by the sets $\{\mu \in M : \mu(B) = k\}$, $k \in \mathbb{N}_0$ and $B \in \mathfrak{B}(\mathbb{R})$. A *random measure* on \mathbb{R} is a measurable mapping Λ_0^* from a measurable space $(\Omega_0, \mathfrak{F}_0)$ to (M, \mathcal{M}) . Let Q be a probability measure on $\{\Omega_0, \mathfrak{F}_0\}$. We write E for expectations under Q . We assume that a group $\tau := \{\tau_t : t \in \mathbb{R}\}$ of transformations on Ω_0 exists such that Λ_0^* is *consistent* with τ , and τ is *stationary* with respect to Q ; i.e.,

- (ia) $\Lambda_0^*(B) \circ \tau_t = \Lambda_0^*(B + t)$ for all $B \in \mathfrak{B}(\mathbb{R})$ and $t \in \mathbb{R}$,
- (iia) $Q\tau_t^{-1} = Q$ for all $t \in \mathbb{R}$.

Hence, Λ_0^* is stationary under Q . It can be characterized by the random time change Λ_0 defined by

$$\Lambda_0(t) := \begin{cases} \Lambda_0^*((0, t]) & \text{if } t \geq 0 \\ -\Lambda_0^*((t, 0]) & \text{if } t < 0. \end{cases} \quad (5.1)$$

Note that $\Lambda_0(0) = 0$ and that Λ_0 generates Λ_0^* ; see (2.3). In case Λ_0^* is an integer-valued random measure, the RTC Λ_0 is also integer-valued and can never satisfy part (a) of Assumption (i), notwithstanding the choice of the family Θ . So, we must choose the RTC generating Λ_0^* in a more clever way.

Furthermore, we assume that

$$(iiia) \quad \underline{Q}(0 < \bar{\Lambda}_0 < \infty) = 1.$$

Here $\bar{\Lambda}_0$ is the long-run average $E(\Lambda_0(1) | \mathfrak{F}_0) = \lim_{t \rightarrow \infty} \Lambda_0(t)/t$ with \mathfrak{F}_0 the invariant σ -field of τ . Similar to [10], we define the Palm distribution Q^0 of Q with respect to Λ_0^* by

$$Q^0(A) := E \left(\frac{1}{\Lambda_0^*(0, 1]} \int 1_A \circ \tau_t \Lambda_0^*(dt) \right), \quad A \in \mathfrak{F}_0. \quad (5.2)$$

As in (3.7), we use the random intensity; see also [11] and [9]. For a fixed Λ_0^* , this PD does not really depend on the choice made for the RTC which generates Λ_0^* . So Λ_0 may be replaced by another RTC which generates Λ_0^* . By Lemma 2.4 we can also consider $Q^0(A)$ along the vertical axis:

$$Q^0(A) = E \left(\frac{1}{\Lambda_0^*(0, 1]} \int_0^{\Lambda_0^*(1)} 1_A \circ \tau_{\Lambda_0^*(x)} dx \right), \quad A \in \mathfrak{F}_0. \quad (5.3)$$

Since, for fixed $\omega_0 \in \Omega_0$, $\Lambda_0^*(x)$ and $\Lambda_0^*(x-)$ can be unequal for at most countably many $x \in \mathbb{R}$, we may equivalently use $\Lambda_0^*{}^{-1}(x) := \Lambda_0^*(x-)$ in (5.3) instead of $\Lambda_0^*(x)$, i.e., we may also use the left-continuous version $\Lambda_0^*{}^{-1}$ of Λ_0^* .

As mentioned above, a family Θ of transformations not necessarily satisfies Assumption (i), not even if (ia) holds. We have to make the measurable space $(\Omega_0, \mathfrak{F}_0)$ richer. Assume that (ia), (iia) and (iiia) are satisfied, and define

$$\begin{aligned} \tilde{\Omega} &:= \Omega_0 \times \mathbb{R} \text{ and } \tilde{\mathfrak{F}} := \mathfrak{F}_0 \times \mathfrak{B}(\mathbb{R}), \\ \Omega &:= \{(\omega_0, z) \in \tilde{\Omega} : 0 \leq z \leq \Lambda_0^*(\{0\}, \omega_0)\}, \\ \mathfrak{F} &:= \tilde{\mathfrak{F}} \cap \Omega. \end{aligned}$$

Let $\omega = (\omega_0, z)$ be an element of $\tilde{\Omega}$. For $s, t, x \in \mathbb{R}$ we define:

$$\begin{aligned} \Theta_{(t, x)} \omega &:= (\tau_t \omega_0, \Lambda_0(t, \omega_0) + z - x) \in \tilde{\Omega}, \\ \Lambda(t, \omega) &:= \Lambda_0(t, \omega_0) + z, \\ \Lambda^*((s, t], \omega) &:= \Lambda(t, \omega) - \Lambda(s, \omega) \text{ for } s \leq t. \end{aligned} \quad (5.4)$$

Next, we identify Ω_0 and $\Omega_0 \times \{0\}$. With this identification, Λ and Λ^* are extensions of Λ_0 and Λ_0^* . Note, however, that $\Omega = \Omega_0$ if $\Lambda_0(\cdot, \omega)$ is continuous on \mathbb{R} for all $\omega \in \Omega_0$. Note also that the last definition above implies a measure $\Lambda^*(\cdot, \omega)$ on

$\mathfrak{B}(\mathbb{R})$ with $\Lambda^*(B, \omega) = \Lambda_0^*(B, \omega_0)$ for all $B \in \mathfrak{B}(\mathbb{R})$, and that the random function Λ , defined on (Ω, \mathfrak{F}) is indeed a random time change since $\Lambda(\cdot, \omega) \in G$ for all $\omega \in \Omega$. The family Θ of transformations on $(\tilde{\Omega}, \tilde{\mathfrak{F}})$ satisfies part (b) of Assumption (i), even for all $\omega = (\omega_0, z)$ in $\tilde{\Omega}$ and for all $t, x, s, y \in \mathbb{R}$:

$$\begin{aligned} \Theta_{(s,y)}(\Theta_{(t,x)}\omega) &= \Theta_{(s,y)}(\tau_t\omega_0, \Lambda_0(t, \omega_0) + z - x) \\ &= (\tau_s(\tau_t\omega_0), \Lambda_0(s, \tau_t\omega_0) + \Lambda_0(t, \omega_0) + z - x - y) \\ &= (\tau_{s+t}\omega_0, \Lambda_0(s+t, \omega_0) + z - (x+y)), \end{aligned}$$

which equals $\Theta_{(s+t, x+y)}\omega$. (In the last equality, we used (ia) and the group property of the family τ on $(\Omega_0, \mathfrak{F}_0)$.) Again with (ia), it is an easy exercise to prove that part (a) of (i) also holds. Hence, we can define groups θ and η of transformations on Ω as in Section 2. Note that, with the identification $\omega_0 = (\omega_0, 0)$, we have for $\omega = (\omega_0, z)$:

$$\theta_t\omega = (\tau_t\omega_0, 0) = \tau_t\omega_0 \in \Omega_0. \tag{5.5}$$

Especially, τ_t is just the restriction of θ_t to Ω_0 (as it should be). We can extend $(\Omega_0, \mathfrak{F}_0, Q)$ to $(\Omega, \mathfrak{F}, P)$ by the definition:

$$P(A) = Q(A \cap \Omega_0), \quad A \in \mathfrak{F}. \tag{5.6}$$

The pair (θ, P) also satisfies (ii). So, θ is stationary with respect to P . concerning the invariant σ -fields \mathfrak{J}_0 and \mathfrak{J} of τ and θ , respectively, we note that: $A \cap \Omega_0 \in \mathfrak{J}_0$ if $A \in \mathfrak{J}$. Hence, Assumption (iiia), with E denoting expectation under Q , implies Assumption (iii), with E denoting expectation under P .

We conclude that a random measure Λ_0^* satisfying (ia), (iia) and (iiia) can (in a natural way) be extended to a random measure Λ^* and a corresponding random time change Λ which satisfies Assumptions (i)-(iii); without additional assumptions. Conversely, a random time change Λ satisfying (i)-(iii) implies a random time change $\Lambda_0 := \Lambda \circ \theta_0$ which satisfies (ia), (iia), and (iiia).

Having extended $(\Omega_0, \mathfrak{F}_0, Q, \tau, \Lambda_0^*, \Lambda_0)$ to $(\Omega, \mathfrak{F}, P, \theta, \Lambda^*, \Lambda)$, the definition of Q^0 in (5.2) transforms into the definition of P^0 - the OPD of P w.r.t. Λ - in (3.7). Note that $P^0(A) = Q^0(A \cap \Omega_0)$. We will interpret P^0 as the PD of Q w.r.t. the random measure Λ_0^* . Similarly, we will call P_Λ the DPD of Q w.r.t. Λ_0^* . The relationship between these two distributions of Palm type is described in Theorem 3.3.

6. PDs in the Point Process Case

In the context of point processes, the corresponding random time change is a stepfunction with integer-valued stepsizes. The jumps occur precisely at the arrival times. The DPD treats the vertical jumps in a continuous fashion (recall Corollary 3.1) while only discrete positions are of interest (customers for example). For applications, a modification of the DPD is thus desirable. For example, in a batch point process representing customers arriving in busloads to a queue, we should modify our DPD to account for individuals within a bus. This distinction is characterized by Theorem 6.1(d) and (e) below. The Palm type distribution that we will obtain is equivalent to the PD in [2] for the sequence approach. Several distributions of Palm

type are then compared.

Recalling (5.1) (with discussion right after) and (5.4), we can start with any time stationary point process as defined by a stationary random counting measure and construct from it (via an extension) a special random time change $\Lambda = \Phi$ on \mathbb{R} satisfying (i)-(iii). That is, Φ is an RTC with $\Phi(t) - \Phi(s) \in \mathbb{Z}$ for all $\omega \in \Omega$ and $s, t \in \mathbb{R}$. Note that whereas there are sample paths of Φ such that $\Phi(0)$ can be non-zero and non-integer valued, under time stationary P this occurs with probability zero (but under DPD P_Φ this probability is one). Motivated by this RTC construction, we shall refer to any RTC with integer-valued increments as a *random point process* and denote it by Φ .

Recall that P_Φ and P^0 , the DPD and the OPD of P w.r.t. Φ , are defined by

$$P_\Phi(A) = E \left(\frac{1}{\Phi} \int_0^{\Phi(1)} 1_A \circ \eta_x dx \right), P^0(A) = E \left(\frac{1}{\Phi} \int_0^{\Phi(1)} 1_A \circ \theta_{\Phi'(x)} dx \right), A \in \mathfrak{F} \quad (6.1)$$

(cf. (3.1) and (3.8)), and that

$$P_\Phi(A) = \lim_{y \rightarrow \infty} \frac{1}{y} \int_0^y P(\eta_x^{-1} A) dx, P^0(A) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n P(\theta_{T_i}^{-1} A), A \in \mathfrak{F} \quad (6.2)$$

(cf. Theorem 3.2 and (3.9)). Here $\bar{\Phi} = E(\Phi(1) | \mathfrak{J})$, and T_i is the *time of i th occurrence (arrival)* defined by $T_i = \Phi'(i - 1 + \Phi(0)) = \Phi'(i - 1) \circ \theta_0$. So,

$$\dots \leq T_{-2} \leq T_{-1} \leq T_0 \leq 0 < T_1 \leq T_2 \leq \dots \quad (6.3)$$

Recall, for the canonical settings, the intuitive interpretations of P_Φ and P^0 following (3.4) and (3.6), respectively. Obviously, P^0 cannot discriminate at all between two simultaneous occurrences within one batch. On the other hand, while the DPD does distinguish among positions within a batch, it does so continuously. A modified version of the DPD overcomes these difficulties. Let m denote the lattice-measure concentrated on the set \mathbb{Z} of integers. We will use m (instead of Lebesgue-measure) to force selection along the vertical axis to be restricted to the integers. Define the distribution \bar{P}_Φ by

$$\begin{aligned} \bar{P}_\Phi(A) &= E \left(\frac{1}{\bar{\Phi}_0} \int_0^{\bar{\Phi}(1)} 1_A \circ \eta_x m(dx) \right) \\ &= E \left(\frac{1}{\bar{\Phi}_0} \sum_{i=1}^{\bar{\Phi}(1)} 1_A \circ \eta_i \right), A \in \mathfrak{F}. \end{aligned} \quad (6.4)$$

Here $\bar{\Phi}_0 := E(\Phi(1) | \mathfrak{J}_{00})$ with \mathfrak{J}_{00} the invariant σ -field of the group $\{\eta_i : i \in \mathbb{Z}\}$ of transformations on Ω . Note that $\mathfrak{J} \subset \mathfrak{J}_{00}$ by Lemma 2.3, that $\mathfrak{J} \neq \mathfrak{J}_{00}$ since the ω -set $(\Phi(0) \in \mathbb{N})$ does not belong to \mathfrak{J} , and that

$$\bar{P}_\Phi(\eta_i^{-1} A) = \bar{P}_\Phi(A), A \in \mathfrak{F} \text{ and } i \in \mathbb{Z}. \quad (6.5)$$

So, $\{\eta_i\}$ is stationary w.r.t. $\overline{P_\Phi}$, and

$$\frac{1}{n} \sum_{i=1}^n 1_A \circ \eta_i \rightarrow \overline{P_\Phi}(A | \mathfrak{J}_{00}) \quad \overline{P_\Phi}\text{- and } P\text{-a.s.}$$

since $P = \overline{P_\Phi}$ on \mathfrak{J}_{00} . Hence,

$$\frac{1}{n} \sum_{i=1}^n P(\eta_i^{-1} A) \rightarrow \overline{P_\Phi}(A), \quad A \in \mathfrak{F}. \quad (6.6)$$

In the canonical setting, we can interpret $\overline{P_\Phi}$ as arising from P by randomly choosing a positive integer i on the vertical axis and shifting the origin to $(\Phi'(i), i) = (T_{i+1}, i)$ on the *extended* graph of Φ . In case of a non-simple PP, relation (6.6) makes clear that $\overline{P_\Phi}$ gives the opportunity to discriminate between the arrivals within a batch and that it is equivalent to the distribution \mathbf{P} on page 82 of [2].

Let $\beta_i := T_{i+1} - T_i$, $i \in \mathbb{Z}$, be the sequence of *interval lengths (interarrivals)* of the PP. It can easily be proved that

$$\beta_j \circ \theta_t = \beta_{j + \Phi(t) - \Phi(0)}, \quad j \in \mathbb{Z} \text{ and } t \in \mathbb{R}, \quad (6.7)$$

and that in general, $\beta_j \circ \eta_1$ is not equal to β_{j+1} . However, by renumbering the interarrivals by making use of the special character of our framework, we can regain this property. Set

$$\alpha := \max\{\Phi(0) - i : i \in \mathbb{N}_0 \text{ and } \Phi(0) - i \leq 0\}. \quad (6.8)$$

The magnitude of α represents the minimal amount required to add to $\Phi(0)$ to make it integer-valued. In a canonical setting, $\Theta_{(0, \alpha(\omega))} \omega$ moves the origin $(0, 0)$ of $\Gamma(\omega)$ downwards to the first position on this extended graph which is integer-distanced from $(0, \Phi(0, \omega))$. (If $\Phi(0, \omega) = 0$, nothing happens.) With $\Phi_\alpha := \Phi - \alpha$, we define

$$\tilde{T}_j := \Phi'(j + \alpha) = T_{j+1 - \Phi_\alpha(0)} \text{ and } \tilde{\beta}_j := \tilde{T}_{j+1} - \tilde{T}_j, \quad j \in \mathbb{Z}. \quad (6.9)$$

It is obvious that $\alpha \circ \eta_1 = \alpha$ since (6.8) does not change by adding an integer to the $\Phi(0) - i$. Consequently,

$$\tilde{\beta}_j \circ \eta_1 = \tilde{\beta}_{j+1}, \quad j \in \mathbb{Z}. \quad (6.10)$$

Hence, $(\tilde{\beta}_j)$ is stationary under $\overline{P_\Phi}$.

We next compare the distributions P , P^0 , P_Φ and $\overline{P_\Phi}$. At first, note that

$$\begin{aligned} P(\Phi(0) = 0) &= 1 \text{ and } P^0(\Phi(0) = 0) = 1, \\ P_\Phi(\Phi(0) \in \mathbb{N}_0) &= 0 \text{ and } \overline{P_\Phi}(\Phi(0) \in \mathbb{N}) = 1, \\ P(\Phi(0-) = 0) &= 1 \text{ and } P^0(\Phi(0-) < 0) = 1, \\ P_\Phi(\Phi(0-) = 0) &= 0 \text{ and } \overline{P_\Phi}(\Phi(0-) = 0) > 0. \end{aligned} \quad (6.11)$$

(Here \mathbb{N} does not contain 0, but \mathbb{N}_0 does.) In the following theorem, we write E^0 , E_Φ and $\overline{E_\Phi}$ for expectations under P^0 , P_Φ and $\overline{P_\Phi}$, respectively.

Theorem 6.1: *Let Φ be a PP on $(\Omega, \mathfrak{F}, P)$ which satisfies Assumptions (i)-(iii). Then, for $A \in \mathfrak{F}$,*

$$(a) \quad P_\Phi(A) = \overline{E_\Phi}(\int_0^1 1_A \circ \eta_x dx),$$

- (b) $\overline{P_\Phi}(A) = P_\Phi(\eta_\alpha^{-1}A),$
- (c) $P^0(A) = P_\Phi(\theta_0^{-1}A) = \overline{P_\Phi}(\theta_0^{-1}A),$
- (d) $P_\Phi(A) = E^0(\frac{1}{\Phi\{0\}} \int_{-\Phi\{0\}}^0 1_A \circ \eta_x dx),$
- (e) $\overline{P_\Phi}(A) = E^0(\frac{1}{\Phi\{0\}} \sum_{i=-\Phi\{0\}}^{-1} 1_A \circ \eta_i).$

Proof: Note that, for $n \in \mathbb{N}$ and $A \in \mathcal{F}$,

$$\frac{1}{n} \int_0^n 1_A \circ \eta_y dy = \frac{1}{n} \sum_{i=1}^n \int_0^1 1_A \circ \eta_x \circ \eta_{i-1} dx. \quad (6.12)$$

As $n \rightarrow \infty$, the LHS tends to $P_\Phi(A | \mathcal{J})$, both P_Φ -a.s. and P -a.s. The RHS of (6.12) tends to $\overline{E_\Phi}(\int_0^1 1_A \circ \eta_x dx | \mathcal{J}_{00})$, both $\overline{P_\Phi}$ -a.s. and P -a.s. Since $P = P_\Phi$ on \mathcal{J} and $P = \overline{P_\Phi}$ on \mathcal{J}_{00} , we obtain both sides of (a) as limits of $\frac{1}{n} \int_0^n P(\eta_y^{-1}A) dy$ as $n \rightarrow \infty$. So, the two sides have to be equal. For part (b), note that under $\overline{P_\Phi}$ we have by (6.11) that $\Phi'(x) = 0$ for all $x \in (0, 1)$. Hence, $\overline{P_\Phi}$ -a.s., the composition $\alpha \circ \eta_x$ equals $\alpha - x$ for all $x \in (0, 1)$. With this result, (b) follows from (a). Part (d) and the first equality in (c) follow from Theorem 3.3. The second equality in (c) is a consequence of (a) and the RHS of (6.11). Part (e) follows from (b) and (d). \square

Marked Point Processes

Formally, to distinguish among customers within a batch, they need to be labeled or *marked*. This motivates considering the more general case of marked point processes in which to each arrival time T_j is attached a mark m_j . As we will see, under the new labeling used above, the relabeled sequence $\{(\tilde{\beta}_j, \tilde{m}_j)\}$, of interarrival times and marks, is stationary.

Let K be a metric space, assumed to be complete and separable. $\mathfrak{B}(K)$ denotes the Borel- σ -field on K . A *marked point process* (MPP) on \mathbb{R} with mark space K is a random pair $(\Phi, (m_i)_{i \in \mathbb{Z}})$ where Φ is a point process and $(m_i)_{i \in \mathbb{Z}}$ is a random sequence in K . The two elements of the pair are defined on a common probability space (Ω, \mathcal{F}, P) . We interpret m_i as the *mark* of T_i , $i \in \mathbb{Z}$, and assume that Φ satisfies Assumptions (i)-(iii). Furthermore, we assume that

(iva)

$$m_i(\Theta_{(t,x)}\omega) = m_{i + \Phi(t,\omega) - \Phi(0,\omega)}(\omega), \quad i \in \mathbb{Z}, \quad \omega \in \Omega, \quad (t,x) \in \Gamma(\omega). \quad (6.13)$$

An MPP is indeed a marked time change (cf. Section 3) since the stochastic process S with

$$S((s,y), \omega) := \begin{cases} m_i(\omega) & \text{if } y = \Phi(0,\omega) + i - 1 \\ 0 & \text{otherwise,} \end{cases}$$

$\omega \in \Omega$ and $(s,y) \in \Gamma(\omega)$, is defined on Γ and satisfies Assumption (iv) by (6.13). Note that S is constant on horizontal parts of Γ and that m_i is just the value of S at the position $(T_i, \Phi(0) + i - 1)$ on Γ . As in (3.11), we could create a stochastic process S_2 which is stationary under P_Φ . In view of (6.9), a renumbering of the sequence $(m_i)_{i \in \mathbb{Z}}$ seems to be of more importance. Set

$$\tilde{m}_j := m_{j+1 - \Phi_\alpha(0)} = S(\tilde{T}_j, j + \alpha), \quad j \in \mathbb{Z}.$$

Hence, \tilde{m}_j is the mark of \tilde{T}_j . Since $\alpha \circ \eta_1 = \alpha$, it is an easy exercise to prove that

$$\tilde{m}_j \circ \eta_1 = \tilde{m}_{j+1}, \quad j \in \mathbb{Z}.$$

So, the sequence $(\tilde{m}_j)_{j \in \mathbb{Z}}$ is stationary under $\overline{P_\Phi}$. In view of (6.6) this result is intuitively clear (and can also be proved from it), at least in the canonical setting.

Appendix 1

Proof of Lemma 2.1: Let $g \in G$.

(a) Only the fact that $g'(0-) \leq 0 \leq g'(0)$ needs an argument. For $y < 0$ we have: $g(0) \geq 0 > y$ and hence $g'(y) \leq 0$. By letting y tend to 0 from below, we obtain that $g'(0-) \leq 0$. For $s < 0$ we have: $g(s) \leq g(0-) \leq 0$. So, $g'(0) \geq 0$.

(b) Let $t \in \mathbb{R}$ and $\epsilon > 0$. Then:

$$g'(g(t + \epsilon)) = \sup\{s \in \mathbb{R}: g(s) \leq g(t + \epsilon)\} \geq t + \epsilon > t.$$

So, $g(t + \epsilon) \notin \{y \in \mathbb{R}: g'(y) \leq t\}$, and

$$g(t + \epsilon) \geq \sup\{y \in \mathbb{R}: g'(y) \leq t\} = (g')'(t). \quad (\text{A.1})$$

By letting ϵ tend to 0, we obtain $g(t) \geq (g')'(t)$. Suppose that $g(t)$ is strictly larger than $(g')'(t)$. Then $y \in \mathbb{R}$ would exist such that $y > (g')'(t)$ and $y < g(t)$. On one hand, $g'(y)$ would be larger than t because of (A.1). On the other hand, we could choose a positive ϵ such that $y < g(t) - \epsilon$, and hence $g'(y) \leq g'(g(t) - \epsilon) \leq t$. We conclude that $g(t) = (g')'(t)$ for all $t \in \mathbb{R}$.

(c) Suppose that $(t, x) \in \Gamma(g)$, i.e., $g(t-) \leq x \leq g(t)$. Then

$$g'(x) \geq g'(g(t-)) = \sup\{s \in \mathbb{R}: g(s) \leq g(t-)\} \geq t.$$

For $\epsilon > 0$ we have: $x - \epsilon < g(t) - \frac{1}{2}\epsilon$ and $g'(x - \epsilon) \leq g'(g(t) - \frac{1}{2}\epsilon) \leq t$. Hence, $g'(x-) \leq t \leq g'(x)$ and $(x, t) \in \Gamma(g')$. The reversed implication follows from (b).

(d) Follows from (c). □

Proof of Lemma 2.3: We prove that $\mathfrak{g}^{(\eta)} \subset \mathfrak{g}^{(\theta)}$ for a family $\Theta = \{\Theta_{(t,x)}: (t,x) \in \mathbb{R}^2\}$ of transformations (on $\tilde{\Omega}$) which satisfy Assumption (i). The reversed inclusion follows by similar arguments.

Let $A \in \mathfrak{g}^{(\eta)}$, i.e.,

$$\text{for all } \omega' \in \Omega \text{ and } x \in \mathbb{R}: \omega' \in A \text{ iff } \eta_x \omega' \in A. \quad (\text{A.2})$$

We prove that $\omega \in A$ iff $\theta_s \omega \in A$, for all $\omega \in \Omega$ and $s \in \mathbb{R}$. Let $\omega \in A$ and $s \in \mathbb{R}$. For $x: = -\Lambda(s, \omega)$ we obtain by Lemma 2.2 that

$$\eta_x(\theta_s \omega) = \eta_{x + \Lambda(s, \omega)} \omega = \eta_0 \omega,$$

which belongs to A by (A.2). Again by (A.2), with $\omega' = \theta_s \omega$, we conclude that $\theta_s \omega \in A$. Let $\omega \in \Omega$ be such that $\theta_s \omega \in A$. Note that $\theta_0 \omega = \theta_{-s}(\theta_s \omega)$ belongs to A

because of the above arguments. By (A.2), with $x = -\Lambda(0, \omega)$, we obtain that $\eta_0 \omega = \eta_x(\theta_0 \omega)$ belongs to A . Again by (A.2), with $\omega' = \omega$ and $x = 0$, we conclude that $\omega \in A$. \square

Proof of Lemma 2.4: It is an easy exercise to prove that, for all $s, t \in \mathbb{R}$ with $s < t$,

$$g(s) \geq x \text{ iff } s \geq g^{-1}(x). \quad (\text{A.3})$$

Hence, the integral in the middle is equal to

$$\int_{-\infty}^{+\infty} 1_{(a,b]}(g^{-1}(x))f(g^{-1}(x))dx. \quad (\text{A.4})$$

Note that g^{-1} induces on \mathbb{R} the measure μ defined by

$$\mu((s, t]) = \text{Leb}\{x \in \mathbb{R} : s < g^{-1}(x) \leq t\}, \quad s < t.$$

Here Leb represents Lebesgue measure. Again by (A.3), it follows that $\mu = g^*$. This proves the right-hand equality. The left-hand equality follows immediately since g^{-1} and g' can only differ in countably many points. \square

Proof of Corollary 3.1: It must be shown that $P_\Lambda(\Lambda(0) \in B \mid \Lambda\{0\})$ is P_Λ -a.s. equal to

$$\frac{1}{\Lambda\{0\}} \int_0^{\Lambda\{0\}} 1_B(s) ds 1_{(\Lambda\{0\} > 0)} + 1_B(0) 1_{(\Lambda\{0\} = 0)}, \quad B \in \mathfrak{B}([0, \infty)).$$

(The second piece handles the special case when a sample path does not have a jump at 0.)

To this end, let $T_i, i \geq 1$, be the subsequent times (if any) in $(0, \infty)$ where Λ is discontinuous. Set $D_i := \Lambda(T_i -)$ and $S_i := \Lambda(T_i)$, $i \geq 1$, and note that for $y \in [D_i, S_i)$ we have: $\Lambda\{0\} \circ \eta_y = S_i - D_i$ and $\Lambda(0) \circ \eta_y = S_i - y$. For $y \in [D_1, \infty)$ but y outside the intervals $[D_i, S_i)$ we have: $\Lambda\{0\} \circ \eta_y = 0$.

Let $B, C \in \mathfrak{B}([0, \infty))$. By Theorem 3.2 we obtain on one hand, that

$$\begin{aligned} & P_\Lambda(\Lambda\{0\} \in (C \cap (0, \infty)) \text{ and } \Lambda(0) \in B) \\ &= \lim_{n \rightarrow \infty} E \left(\frac{1}{S_{n_i}} \sum_{i=1}^n \int_{D_i}^{S_i} (1_{(\Lambda\{0\} \in C)} \circ \eta_y \cdot 1_{(\Lambda(0) \in B)} \circ \eta_y) dy \right) \\ &= \lim_{n \rightarrow \infty} E \left(\frac{1}{S_{n_i}} \sum_{i=1}^n 1_{(S_i - D_i \in C)} \cdot \int_0^{S_i - D_i} 1_B(s) ds \right), \end{aligned}$$

while on the other hand,

$$E_\Lambda \left(1_{(\Lambda\{0\} \in (C \cap (0, \infty)))} \cdot \frac{1}{\Lambda\{0\}} \int_0^{\Lambda\{0\}} 1_B(s) ds \right)$$

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} E \left(\frac{1}{S_{n_i}} \sum_{i=1}^n 1_{(S_i - D_i \in C)} \int_{D_i}^{S_i} \frac{1}{S_i - D_i} \int_0^{S_i - D_i} 1_B(s) ds dy \right) \\
 &= \lim_{n \rightarrow \infty} E \left(\frac{1}{S_{n_i}} \sum_{i=1}^n 1_{(S_i - D_i \in C)} \cdot \int_0^{S_i - D_i} 1_B(s) ds \right).
 \end{aligned}$$

The corollary follows immediately. \square

Proof of Theorem 3.3: By (3.1) and the last equality in Lemma 2.2, part (a) follows immediately from (3.8). Since $P_\Lambda \Theta_{(0,0)}^{-1} = P_\Lambda$, we obtain with (a) that

$$P^0(A \cap (\Lambda\{0\} = 0)) = P_\Lambda((\Theta_{(0,0)}^{-1}A) \cap (\Lambda\{0\} = 0)) = P_\Lambda(A \cap (\Lambda\{0\} = 0)).$$

By part (a) and again by Lemma 2.2, we have

$$\begin{aligned}
 &E^0 \left(\frac{1}{\Lambda\{0\}} \int_{-\Lambda\{0\}}^0 1_A \circ \eta_x dx \cdot 1_{(\Lambda\{0\} > 0)} \right) \\
 &= E_\Lambda \left(\int_{\Lambda(0-)}^{\Lambda(0)} \frac{1}{\Lambda\{\Lambda'(y)\}} 1_A \circ \eta_y \cdot 1_{(\Lambda\{\Lambda'(y)\} > 0)} dy \right)
 \end{aligned}$$

since $\Lambda'(y) = 0$ for all $y \in (\Lambda(0-), \Lambda(0))$. By Fubini's theorem and Theorem 3.1, this last expression is equal to

$$\begin{aligned}
 &\int_{-\infty}^{+\infty} E_\Lambda \left(1_{A \cap (\Lambda\{0\} > 0)} \frac{1}{\Lambda\{0\}} 1_{(\Lambda(0-) \circ \eta_{-y} < y < \Lambda(0) \circ \eta_{-y})} \right) dy \\
 &= E_\Lambda \left(\frac{1}{\Lambda\{0\}} 1_{A \cap (\Lambda\{0\} > 0)} \cdot \text{Leb}\{y \in \mathbb{R}: \Lambda(\Lambda'(-y)-) < 0 < \Lambda(\Lambda'(-y))\} \right) \\
 &= P_\Lambda(A \cap (\Lambda\{0\} > 0)).
 \end{aligned}$$

(Here Leb represents Lebesgue measure.) Part (b) follows. \square

Proof of Theorem 4.2: Starting with $P(A)$, we use inversion of P_Λ into P as expressed in Theorem 4.1 and then we apply Theorem 3.3(b). Splitting the resulting P^0 -expectation into two parts according to whether $\Lambda\{0\} = 0$ or $\Lambda\{0\} > 0$, only the second part, i.e.,

$$E^0 \left(\frac{1}{\Lambda\{0\}} \frac{1}{\Lambda'} \int_{-\Lambda\{0\}}^{0-} \int_0^{\Lambda'(1+x)} 1_A \circ \theta_t dt dx \cdot 1_{(\Lambda\{0\} > 0)} \right),$$

needs some arguments. Since $\Lambda'(1+x) = 0$ for all x with $-\Lambda\{0\} < x < -1$ (if any), we can restrict the outer integral. Concerning the inner integral, note that there are

at most countably many t where $\Lambda(\cdot, \omega)$ is discontinuous; we omit them. Note also that the remaining t satisfy $\Lambda'(1+x) \geq t$ iff $\Lambda(t) \leq 1+x$. Applying Fubini's theorem to the resulting expression, we obtain:

$$\begin{aligned}
 E^0 & \left(\frac{1}{\Lambda\{0\}} \frac{1}{\Lambda'} \int_0^{\Lambda'(1-)} 1_A \circ \theta_t \int_{-(\Lambda\{0\} \wedge 1)}^{0-} 1_{[\Lambda(t)-1, \infty)}(x) dx dt \cdot 1_{(\Lambda\{0\} > 0)} \right) \\
 & = E^0 \left(\frac{1}{\Lambda\{0\}} \frac{1}{\Lambda'} \int_0^{\Lambda'(1-)} (\Lambda\{0\} \wedge (1 - \Lambda(t))) \cdot 1_A \circ \theta_t dt \cdot 1_{(\Lambda\{0\} > 0)} \right).
 \end{aligned}$$

The theorem follows by noting that for all $t \in (\Lambda'(1-, \omega), \Lambda'(1, \omega))$, we have: $1 - \Lambda(t, \omega) = 0$. \square

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