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BSDE ASSOCIATED WITH LÉVY PROCESSES AND APPLICATION TO PDIE

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We deal with backward stochastic differential equations (BSDE for short) driven by Teugel's martingales and an independent Brownian motion. We study the existence, uniqueness and comparison of solutions for these equations under a Lipschitz as well as a locally Lipschitz conditions on the coefficient. In the locally Lipschitz case, we prove that if the Lipschitz constant L_N behaves as $\sqrt{\log(N)}$ in the ball $B(0, N)$, then the corresponding BSDE has a unique solution which depends continuously on the on the coefficient and the terminal data. This is done with an unbounded terminal data. As application, we give a probabilistic interpretation for a large class of partial differential integral equations (PDIE for short).

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1 Introduction

Since the paper [8] of Pardoux and Peng, several works have been devoted to the study of BSDEs as well as to their applications. This is due to the connections of BSDEs with stochastic optimal control and stochastic games (Hamadène and Lepeltier [3]) as well as to mathematical finance (El Karoui *et al.* [4]). Backward stochastic differential equations also appear as a powerful tool in partial differential equations where they provide probabilistic formulas for their solutions (Peng [10], Pardoux and Peng [9]).
A solution of a classical BSDE is a pair of adapted processes (Y, Z) satisfying:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s. \tag{1.1}$$

When the coefficient f is uniformly Lipschitz, the BSDE (1.1) has a unique solution. The proof is mainly based on the Itô martingale representation theorem.

In Nualart and Schoutens [6], a martingale representation theorem associated to Lévy processes was proved. It then is natural to extend equations (1.1) to BSDE's driven by a Lévy process (Nualart and Schoutens [7]). In their paper [7], the authors proved the existence and uniqueness of solutions, under Lipschitz conditions on the coefficient.

In this paper, we deal with BSDE driven by both a standard Brownian motion and an independent Lévy process and having a Lipschitz, or more generally, a locally Lipschitz coefficient. In the locally Lipschitz case, we prove that if the Lipschitz constant L_N behaves as $\sqrt{\log(N)}$ in the ball $B(0, N)$, then the corresponding BSDE has a unique solution. We don't impose any boundedness condition on the terminal data. It will be assumed square integrable only. Moreover, a comparison theorem as well as a stability of solutions are established in this setting. Our results extend in particular those of ([1], [2]) to BSDE driven by a Lévy process. As an application, we give a probabilistic interpretation for a large class of partial differential integral equations.

The paper is organized as follows. In Section 2, we introduce some notations and assumptions. Section 3 is devoted to the proof of existence, uniqueness and comparison results for BSDE driven by a Lévy process, under Lipschitz conditions. Those equations are also discussed under locally Lipschitz conditions in Section 4. In Section 5, we include an application to PDIE.

2 Preliminaries and Notations

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, W_t, L_t : t \in [0, T])$ be a complete Wiener–Lévy space in $\mathbb{R} \times \mathbb{R}\backslash\{0\}$, with Lévy measure ν, i.e. $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, $\{\mathcal{F}_t : t \in [0, T]\}$ is a right–continuous increasing family of complete sub σ–algebras of $\mathcal{F}$, $\{W_t : t \in [0, T]\}$ is a standard Wiener process in $\mathbb{R}$ with respect to $\{\mathcal{F}_t : t \in [0, T]\}$ and $\{L_t : t \in [0, T]\}$ is a $\mathbb{R}$–valued Lévy process of the form $L_t = bt + \ell_t$ independent of $\{W_t : t \in [0, T]\}$, corresponding to a standard Lévy measure ν satisfying the following conditions :
i) $\int_{\mathbb{R}}(1 \wedge y^2)\nu(dy) < \infty$,
ii) $\int_{]-\varepsilon,\varepsilon[^c} e^{\lambda|y|}\nu(dy) < \infty$, for every $\varepsilon > 0$ and for some $\lambda > 0$.

We assume that

$$\mathcal{F}_t = \sigma(L_s, s \leq t) \vee \sigma(W_s, s \leq t) \vee \mathcal{N}$$

where $\mathcal{N}$ denotes the totality of $\mathbb{P}$–null sets and $\mathcal{G}_1 \vee \mathcal{G}_2$ denotes the σ–field generated by $\mathcal{G}_1 \cup \mathcal{G}_2$.
Let $\mathcal{H}^2$ denote the space of real valued, square integrable and $\mathcal{F}_t$–progressively measurable processes $\phi = \{\phi_t : t \in [0,T]\}$ such that

$$\|\phi\|^2 = \mathbb{E}\int_0^T |\phi_t|^2 dt \ < \ \infty.$$

and denote by $\mathcal{P}^2$ the subspace of $\mathcal{H}^2$ formed by the predictable processes.
Let l^2 be the space of real valued sequences $(x_n)_{n\geq 0}$ such that $\sum_{i=0}^{\infty} x_i^2$ is finite. We shall denote by $\mathcal{H}^2(l^2)$ and $\mathcal{P}^2(l^2)$ the corresponding spaces of l^2–valued processes equipped with the norm

$$\|\phi\|^2 = \sum_{i=0}^{\infty}\mathbb{E}\int_0^T |\phi_t^{(i)}|^2 dt.$$

Let us define:
(**A**.**1**) a terminal value $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$.
(**A**.**2**) a process f, which is a map $f : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times l^2 \longrightarrow \mathbb{R}$, such that

(i) f is progressively measurable also $f(.,0,0,0) \in \mathcal{H}^2$.

(ii) There exists $L > 0$ such that

$$|f(t,\omega,y,u,z) - f(t,\omega,y',u',z')| \leq L(|y-y'| + |u-u'| + \|z-z'\|).$$

We recall the Itô formula for càdlàg semimartingales.

2.1 Itô's formula

Let $X = \{X_t : t \in [0,T]\}$ be a càdlàg semimartingale, with quadratic variation denoted by $[X] = \{[X]_t : t \in [0,T]\}$ and let F be a $\mathcal{C}^2$ real valued function. Then $F(X)$ is also a semimartingale and the following formula holds:

$$\begin{aligned} F(X_t) &= F(X_0) + \int_0^t F'(X_{s-})dX_s + \frac{1}{2}\int_0^T F''(X_s)d[X]_s^c \qquad (2.1)\\ &\quad + \sum_{0<s\leq t}\{F(X_s) - F(X_{s-}) - F'(X_{s-})\Delta X_s\}. \end{aligned}$$

where $[X]^c$ (sometimes denoted by $\langle X\rangle$) is the continuous part of the quadratic variation $[X]$. We also note that in the case where $F(x) = x^2$, the formula (2.1) takes the form

$$X_t^2 = X_0^2 + \int_0^t 2X_{s-}dX_s + \int_0^t d[X]_s. \qquad (2.2)$$

Moreover if X and Y are two càdlàg semimartingales then we have

$$X_tY_t = X_0Y_0 + \int_0^t X_{s-}dY_s + \int_0^t Y_{s-}dX_s + \int_0^t d[X,Y]_s. \qquad (2.3)$$

where $[X,Y]$ stands for the quadratic covariation of X , Y also called the bracket process. For a complete survey in this topic we refer to Protter [11].

2.2 Predictable representation

We denote by $(H^{(i)})_{i\geq 1}$ the Teugel's Martingales associated with the Lévy process $\{L_t : t \in [0,T]\}$. More precisely

$$H_t^{(i)} = c_{i,i}Y_t^{(i)} + c_{i,i-1}Y_t^{(i-1)} + \ldots + c_{i,1}Y_t^{(1)},$$

where $Y_t^{(i)} = L_t^{(i)} - \mathbb{E}[L_t^{(i)}] = L_t^{(i)} - t\mathbb{E}[L_1^{(i)}]$ for all $i \geq 1$ and $L_t^{(i)}$ are power–jump processes. That is, $L_t^{(1)} = L_t$ and $L_t^{(i)} = \sum_{0<s\leq t}(\Delta L_t)^i$ for $i \geq 2$. It was shown in Nualart and Schoutens [6] that the coefficients $c_{i,k}$ correspond to the orthonormalization of the polynomials $1, x, x^2, \ldots$ with respect to the measure $\mu(dx) = x^2\nu(dx) + \sigma^2\delta_0(dx)$:

$$q_{i-1} = c_{i,i}x^{i-1} + c_{i,i-1}x^{i-2} + \ldots + c_{i,1}.$$

We set

$$p_i(x) = xq_{i-1}(x) = c_{i,i}x^i + c_{i,i-1}x^{i-1} + \ldots + c_{i,1}x.$$

The martingales $(H^{(i)})_{i\geq 1}$ can be chosen to be pairwise strongly orthonormal martingales. More details, in this subject, can be found in Nualart and Schoutens [6].

The main tool in the theory of BSDEs is the martingale representation theorem, which is well known for martingales which are adapted to the filtration of the Brownian motion or that of Poisson point process (e.g Situ [13]) or that of a Poisson random measure (e.g Ouknine [12]). A more general and interesting martingale representation theorem (proven by different ways) appeared recently in Løkka [5] and in Nualart and Schoutens [7].

Proposition 2.1: *Let $\{M_t : t \in [0,T]\}$ be a square integrable martingale which is adapted to the filtration $\mathcal{F}_t$ defined above. Then, there exist $U \in \mathcal{P}^2$ and $Z \in \mathcal{P}^2(l^2)$ such that*

$$M_t = \mathbb{E}[M_t] + \int_0^t U_s dW_s + \sum_{i=1}^{\infty}\int_0^t Z_s^{(i)} dH_s^{(i)}.$$

Proof. The Proof follows by combining the result of Løkka [5] (Theorem 5) and that of Nualart and Schoutens [6]. ■

We denote by $\mathcal{E}$ the set of $\mathbb{R}\times\mathbb{R}\times l^2$–valued processes (Y,U,Z) defined on $\mathbb{R}_+\times\Omega$ which are $\mathcal{F}_t$–adapted and such that:

$$\|(Y,U,Z)\|^2 = \mathbb{E}\left(\sup_{0\leq t\leq T}|Y_t|^2 + \int_0^T |U_s|^2\, ds + \int_0^T \|Z_s\|^2 ds\right) < +\infty.$$

The couple $(\mathcal{E}, \|.\|)$ is then a Banach space.

We now introduce our BSDE. Given a data (f,ξ) we want to solve the following stochastic integral equation, which we denote by Equation (f,ξ):

$$Y_t = \xi + \int_t^T f(s, Y_{s-}, U_s, Z_s)ds - \int_t^T U_s dW_s - \sum_{i=1}^{\infty}\int_t^T Z_s^{(i)} dH_s^{(i)}.$$

Definition 2.2: A solution of equation $Eq(f,\xi)$ is a triple (Y,U,Z) which belongs to the space $(\mathcal{E}, \|.\|)$ and satisfies $Eq(f,\xi)$.

3 BSDE Driven by Lévy Processes

3.1 Existence and uniqueness of solutions

Theorem 3.1: *Let the assumptions* (**A.1**), (**A.2**) *hold. Assume moreover that* ξ *is a square integrable random variable which is* $\mathcal{F}_T$*–measurable. Then* $Eq(f,\xi)$ *has a unique solution.*

Proof: Uniqueness. Let (Y,U,Z) and $(\widetilde{Y},\widetilde{U},\widetilde{Z})$ be two solutions of equation $Eq(f,\xi)$. By Itô's formula 2.2, we have

$$\begin{aligned}&\mathbb{E}|Y_t-\widetilde{Y}_t|^2+\mathbb{E}\int_t^T|U_s-\widetilde{U}_s|^2ds+\mathbb{E}\int_t^T\|Z_s-\widetilde{Z}_s\|^2ds\\ =\;& 2\mathbb{E}\int_t^T\left(Y_{s-}-\widetilde{Y}_{s-}\right)\left[f(s,Y_{s-},U_s,Z_s)-f(s,\widetilde{Y}_{s-},\widetilde{U}_s,\widetilde{Z}_s)\right]ds,\end{aligned}$$

Since f is L–Lipschitz, we get

$$\begin{aligned}\mathbb{E}|Y_t-\widetilde{Y}_t|^2 \quad &+ \quad \left(1-\frac{2L}{\beta^2}\right)\mathbb{E}\int_t^T|U_s-\widetilde{U}_s|^2ds+\left(1-\frac{2L}{\beta^2}\right)\mathbb{E}\int_t^T\|Z_s-\widetilde{Z}_s\|^2ds\\ &\leq \quad L(\beta^2+2)\mathbb{E}\int_t^T|Y_{s-}-\widetilde{Y}_{s-}|^2ds,\end{aligned}$$

where we have used the inequality $2xy\leq\beta^2x^2+\frac{y^2}{\beta^2}$. If we choose $\frac{2L}{\beta^2}=\frac{1}{2}$, we obtain

$$\mathbb{E}|Y_t-\widetilde{Y}_t|^2+\mathbb{E}\int_t^T|U_s-\widetilde{U}_s|^2ds+\mathbb{E}\int_t^T\|Z_s-\widetilde{Z}_s\|^2ds\leq C\mathbb{E}\int_t^T|Y_s-\widetilde{Y}_s|^2ds.$$

Uniqueness now follows from Gronwall's lemma.

Existence. Using the martingale representation theorem (Proposition 2.1), one can prove that the following BSDE

$$Y_t=\xi+\int_t^Tf(s,0,0,0)ds-\int_t^TU_sdW_s-\int_t^T\langle Z_s,dH_s\rangle,$$

has a solution.

Now, define (Y^n,U^n,Z^n) as follows:
$Y^0=Z^0=U^0=0$ and $(Y^{n+1},U^{n+1},Z^{n+1})$ is the unique solution to the BSDE

$$Y_t^{n+1}=\xi+\int_t^Tf(s,Y_{s-}^n,U_s^n,Z_s^n)ds-\int_t^TU_s^{n+1}dW_s-\int_t^T\langle Z_s^{n+1},dH_s\rangle,$$

We shall prove that (Y^n,U^n,Z^n) is a Cauchy sequence in the Banach space $\mathcal{E}$.
To simplify the notations, put :

$$\overline{Y}_s^{n,m}:=Y_s^n-Y_s^m,\;\;\overline{U}_s^{n,m}:=U_s^n-U_s^m\quad\text{and}\quad\overline{Z}_s^{n,m}:=Z_s^n-Z_s^m$$

and

$$\overline{f}_s^{n,m}:=f(s,Y_{s-}^n,U_s^n,Z_s^n)-f(s,Y_{s-}^m,U_s^m,Z_s^m).$$

Itô's formula (2.2), shows that for every $n < m$

$$\begin{aligned} e^{\alpha t}|\overline{Y}_t^{n+1,m+1}|^2 \quad & + \int_t^T e^{\alpha s}|\overline{U}_s^{n+1,m+1}|^2 ds \\ & + \int_t^T e^{\alpha s}\|\overline{Z}_s^{n+1,m+1}\|^2 ds + \alpha \int_t^T e^{\alpha s}|\overline{Y}_{s-}^{n+1,m+1}|^2 ds \\ & = 2\int_t^T e^{\alpha s}\overline{Y}_{s-}^{n+1,m+1}\overline{f}_s^{n,m} ds - 2\int_t^T e^{\alpha s}\overline{Y}_{s-}^{n+1,m+1}\overline{U}_s^{n,m} dW_s \\ & -2\int_t^T e^{\alpha s}\overline{Y}_{s-}^{n+1,m+1}\left\langle \overline{Z}_s^{n,m}, dH_s \right\rangle - (N_T - N_t), \end{aligned}$$

where $\{N_t : t \in [0,T]\}$ is a martingale given by

$$N_t = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_0^t e^{\alpha s}\overline{Z}_s^{n+1,m+1,(i)}\overline{Z}_s^{n+1,m+1,(j)}(d[H^{(i)}, H^{(j)}]_s - d\left\langle H^{(i)}, H^{(j)}\right\rangle_s).$$

Taking the expectation and using the fact that $\langle H^{(i)}, H^{(j)}\rangle = \delta_{i,j}t$, we get

$$\begin{aligned} \mathbb{E}\, e^{\alpha t}|\overline{Y}_t^{n+1,m+1}|^2 \quad & +\mathbb{E}\int_t^T e^{\alpha s}|\overline{U}_s^{n+1,m+1}|^2 ds \\ & +\mathbb{E}\int_t^T e^{\alpha s}\|\overline{Z}_s^{n+1,m+1}\|^2 ds + \alpha\mathbb{E}\int_t^T e^{\alpha s}|\overline{Y}_{s-}^{n+1,m+1}|^2 ds \\ & = 2\mathbb{E}\int_t^T e^{\alpha s}\overline{Y}_{s-}^{n+1,m+1}\overline{f}_s^{n,m} ds \end{aligned}$$

Since f is L-Lipschitz, we get

$$\begin{aligned} & e^{\alpha t}\mathbb{E}|\overline{Y}_t^{n+1,m+1}|^2 + \int_t^T e^{\alpha s}\mathbb{E}|\overline{U}_s^{n+1,m+1}|^2 ds \\ & + \int_t^T e^{\alpha s}\mathbb{E}\|\overline{Z}_s^{n+1,m+1}\|^2 ds + \alpha\int_t^T e^{\alpha s}\mathbb{E}|\overline{Y}_{s-}^{n+1,m+1}|^2 ds \\ & \le 2L\mathbb{E}\int_t^T e^{\alpha s}|\overline{Y}_{s-}^{n+1,m+1}|\left[|\overline{Y}_{s-}^{n,m}| + |\overline{U}_s^{n,m}| + \|\overline{Z}_s^{n,m}\|\right] ds, \end{aligned}$$

and then

$$\begin{aligned} e^{\alpha t}\mathbb{E}|\overline{Y}_t^{n+1,m+1}|^2 \quad & + \int_t^T e^{\alpha s}\mathbb{E}|\overline{U}_s^{n+1,m+1}|^2 ds + \int_t^T e^{\alpha s}\mathbb{E}\|\overline{Z}_s^{n+1,m+1}\|^2 ds \\ & +(\alpha - L^2\beta^2)\int_t^T e^{\alpha s}\mathbb{E}|\overline{Y}_{s-}^{n+1,m+1}|^2 ds \\ & \le \frac{3}{\beta^2}\mathbb{E}\int_t^T e^{\alpha s}\left(|\overline{Y}_{s-}^{n,m}|^2 + |\overline{U}_s^{n,m}|^2 + \|\overline{Z}_s^{n,m}\|^2\right) ds. \end{aligned}$$

Choosing β and α such that $\frac{3}{\beta^2}=\frac{1}{2}$ and $\alpha-6L^2=1$, we get

$$e^{\alpha t}\mathbb{E}|\overline{Y}_t^{n+1,m+1}|^2 \quad + \int_t^T e^{\alpha s}\mathbb{E}|\overline{U}_s^{n+1,m+1}|^2 ds + \int_t^T e^{\alpha s}\mathbb{E}\|\overline{Z}_s^{n+1,m+1}\|^2 ds$$
$$\leq \frac{1}{2}\mathbb{E}\int_t^T e^{\alpha s}\left(|\overline{Y}_{s-}^{n,m}|^2+|\overline{U}_s^{n,m}|^2+\|\overline{Z}_s^{n,m}\|^2\right)ds$$

It follows immediately, for all $m>n$, that

$$\mathbb{E}\int_0^T e^{\alpha s}|\overline{Y}_{s-}^{n,m}|^2 ds+\mathbb{E}\int_0^T e^{\alpha s}|\overline{U}_s^{n,m}|^2 ds+\mathbb{E}\int_0^T e^{\alpha s}\|\overline{Z}_s^{n,m}\|^2 ds \leq \frac{C}{2^n}.$$

Using again Itô's formula and Doob's inequality, it follows that there exists a universal constant C such that

$$\mathbb{E}\sup_{0\leq s\leq T}|\overline{Y}_s^{n,m}|^2+\mathbb{E}\int_0^T e^{\alpha s}|\overline{U}_s^{n,m}|^2 ds+\mathbb{E}\int_0^T e^{\alpha s}\|\overline{Z}_s^{n,m}\|^2 ds \leq \frac{C}{2^n}.$$

Consequently, (Y^n,U^n,Z^n) is a Cauchy sequence in the Banach space $\mathcal{E}$. It is not difficult to show that

$$(Y,U,Z)=\lim_{n\to\infty}(Y^n,U^n,Z^n),$$

solves our BSDE. ■

The following theorem gives a bound for the difference between two solutions of $Eq(f,\xi)$. It can be proved by using Itô's formula, the Lipschitz property of f and Gronwall's lemma.

Theorem 3.2: *Given standard data (f,ξ) and $(\widetilde{f},\widetilde{\xi})$, let (Y,U,Z) and $(\widetilde{Y},\widetilde{U},\widetilde{Z})$, be the unique solution the equation $Eq(f,\xi)$ and $Eq(\widetilde{f},\widetilde{\xi})$ respectively. Then*

$$\mathbb{E}\int_0^T\left(|\widetilde{Y}_{s-}-Y_{s-}|^2+|\widetilde{U}_s-U_s|^2+\|\widetilde{Z}_s-Z_s\|^2\right)ds$$
$$\leq C\left(\mathbb{E}|\widetilde{\xi}-\xi|^2+\mathbb{E}\int_0^T|\widetilde{f}(s,Y_{s-},U_s,Z_s)-f(s,Y_{s-},U_s,Z_s)|^2 ds\right).$$

3.2 Comparison theorem

In this subsection, we prove a comparison theorem for BSDE driven by Lévy process. This is an important tool in the probabilistic interpretation of viscosity solutions of partial differential equations.

Theorem 3.3: *Given standard data (f_1,ξ_1) and (f_2,ξ_2), suppose that $\xi_1\leq\xi_2$ and $f_1(t,y,u,z)\leq f_2(t,y,u,z)$ for all $(y,u,z)\in\mathbb{R}\times\mathbb{R}\times l^2$, $d\mathbb{P}\times dt$–a.s. Then $Y_t^{f_1}\leq Y_t^{f_2}$, $t\in[0,T]$.*

Proof: Set

$$\overline{Y}_s:=Y_s^{f_2}-Y_s^{f_1},\ \overline{U}_s:=U_s^{f_2}-U_s^{f_1},\ \overline{Z}_s:=Z_s^{f_2}-Z_s^{f_1},\ \overline{\xi}:=\xi_2-\xi_1,$$

and

$$\overline{f}_s := f_2(s, Y^{f_2}_{s-}, U^{f_2}_s, Z^{f_2}_s) - f_1(s, Y^{f_2}_{s-}, U^{f_2}_s, Z^{f_2}_s).$$

We define three stochastic processes as follows

$$\alpha_s = \begin{cases} \overline{Y}^{-1}_{s-}\left(f_1(s, Y^{f_2}_{s-}, U^{f_2}_s, Z^{f_2}_s) - f_1(s, Y^{f_1}_{s-}, U^{f_2}_s, Z^{f_2}_s)\right) & \text{if } \overline{Y}_{s-} \neq 0 \\ 0 & \text{if } \overline{Y}_{s-} = 0, \end{cases}$$

$$\beta_s = \begin{cases} \overline{U}^{-1}_{s}\left(f_1(s, Y^{f_1}_{s-}, U^{f_2}_s, Z^{f_2}_s) - f_1(s, Y^{f_1}_{s-}, U^{f_1}_s, Z^{f_2}_s)\right) & \text{if } \overline{U}_{s} \neq 0 \\ 0 & \text{if } \overline{U}_{s} = 0, \end{cases}$$

and for all $i \in \mathbb{N}^*$ let $\widetilde{Z}^{(i)}$ denote the l^2–valued stochastic process such that its i first components are equal to those of Z^{f_2} and its $\mathbb{N}^* \setminus \{1, 2, \ldots, i\}$ last components are equal to those of Z^{f_1}. With this notation, we define for $i \in \mathbb{N}^*$

$$\gamma^{(i)}_s = \begin{cases} (\overline{Z}^{(i)}_s)^{-1}\left(f_1(s, Y^{f_1}_{s-}, U^{f_1}_s, \widetilde{Z}^{(i)}_s) - f_1(s, Y^{f_1}_{s-}, U^{f_1}_s, \widetilde{Z}^{(i-1)}_s)\right) & \text{if } \overline{Z}^{(i)}_s \neq 0 \\ 0 & \text{if } \overline{Z}^{(i)}_s = 0. \end{cases}$$

It is clear that

$$\left\langle \gamma_s, \overline{Z}_s \right\rangle = \left(f_1(s, Y^{f_1}_{s-}, U^{f_1}_s, Z^{f_2}_s) - f_1(s, Y^{f_1}_{s-}, U^{f_1}_s, Z^{f_1}_s)\right),$$

and the processes $\{\alpha_t : t \in [0, T]\}$, $\{\beta_t : t \in [0, T]\}$ and $\{\gamma_t : t \in [0, T]\}$ are progressively measurable and bounded.
For $0 \leq s \leq t \leq T$, let

$$M^{H,W}_t := \int_0^t \beta_s dW_s + \int_0^t \langle \gamma_s, dH_s \rangle$$

$$\Gamma_{s,t} := \exp\left[\int_s^t \left(\alpha_r dr - d[M^{H,W}_{\cdot}]^c_r + dM^{H,W}_r\right)\right] \prod_{s<r\leq t}\left(1 + \Delta M^{H,W}_r\right)\exp\left(-\Delta M^{H,W}_r\right).$$

Using Itô's formula (2.1) one can see that $\{\Gamma_{s,r} : r \in [s, T]\}$ satisfies the stochastic linear equation

$$\Gamma_{s,t} = 1 + \int_s^t \Gamma_{s,r-} dM^{H,W}_r + \int_s^t \Gamma_{s,r-}\alpha_r dr. \tag{3.1}$$

Since

$$\begin{aligned} \overline{Y}_t &= \overline{\xi} + \int_t^T \left(\alpha_r \overline{Y}_{r-} + \beta_r \overline{U}_r + \left\langle \gamma_r, \overline{Z}_r \right\rangle\right) dr \\ &\quad + \int_t^T \overline{f}_r dr - \int_t^T \overline{U}_r dW_s - \int_t^T \left\langle \overline{Z}_r, dH_r \right\rangle, \end{aligned}$$

we use formula (2.3) and relation (3.1) to show that for all $0 \leq s \leq t \leq T$

$$\begin{aligned} \overline{Y}_s &= \Gamma_{s,t}\overline{Y}_t + \int_s^t \Gamma_{s,r-}\overline{f}_r dr \\ &\quad - \int_s^t \Gamma_{s,r-}\left(\overline{U}_r + \beta_r \overline{Y}_{r-}\right) dW_s - \int_s^t \Gamma_{s,r-}\left\langle \gamma_r \overline{Y}_{r-} + \overline{Z}_r, dH_r \right\rangle \\ &\quad + \sum_{i=1}^{\infty} \int_s^t \Gamma_{s,r-}\gamma^{(i)}_r \overline{Z}^{(i)}_r (d[H^{(i)}]_r - d < H^{(i)} >_r). \end{aligned}$$

Since the last three terms in the right–hand of the above equation are martingales, we deduce that

$$\overline{Y}_s = \mathbb{E}\left(\Gamma_{s,t}\overline{Y}_t + \int_s^t \Gamma_{s,r-}\overline{f}_r dr \ / \ \mathcal{F}_s\right).$$

Hence, the result follows, for $t = T$, by the positivity of $\overline{\xi}$ and $\overline{f}$. ■

4 BSDE with Locally Lipschitz Coefficient

The aim of this section is to prove the existence and uniqueness of solutions for BSDE with locally Lipschitz generator. More precisely, we assume that the following conditions hold:

H.1) f is continuous in (y, u, z) for almost all (t, ω),

H.2) there exists $K > 0$ and $0 \leq \alpha < 1$ such that $|f(t, \omega, u, y, z)| \leq K(1 + |y|^\alpha + |u|^\alpha + \|z\|^\alpha)$.

H.3) for every $N \in \mathbb{N}$, there exists a constant $L_N > 0$ such that
$|f(t, \omega, y, u, z) - f(t, \omega, y', u', z')| \leq L_N(|y - y'| + |u - u'| + \|z - z'\|)$, $\mathbb{P}$–*a.s.*, *a.e.* $t \in [0, T]$
and $\forall$ y, y', u, u', z, z' such that $|y| \leq N$, $|y'| \leq N$, $|u| \leq N$, $|u'| \leq N$, $\|z\| \leq N$, $\|z'\| \leq N$.

When the assumptions H.1) and H.2) are satisfied, we can define the family of semi–norms $(\rho_n(f))_n$

$$\rho_n(f) = \left(\mathbb{E}\int_0^T \sup_{|y|,|u|,\|z\| \leq n} |f(s, y, u, z)|^2 ds\right)^{\frac{1}{2}}.$$

We denote by Lip_{loc} (resp. Lip) the set of processes f satisfying H.1)–H.2) which are locally Lipschitz, i.e. satisfy the assumption H.3), (resp. globally Lipschitz) with respect to (y, u, z).
$Lip_{loc,\alpha}$ denotes the subset of those processes f which belong to Lip_{loc} and which satisfy H.2).
The main results are the following

Theorem 4.1: (Existence and uniqueness). *Let $f \in Lip_{loc,\alpha}$ and ξ be a square integrable random variable. Then equation $Eq(f, \xi)$ has a unique solution if $L_N \leq L + \sqrt{\log(N)}$, where L is some positive constant.*

We give now a stability result for the solution with respect to the data (f, ξ). Roughly speaking, if f_n converges to f in the metric defined by the family of semi–norms (ρ_N) and ξ_n converges to ξ in $L^2(\Omega)$ then (Y^n, U^n, Z^n) converges to (Y, U, Z) in $\mathcal{E}$. Let (f_n) be a sequence of functions which are $\mathcal{F}_t$–progressively measurable for each n. Let $(\xi_n)_{n\geq 1}$ be a sequence of random variables which are $\mathcal{F}_T$–measurable for each n and such that $\mathbb{E}|\xi_n|^2 < \infty$. We will assume that for each n, the BSDE $Eq(f_n, \xi_n)$ corresponding to the data (f_n, ξ_n) has a (not necessarily unique) solution. Each solution of the equation $Eq(f_n, \xi_n)$ will be denoted by (Y^{f_n}, Z^{f_n}).

We suppose also that the following assumptions H.4), H.5) and H.6) are fulfilled,

H.4) *For every N, $\rho_N(f_n - f) \longrightarrow 0$ as $n \to \infty$.*

H.5) $\mathbb{E}|\xi_n - \xi|^2 \longrightarrow 0$ as $n \to \infty$.

H.6) There exist $K > 0$ such that,

$$\sup_n |f_n(t,\omega,y,u,z)| \leq K(1 + |y|^\alpha + |u|^\alpha + \|z\|^\alpha) \quad \mathbb{P}\text{–}a.s., \ a.e. \ t \in [0,T].$$

Theorem 4.2:(Stability). *Let f and ξ be as in Theorem 4.1. Assume that (f_n, ξ_n) satisfies H.4), H.5) and H.6). Then we have*

$$\lim_{n\to+\infty} \left(\mathbb{E} \sup_{0\leq t\leq T} |Y_t^{f_n} - Y_t|^2 + \mathbb{E}\int_0^T |U_s^{f_n} - U_s|^2 ds + \mathbb{E}\int_0^T \|Z_s^{f_n} - Z_s\|^2 ds \right) = 0.$$

To prove Theorems 4.1 and 4.2 we need the two following lemmas.

Lemma 4.3: *Let ξ^1, ξ^2 be two d–dimensional square integrable random variables which are $\mathcal{F}_T$–measurable. Let f_1 and f_2 be two functions which satisfy H.1), H.2). Let $(Y^{f_1}, U^{f_1}, Z^{f_1})$ [resp. $(Y^{f_2}, U^{f_2}, Z^{f_2})$] be a solution of the BSDE $Eq(f_1, \xi^1)$ [resp. $Eq(f_2, \xi^2)$]. Then for every locally Lipschitz function f and every $N > 1$, the following estimates hold*

$$\begin{aligned} &\mathbb{E}\int_0^T |U_s^{f_1} - U_s^{f_2}|^2 ds + \mathbb{E}\int_0^T \|Z_s^{f_1} - Z_s^{f_2}\|^2 ds \\ \leq \quad & C(K,\xi^1,\xi^2)\left\{ \mathbb{E}(|\xi^1 - \xi^2|^2) + \left[\mathbb{E}\int_0^T |Y_s^{f_1} - Y_s^{f_2}|^2 ds\right]^{\frac{1}{2}} \right\} \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}(|Y_s^{f_1} - Y_s^{f_2}|^2) \quad \leq \quad & C_1 \big[\mathbb{E}(|\xi^1 - \xi^2|^2) + \rho_N^2(f_1 - f) + \rho_N^2(f - f_2) \\ & + \frac{C(K,\xi^1,\xi^2)}{(2L_N + 2L_N^2)N^{2(1-\alpha)}}\big] \exp\big((2L_N + 2L_N^2)(T-s)\big), \end{aligned}$$

where $C(K,\xi^1,\xi^2)$ is a constant which depends on K, $\mathbb{E}|\xi^1|^2$ and $\mathbb{E}|\xi^2|^2$, and C_1 is a universal constant.

Proof: The first inequality follows from Itô's formula and Schwarz inequality. We shall prove the second one. Let $<,>$ denote the inner product in $\mathbb{R}^d$.

We set

$$\overline{Y}_s := Y_s^{f_1} - Y_s^{f_2}, \ \ \overline{U}_s := U_s^{f_1} - U_s^{f_2} \quad \text{and} \quad \overline{Z}_s := Z_s^{f_1} - Z_s^{f_2},$$

and

$$\overline{f}_s := f_1(s, Y_{s-}^{f_1}, U_s^{f_1}, Z_s^{f_1}) - f_2(s, Y_{s-}^{f_2}, U_s^{f_2}, Z_s^{f_2}).$$

By Itô's formula we have

$$\begin{aligned} |\overline{Y}_t|^2 \quad & + \int_t^T |\overline{U}_s|^2 ds + \int_t^T \|\overline{Z}_s\|^2 ds = |\xi^1 - \xi^2|^2 + 2\int_t^T \overline{Y}_{s-}\overline{f}_s ds - 2\int_t^T \overline{Y}_{s-}\overline{U}_s dW_s \\ & -2\sum_{i=1}^\infty \int_t^T \overline{Y}_{s-}\overline{Z}_s^{(i)} dH_s^{(i)} - \sum_{i=1}^\infty \sum_{j=1}^\infty \int_t^T \overline{Z}_s^{(i)}\overline{Z}_s^{(j)} d\left([H^{(i)}, H^{(j)}]_s - < H^{(i)}, H^{(j)} >_s\right). \end{aligned}$$

Using the fact that $\int_0^t \overline{Z}_s^{(i)}\overline{Z}_s^{(j)} d\left([H^{(i)},H^{(j)}]_s - < H^{(i)},H^{(j)} >_s\right)$ is a martingale and taking the expectation we get

$$\mathbb{E}\left|\overline{Y}_t\right|^2 + \mathbb{E}\int_t^T \left|\overline{U}_s\right|^2 ds + \mathbb{E}\int_t^T \|\overline{Z}_s\| ds = \mathbb{E}|\xi^1-\xi^2|^2 + 2\mathbb{E}\int_t^T \langle \overline{Y}_{s-}, \overline{f}_s\rangle ds.$$

Let β and γ be strictly positive numbers. For a given $N>1$, let L_N be the Lipschitz constant of f in the ball $B(0,N)$,

$$A^N := \left\{(s,\omega); \quad |Y_{s-}^{f_1}|^2 + |U_s^{f_2}|^2 + \|Z_s^{f_1}\|^2 + |U_s^{f_1}|^2 + |Y_{s-}^{f_2}|^2 + \|Z_s^{f_2}\|^2 \geq N^2\right\},$$

$A^{N,c} := \Omega \setminus A^N$ and denote by $\mathbb{1}_A$ the indicator function of the set A. We have

$$\begin{aligned}\mathbb{E}|\overline{Y}_t|^2 + \mathbb{E}\int_t^T |\overline{U}_s|^2 ds + \mathbb{E}\int_t^T \|\overline{Z}_s\|^2 ds &= \mathbb{E}|\xi^1-\xi^2|^2 + 2\mathbb{E}\int_t^T \langle \overline{Y}_{s-}, \overline{f}_s\rangle(\mathbb{1}_{A^N} + \mathbb{1}_{A^{N,c}})ds \\ &:= \mathbb{E}|\xi^1-\xi^2|^2 + I_1 + I_2 + I_3 + I_4,\end{aligned}$$

where

$$\begin{aligned}
I_1 &:= 2\mathbb{E}\int_t^T \langle \overline{Y}_{s-}, \overline{f}_s\rangle \mathbb{1}_{A^N} ds \\
I_2 &:= 2\mathbb{E}\int_t^T \langle \overline{Y}_{s-}, f_1(s, Y_{s-}^{f_1}, U_s^{f_1}, Z_s^{f_1}) - f(s, Y_{s-}^{f_1}, U_s^{f_1}, Z_s^{f_1})\rangle \mathbb{1}_{A^{N,c}} ds \\
I_3 &:= 2\mathbb{E}\int_t^T \langle \overline{Y}_{s-}, f(s, Y_{s-}^{f_1}, U_s^{f_1}, Z_s^{f_1}) - f(s, Y_{s-}^{f_2}, U_s^{f_2}, Z_s^{f_2})\rangle \mathbb{1}_{A^{N,c}} ds. \\
I_4 &:= 2\mathbb{E}\int_t^T \langle \overline{Y}_{s-}, f(s, Y_{s-}^{f_2}, U_s^{f_2}, Z_s^{f_2}) - f_2(s, Y_{s-}^{f_2}, U_s^{f_2}, Z_s^{f_2})\rangle \mathbb{1}_{A^{N,c}} ds.
\end{aligned}$$

It is not difficult to check that

$$\begin{aligned}
I_2 &\leq \mathbb{E}\int_t^T |\overline{Y}_{s-}|^2 \mathbb{1}_{A^{N,c}} ds + \rho_N^2(f_1 - f) \\
I_4 &\leq \mathbb{E}\int_t^T |\overline{Y}_{s-}|^2 \mathbb{1}_{A^{N,c}} ds + \rho_N^2(f - f_2).
\end{aligned}$$

Since f is L_N-Lipschitz in the ball $B(0,N)$, we get

$$I_3 \leq (2L_N + \gamma^2)\mathbb{E}\int_t^T |\overline{Y}_{s-}|^2 \mathbb{1}_{A^{N,c}} ds + \frac{2L_N^2}{\gamma^2}\mathbb{E}\int_t^T |\overline{U}_s|^2 ds + \frac{2L_N^2}{\gamma^2}\mathbb{E}\int_t^T \|\overline{Z}_s\|^2 ds.$$

To estimate I_1, we use Hölder's inequality and the fact that

$$\mathbb{1}_{A^N} \leq \frac{|Y_{s-}^{f_1}|^2 + |U_s^{f_2}|^2 + \|Z_s^{f_1}\|^2 + |U_s^{f_1}|^2 + |Y_{s-}^{f_2}|^2 + \|Z_s^{f_2}\|^2}{N^2}$$

to obtain

$$I_1 \leq \beta^2\mathbb{E}\int_t^T |\overline{Y}_s|^2 \mathbb{1}_{A^N} ds + \frac{1}{\beta^2}\mathbb{E}\int_t^T |\overline{f}_s|^2 \mathbb{1}_{A^N} ds$$

$$\begin{aligned}
&\leq \beta^2\mathbb{E}\int_t^T |\overline{Y}_{s-}|^2 1\!\!1_{A^N} ds + \frac{8K^2}{\beta^2}\mathbb{E}\int_t^T (1+|Y_{s-}^{f_2}|^{2\alpha}+|U_s^{f_2}|^{2\alpha}+\|Z_s^{f_2}\|^{2\alpha})1\!\!1_{A^N} ds \\
&\quad + \frac{8K^2}{\beta^2}\mathbb{E}\int_t^T (1+|Y_{s-}^{f_1}|^{2\alpha}+|U_s^{f_1}|^{2\alpha}+\|Z_s^{f_1}\|^{2\alpha})1\!\!1_{A^N} ds \\
&\leq \beta^2\mathbb{E}\int_t^T |\overline{Y}_{s-}|^2 1\!\!1_{A^N} ds + \frac{C(K,\xi^1,\xi^2)}{\beta^2 N^{2(1-\alpha)}}.
\end{aligned}$$

If we choose $\beta^2 = 2L_N^2 + 2L_N$ and $\gamma^2 = 2L_N^2$ then we use the above estimates we have

$$\begin{aligned}
\mathbb{E}|\overline{Y}_t|^2 \quad &+\mathbb{E}\int_t^T |\overline{U}_s|^2 ds + \mathbb{E}\int_t^T \|\overline{Z}_s\|^2 ds \\
&\leq \mathbb{E}|\xi^1-\xi^2|^2 + (2L_N^2+2L_N+2)\mathbb{E}\int_t^T |\overline{Y}_s|^2 ds \\
&\quad +\rho_N^2(f_1-f)+\rho_N^2(f-f_2)+\frac{C(K,\xi^1,\xi^2)}{\beta^2 N^{2(1-\alpha)}}.
\end{aligned}$$

Using Gronwall's lemma, we get

$$\begin{aligned}
\mathbb{E}|\overline{Y}_t|^2 \quad &\leq \left[\mathbb{E}|\xi^1-\xi^2|^2 + \frac{C(K,\xi^1,\xi^2)}{(2L_N^2+2L_N)N^{2(1-\alpha)}}\right]\exp\big([L_N^2+2L_N](T-t)+2\big). \\
&\quad + \left[\rho_N^2(f_1-f)+\rho_N^2(f-f_2)\right]\exp\big([L_N^2+2L_N](T-t)+2\big).
\end{aligned}$$

Lemma 4.3 is proved. ■

Lemma 4.4: *Let f be a function which satisfies* H.1), H.2) and H.3). *Then there exists a sequence of functions f_n such that,*

(i) a) *For each n, $f_n \in Lip_\alpha$.*

b) $\sup_n |f_n(t,\omega,y,u,z)| \leq |f(t,\omega,y,u,z)| \leq K(1+|y|^\alpha+|u|^\alpha+\|z\|^\alpha)$ $\mathbb{P}$–a.s., a.e. $t\in[0,T]$.

(ii) *For every N, $\rho_N(f_n-f)\longrightarrow 0$ as $n\longrightarrow\infty$.*

Proof: Let ψ_n be a sequence of smooth functions with support in the ball $B(0,n+1)$ and such that $\psi_n = 1$ in the ball $B(0,n)$. It is not difficult to see that the sequence (f_n) of truncated functions, defined by $f_n = f\psi_n$, satisfies all the properties quoted in Lemma 4.4. ■

Lemma 4.5: *Let f and ξ be as in Theorem* 4.1. *Let (f_n) be the sequence of functions associated to f by Lemma* 4.4 *and denote by $(Y^{f_n},U^{f_n},Z^{f_n})$ the solution of equation $Eq(f_n,\xi)$. Then,*

(i) a) $\sup_n \mathbb{E}|Y_t^{f_n}|^2 \leq \left[\mathbb{E}\,|\xi|^2+K\right]\exp(K) = K_1$.

b) $\sup_n \mathbb{E}\left(\int_0^T |U_s^{f_n}|^2 ds + \int_0^T \|Z_s^{f_n}\|^2 ds\right) \leq \left[\mathbb{E}\,|\xi|^2+K\right][2+(K)\exp(K)] = K_2$.

(ii) There exists a process $(Y,U,Z)\in\mathcal{E}$ such that

$$\lim_{n\to\infty}\left\|\left(Y^{f_n},U^{f_n},Z^{f_n}\right)-(Y,U,Z)\right\| = 0.$$

Proof of Lemma 4.5: For simplicity,we assume $L = 0$. Assertion (i) follows from standard arguments of BSDE. Let us prove (ii). First, assume that $L_N \leq \sqrt{\frac{(1-\alpha)}{2(T-t)}\log(N)}$. Then applying Lemma 4.3 to $(Y^{f_1}, U^{f_1}, Z^{f_1}, f_1, \xi^1) = (Y^{f_n}, U^{f_n}, Z^{f_n}, f_n, \xi)$, $(Y^{f_2}, U^{f_2}, Z^{f_2}, f_2, \xi^2) = (Y^{f_m}, U^{f_m}, Z^{f_m}, f_m, \xi)$ and next passing to the limits successively on n, m, N one gets Lemma 4.5. Assume now that $L_N \leq \sqrt{\log(N)}$. Let δ be a strictly positive number such that $\delta < \frac{(1-\alpha)}{2}$. Let $([t_{i+1}, t_i])$ be a subdivision of $[0, T]$ such that $|t_{i+1} - t_i| \leq \delta$. Applying Lemma 4.3 in all the subintervals $[t_{i+1}, t_i]$ we get Lemma 4.5. ■

Proof of Theorems 4.1 and 4.2:. The uniqueness follows from Lemma 4.3 by letting $f_1 = f_2 = f$ and $\xi^1 = \xi^2 = \xi$). We shall prove the existence of solutions. By Lemma 4.5, there exists $(Y, U, Z) \in \mathcal{E}$ such that $||(Y^{f_n}, U^{f_n}, Z^{f_n}) - (Y, U, Z)|| \to 0$ as $n \to \infty$. Thus, we immediately have

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\left(\sup_{0\leq s\leq T}\left|Y_s^{f_n} - Y_s\right|^2\right) &= \lim_{n\to\infty} \mathbb{E}\int_0^T \left|U_s^{f_n} - U_s\right|^2 ds \\ &= \lim_{n\to\infty} \mathbb{E}\int_0^T \left\|Z^{f_n}{}_s - Z_s\right\|^2 ds = 0.\end{aligned}$$

It remains to prove that $\int_t^T f_n(s, Y_{s-}^{f_n}, U_s^{f_n}, Z_s^{f_n})ds$ converges to $\int_t^T f(s, Y_{s-}, U_s, Z_s)ds$ *in probability.* Let $N > 1$ and denote by L_N the Lipschitz constant of f in the ball $B(0, N)$. We put $A_n^N := \{(s, \omega) : |Y_{s-}^{f_n}| + |U_s^{f_n}| + \|Z_s^{f_n}\| + |Y_{s-}| + |U_s| + \|Z_s\| \geq N\}$ and $A_n^{N,c} := \Omega \setminus A_n^N$. Since f is L_N–locally Lipschitz, we use the triangle inequality and Lemma 4.4 to obtain

$$\begin{aligned}\mathbb{E}\left|\int_t^T \overline{f}_s^n ds\right| &\leq \mathbb{E}\int_t^T |f_n(s, Y_{s-}^{f_n}, U_s^{f_n}, Z_s^{f_n}) - f(s, Y_{s-}^{f_n}, U_s^{f_n}, Z_s^{f_n})|ds \\ &\quad + \mathbb{E}\int_0^T |f(s, Y_{s-}^{f_n}, U_s^{f_n}, Z_s^{f_n}) - f(s, Y_{s-}, U_s, Z_s)|ds \\ &\leq \mathbb{E}\int_0^T \sup_{|y|,|u|,\|z\|\leq N} |f_n(s, y, u, z) - f(s, y, u, z)|ds \\ &\quad + 2K\sup_n \mathbb{E}\int_0^T (1 + |Y_{s-}^{f_n}|^\alpha + |U_s^{f_n}|^\alpha + \|Z_s^{f_n}\|^\alpha)1\!\!1_{A_n^{N,c}} ds \\ &\quad + L_N\left(\mathbb{E}\int_0^T |\overline{Y}_{s-}^{f_n}|ds + \mathbb{E}\int_0^T |\overline{U}_s^{f_n}|ds + \mathbb{E}\int_0^T \|\overline{Z}_s^{f_n}\|ds\right)\end{aligned}$$

$$+K\sup_n \mathbb{E}\int_0^T (2 + |Y_{s-}^{f_n}|^\alpha + |U_s^{f_n}|^\alpha + \|Z_s^{f_n}\|^\alpha + |Y_{s-}|^\alpha + |U_s|^\alpha + \|Z_s\|^\alpha)1\!\!1_{A_n^{N,c}} ds.$$

Since $|x|^\alpha \leq 1 + |x|$ for each $\alpha \in [0, 1[$, we successively use Lemma 4.5 (i) -b), Schwarz inequality, Chebychev inequality, Lemma 4.5 (i) and Fatou's lemma to get

$$\mathbb{E}|\int_t^T \overline{f}_s^n ds| \leq I_1(n) + L_N I_2(n) + \frac{24K}{N}(1 + 2(K_1 + K_2)),$$

where K_1, K_2 denote the two constant defined in Lemma 4.5 (ii) and where

$$I_1(n) := \mathbb{E}\int_0^T \sup_{|y|,|u|,\|z\|\leq N} |f_n(s, y, u, z) - f(s, y, u, z)|ds$$

and

$$I_2(n) := \mathbb{E}\int_0^T |\overline{Y}^{f_n}_{s-}|ds + \mathbb{E}\int_0^T |\overline{U}^{f_n}_s|ds + \mathbb{E}\int_0^T \|\overline{Z}^{f_n}_s\|ds.$$

Lemma 4.5 (ii) shows that $\lim_{n\to\infty} I_1(n) = 0$. We shall prove that $\lim_{n\to\infty} I_2(n) = 0$. From Lemma 4.5 we have

$$\lim_{n\to\infty} \mathbb{E}\int_0^T \left(|\overline{U}^n_s| + \|\overline{Z}^n_s\|\right) ds = 0.$$

We use Lemma 4.5, Fatou's Lemma and the Lebesgue dominated convergence theorem to show that $\lim_{n\to\infty} \mathbb{E}\int_0^T |\overline{Y}^{f_n}_s|ds = 0$ which shows that equation $Eq(f,\xi)$ has at least one solution. Theorem 4.1 is proved. We get Theorem 4.2 by applying Lemma 4.3 to $(Y^{f_1}, U^{f_1}, Z^{f_1}, f_1, \xi^1) = (Y, U, Z, f, \xi)$, $(Y^{f_2}, U^{f_2}, Z^{f_2}, f_2, \xi^2) = (Y^{f_n}, U^{f_n}, Z^{f_n}, f_n, \xi)$ and by passing to the limits, first on n and next on N. The proofs are finished. ■

5 Applications to PDIE

In this section, we give the links between BSDE driven by Lévy process and a family of partial differential integral equation (PDIE). Let $X_t = \int_0^t \sigma(X_s)dW_s + L_t$, recall that L_t is a Lévy process with Lévy measure ν, which takes the form $L_t = bt + \ell_t$.

We give a technical lemma which will be needed later on.

Lemma 5.1: *Let $h : \Omega \times [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a measurable function such that*

$$|h(s,y)| \le \theta_s(y^2 \wedge |y|) \quad a.s.$$

where $\{\theta_s : 0 \le s \le T\}$ is a nonnegative predictable process such that $\mathbb{E}\int_0^T \theta_s ds < \infty$. Then for each $t \in [0,T]$ we have

$$\sum_{t<s\le T} h(s, \Delta X_s) = \sum_{i=1}^{\infty}\int_t^T \langle h(s,.), p_i(.)\rangle_{L^2(\nu)} dH^{(i)}_s + \int_t^T\int_{\mathbb{R}} h(s,y)\nu(dy)ds.$$

Proof: Since $\Delta X_t = \Delta L_t$, the proof can be performed as that of Nualart and Schoutens [6]. ■

Now, we apply our result to give a version of Clark–Ocone formula for functions of a Lévy process. Consider the following BSDE

$$Y_t = g(X_T) + \int_t^T f(s, Y_{s-}, U_s, Z_s)ds - \int_t^T U_s dW_s - \int_t^T \langle Z_s, dH_s\rangle, \tag{5.1}$$

where $\mathbb{E}|g(X_T)^2| < \infty$.
Define

$$u^1(t,x,y) = u(t,x+y) - u(t,x) - \frac{\partial u}{\partial x}(t,x)y,$$

where u is the solution of the following PDIE

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + \frac{1}{2}\sigma^2(x)\frac{\partial^2 u}{\partial x^2}(t,x) + f\left(t, u(t,x), \sigma(x)\frac{\partial u}{\partial x}(t,x), \left(u^{(i)}\right)_{i=1}^{\infty}\right) \\ \qquad\qquad + \int_{\mathbb{R}} u^1(t,x,y)\nu(dy) + \overline{b}\frac{\partial u}{\partial x}(t,x) = 0 \\ u(T,x) = g(x), \end{cases}$$

$\bar{b} = b + \int_{\{|y|\geq 1\}} y\nu(dy)$ and

$$u^{(1)}(t,x) = \int_{\mathbb{R}} u^1(t,x,y)p_1(y)\nu(dy) + \frac{\partial u}{\partial x}(t,x)\left(\int_{\mathbb{R}} y^2\nu(dy)\right)^{\frac{1}{2}},$$

and for $i \geq 2$

$$u^{(i)}(t,x) = \int_{\mathbb{R}} u^1(t,x,y)p_i(y)\nu(dy).$$

Suppose that u is $\mathcal{C}^{1,2}$ function such that $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$ is bounded by polynomial function of x. Then we have the following

Theorem 5.2: *The unique adapted solution of* (5.1) *is given by*

$$\begin{aligned}
Y_t &= u(t,X_t) \\
U_t &= \sigma(X_t)\frac{\partial u}{\partial x}(t,X_t) \\
Z_t^{(i)} &= \int_{\mathbb{R}} u^1(t,X_{t-},y)p_i(y)\nu(dy) \quad \text{for } i \geq 2, \\
Z_t^{(1)} &= \int_{\mathbb{R}} u^1(t,X_{t-},y)p_1(y)\nu(dy) + \frac{\partial u}{\partial x}(t,X_{t-})\left(\int_{\mathbb{R}} y^2\nu(dy)\right)^{\frac{1}{2}}.
\end{aligned}$$

Proof: Applying Itô's formula to $u(s,X_s)$ we have

$$\begin{aligned}
u(T,X_T) - u(t,X_t) &= \int_t^T \frac{\partial u}{\partial s}(s,X_s)ds + \frac{1}{2}\int_t^T \sigma^2(X_s)\frac{\partial^2 u}{\partial x^2}(s,X_s)ds \\
&\quad + \int_t^T \sigma(X_s)\frac{\partial u}{\partial x}(s,X_s)dW_s + \int_t^T \frac{\partial u}{\partial x}(s,X_{s-})dL_s \\
&\quad + \sum_{t<s\leq T}\left[u(s,X_s) - u(s,X_{s-}) - \frac{\partial u}{\partial x}(s,X_{s-})\Delta X_s\right].
\end{aligned}$$

Lemma 5.1 applied to $u(s,X_{s-}+y) - u(s,X_{s-}) - \frac{\partial u}{\partial x}(s,X_{s-})y$ gives

$$\begin{aligned}
&\sum_{t<s\leq T}\left[u(s,X_s) - u(s,X_{s-}) - \frac{\partial u}{\partial x}(s,X_{s-})\Delta X_s\right] \\
&= \sum_{i=1}^{\infty}\int_t^T\left(\int_{\mathbb{R}} u^1(s,X_{s-},y)p_i(y)\nu(dy)\right)dH_s^{(i)} \\
&\quad + \int_t^T\int_{\mathbb{R}} u^1(s,X_{s-},y)\nu(dy)ds.
\end{aligned}$$

Exploiting this last identity we obtain

$$g(X_T) - u(t,X_t) = \frac{1}{2}\int_t^T \sigma^2(X_s)\frac{\partial^2 u}{\partial x^2}(s,X_s)ds$$

$$
\begin{aligned}
&- \int_t^T f\left(s, u(s, X_s), \left(u^{(i)}(s, X_s)\right)_{i=1}^{\infty}\right) ds \\
&+ \sum_{i=2}^{\infty} \int_t^T \left(\int_{\mathbb{R}} u^1(s, X_{s-}, y) p_i(y) \nu(dy)\right) dH_s^{(i)} \\
&+ \int_t^T \left(\int_{\mathbb{R}} u^1(s, X_{s-}, y) p_1(y) \nu(dy)\right. \\
&\left. + \left(\frac{\partial u}{\partial x}(s, X_{s-})\right) \left(\int_{\mathbb{R}} y^2 \nu(dy)\right)^{\frac{1}{2}}\right) dH_s^{(1)},
\end{aligned}
$$

from which we get the desired result. ■

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