APPROXIMATE CONTROLLABILITY OF SEMILINEAR NEUTRAL SYSTEMS IN HILBERT SPACES

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The approximate controllability of semilinear neutral systems in Hilbert spaces is studied using the Schauder fixed point theorem. It is shown that the approximate controllability of the semilinear system under some conditions is implied by the approximate controllability of its linear part.
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1 Introduction

The theory of controllability of differential equations has been extensively studied in the literature. Zhou [9] studied the approximate controllability for a class of semilinear abstract equations. Naito [7] established the controllability for nonlinear Volterra integrodifferential systems. Chukwu and Lenhart [2] have studied controllability of nonlinear systems in abstract spaces. Hernández and Henriquez [4] obtained some existence results for neutral functional differential equations in Banach spaces by using semigroup theory and the Sadovski fixed point principle. Recently, Dauer and Balachandran [3], Balachandran and Sakthivel [1] studied exact controllability of neutral integrodifferential equations in Banach spaces using the Schaefer fixed point theorem. The purpose of this paper is to study the approximate controllability of semilinear neutral systems in Hilbert spaces by using the Schauder fixed point theorem.

Consider the following semilinear neutral system

$$\frac{d}{dt}\left[x\left(t\right) + g\left(t, x\left(t\right)\right)\right] = Ax\left(t\right) + Bu\left(t\right) + f(t, x\left(t\right)) \tag{1.1}$$

$$x\left(0\right) = x_0, \quad t \in [0, T]$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operators $S\left(t\right)$ in Hilbert space X, $B : U \to X$ is a bounded linear operator, the control $u\left(\cdot\right) \in L_2\left([0, T], U\right)$ takes values in a Hilbert space U and the state $x\left(\cdot\right)$ takes values in a space X. $f, g : [0, T] \times X \to X$ are continuous functions.

The paper consists of 4 sections. Section 2 contains the definition of approximate controllability, the definition of mild solution and some basic assumptions on the data of the problem. Moreover, a formula for a control steering the initial state x_0 to a neighborhood of arbitrary final state is constructed. In Section 3, the main result on approximate controllability of semilinear neutral systems using the Schauder fixed point theorem is obtained. As an application of this result, the initial boundary-value problem is considered in Section 4.

2   Preliminaries

Definition 2.1: System (1.1) is said to be approximately controllable on the interval $[0, T]$ if

$$\overline{\mathcal{R}(T, x_0)} = X$$

where

$$\mathcal{R}(T, x_0) = \{x(x_0; u)(T) : u(\cdot) \in L_2([0, T], U)\}.$$

Definition 2.2: A function $x : [0, T] \to X$ is a mild solution of the abstract Cauchy problem (1.1) if $x(\cdot)$ is continuous on $[0, T]$; for each $0 \le t < T$ the function $AS(t - s) g(s, x(s))$, $s \in [0, t)$, is integrable and

$$x(t) = S(t)(x_0 + g(0, x_0)) - g(t, x(t)) - \int_0^t AS(t - s) g(s, x(s))\, ds$$

$$+ \int_0^t S(t - s) f(s, x(s))\, ds + \int_0^t S(t - s) Bu(s)\, ds \tag{2.1}$$

for $t \in [0, T]$.

The existence of mild solutions for the abstract Cauchy problem (1.1) follows from Theorem 2.1 and 2.2 of [4].

Recall that A is the infinitesimal generator of an analytic semigroup of bounded linear operators $S(t)$ in X such that

$$\|S(t)\| \le M_1 \text{ for some } M_1 \ge 1.$$

At this point, it is possible to define the fractional power $(-A)^\beta$ of $(-A)$ for $0 < \beta < 1$ as a closed linear operator with domain $D\left((-A)^\beta\right)$. Furthermore the subspace $D\left((-A)^\beta\right)$ is dense in X and the expression

$$\|x\|_\beta = \left\|(-A)^\beta x\right\|$$

defines a norm on $D\left((-A)^\beta\right) = X_\beta$ and there exists $C > 0$ such that $\left\|(-A)^{-\beta}\right\| \le C$ for $0 \le \beta \le 1$.

Lemma 2.1: [8] *Let A be the infinitesimal generator of an analytic semigroup $S(t)$. If $0 \in \rho(A)$ then for every $t > 0$ the operator $(-A)^\beta S(t)$ is bounded and*

$$\left\|(-A)^\beta S(t)\right\| \le C_\beta t^{-\beta} e^{-\delta t}, \delta > 0.$$

We introduce two relevant operators and the basic assumption on these operators.

$$\Gamma_0^T = \int_0^T S\left(T - s\right) BB^* S^* \left(T - s\right) ds$$

$$R\left(\alpha, \Gamma_0^T\right) = \left(\alpha I + \Gamma_0^T\right)^{-1}.$$

$(BA1)$ $\alpha R\left(\alpha, \Gamma_0^T\right) \to 0$ as $\alpha \to 0^+$ in the strong operator topology.
It is known that the assumption $(BA1)$ holds if and only if the linear system

$$\dot{x}\left(t\right) = Ax\left(t\right) + Bu\left(t\right) \tag{2.2}$$
$$x\left(0\right) = x_0,$$

is approximately controllable on $[0, T]$, see [5, 6].

The following theorem gives a formula for a control transferring the initial state x_0 to some neighborhood of h at time T.

Theorem 2.1: *For arbitrary $h \in X$, the control*

$$u\left(t\right) = B^* S^* \left(T - t\right) R\left(\alpha, \Gamma_0^T\right) p\left(x\right) \tag{2.3}$$

where

$$p\left(x\right) = h - S\left(T\right)\left[x_0 + g\left(0, x_0\right)\right] + g\left(T, x\left(T\right)\right)$$
$$- \int_0^T S\left(T - s\right) f\left(s, x\left(s\right)\right) ds + \int_0^T AS\left(T - s\right) g\left(s, x\left(s\right)\right) ds$$

transfers the initial state x_0 to

$$x\left(T\right) = h - \alpha\left(\alpha I + \Gamma_0^T\right)^{-1} \left(h - S\left(T\right)\left[x_0 + g\left(0, x_0\right)\right] + g\left(T, x\left(T\right)\right) \right.$$
$$\left. - \int_0^T S\left(T - r\right) f\left(r, x\left(r\right)\right) dr + \int_0^T AS\left(T - r\right) g\left(r, x\left(r\right)\right) dr \right). \tag{2.4}$$

Proof: By substituting (2.3) in (2.1) and writing the obtained equation at $t = T$, one gets the following:

$$x\left(T\right) = S\left(T\right)\left[x_0 + g\left(0, x_0\right)\right] - g(T, x\left(T\right))$$
$$- \int_0^T AS\left(T - s\right) g\left(s, x\left(s\right)\right) ds + \int_0^T S\left(T - s\right) f\left(s, x\left(s\right)\right) ds$$
$$+ \int_0^T S\left(T - s\right) BB^* S^* \left(T - s\right) \left(\alpha I + \Gamma_0^T\right)^{-1} p\left(x\right) ds$$
$$= S\left(T\right)\left[x_0 + g\left(0, x_0\right)\right] - g(T, x\left(T\right))$$
$$- \int_0^T AS\left(T - s\right) g\left(s, x\left(s\right)\right) ds + \int_0^T S\left(T - s\right) f\left(s, x\left(s\right)\right) ds$$
$$+ \Gamma_0^T \left(\alpha I + \Gamma_0^T\right)^{-1} p\left(x\right).$$

Using the identity $\Gamma_0^T \left(\alpha I + \Gamma_0^T\right)^{-1} = I - \alpha\left(\alpha I + \Gamma_0^T\right)^{-1}$ one can obtain equality (2.4).

The following notations are introduced for convenience:

$$K = \max\left\{\left\|\Gamma_0^t\right\| : 0 \le t \le T\right\},$$
$$M = \|B\|, \quad M_0 = \max\left\{\|S(t)x_0\| : 0 \le t \le T\right\}$$
$$M_2 = \max\left\{\|S(t)g(0,x_0)\| : 0 \le t \le T\right\}$$

We assume the following hypotheses:

($H1$) The semigroup $S(t)$, $t > 0$ is analytic.

($H2$) The function $f : I \times X \to X$ is continuous and there exists a constant $L > 0$ such that

$$\|f(t,x)\| \le L \text{ for all } (t,x) \in I \times X.$$

($H3$) The function g is $X_\beta-$ valued, $(-A)^\beta g$ is continuous and there exists a constant $C_1 > 0$ such that

$$\left\|(-A)^\beta g(t,x)\right\| \le C_1 \text{ for all } (t,x) \in I \times X.$$

($H4$) For every $r > 0$, the set of functions

$$\left\{s \to (-A)^\beta g(s,x(s)) : x \in C(0,T;X), \sup_{t \in [0,T]} \|x(t)\| \le r\right\}$$

is equicontinuous on $[0,T]$.

3   Approximate Controllability

In this section, we prove the main result on approximate controllability of semilinear neutral systems. To do this, we first prove the existence of a fixed point of the operator $\mathfrak{F}$ defined below, using the Schauder fixed point theorem. Second, in Theorem 3 we show that under certain assumptions the approximate controllability of (1.1) is implied by the approximate controllability of the linear system (2.2).

Now, define the operator $\mathfrak{F}$ on $C([0,T],X)$ by means of

$$(\mathfrak{F}x)(t) = S(t)[x_0 + g(0,x_0)] - g(t,x(t)) - \int_0^t AS(t-s)g(s,x(s))ds$$
$$+ \int_0^t S(t-s)f(s,x(s))ds + \int_0^t S(t-s)Bu(s)ds, \tag{3.1}$$

where

$$u(t) = B^*S^*(T-t)\left(\alpha I + \Gamma_0^T\right)^{-1}\left(h - S(T)(x_0 + g(0,x_0))\right.$$
$$+ g(T,x(T)) + \int_0^T AS(T-s)g(s,x(s))ds$$
$$\left. - \int_0^T S(T-s)f(s,x(s))ds\right). \tag{3.2}$$

Introduce the set

$$Y_r = \{x \in C\left([0,T], X\right) : x\left(0\right) = x_0, \|x\| \leq r\},$$

where $r > 0$.

Theorem 3.1: *Assume hypotheses* $(H1), (H2), (H3), (H4)$ *and* $(BA1)$ *are satisfied. Then for each* $\alpha > 0$, *the operator* $\mathfrak{F}$ *has a fixed point.*

Proof: The proof of the theorem is long and technical. Therefore it is convenient to divide it into several steps.

Step 1 : For all $\alpha > 0$, there exists $r = r\left(\alpha\right)$ such that $\mathfrak{F}$ maps Y_r into Y_r.

Using the fractional powers of $(-A)$ the operator (3.1) can be rewritten as follows.

$$\left(\mathfrak{F}x\right)\left(t\right) = S\left(t\right)\left(x_0 + g\left(0, x_0\right)\right) - \left(-A\right)^{-\beta}\left(-A\right)^{\beta} g\left(t, x\left(t\right)\right)$$
$$- \int_0^t \left(-A\right)^{1-\beta} S\left(t - s\right)\left(-A\right)^{\beta} g\left(s, x(s)\right)ds$$
$$+ \int_0^t S\left(t - s\right) f\left(s, x\left(s\right)\right)ds + \int_0^t S\left(t - s\right) Bu\left(s\right)ds, \qquad (3.3)$$

where

$$u\left(t\right) = B^* S^*\left(T - t\right)\left(\alpha I + \Gamma_0^T\right)^{-1}\left(h - S\left(T\right)\left(x_0 + g\left(0, x_0\right)\right)\right.$$
$$+ \left(-A\right)^{-\beta}\left(-A\right)^{\beta} g\left(T, x\left(T\right)\right) - \int_0^T S\left(T - s\right) f\left(s, x(s)\right)ds$$
$$+ \left.\int_0^T \left(-A\right)^{1-\beta} S\left(T - s\right)\left(-A\right)^{\beta} g\left(s, x(s)\right)ds\right). \qquad (3.4)$$

Taking norm on (3.3) and (3.4), yields

$$\left\|\left(\mathfrak{F}x\right)\left(t\right)\right\| \leq \left\|S\left(t\right) x_0\right\| + \left\|S\left(t\right)\right\| \left\|g(0, x_0)\right\| + \left\|\left(-A\right)^{-\beta}\left(-A\right)^{\beta} g(t, x\left(t\right))\right\|$$
$$+ \int_0^t \left\|\left(-A\right)^{1-\beta} S\left(t - s\right)\right\| \left\|\left(-A\right)^{\beta} g\left(s, x\left(s\right)\right)\right\| ds$$
$$+ \int_0^t \left\|S\left(t - s\right)\right\| \left\|f\left(s, x\left(s\right)\right)\right\| ds + \int_0^t \left\|S\left(t - s\right)\right\| \left\|B\right\| \left\|u\left(s\right)\right\| ds$$
$$\leq M_0 + M_2 + CC_1 + \int_0^t \frac{C_\beta C_1}{\left(t - s\right)^{1-\beta}} ds + M_1 LT + MM_1\sqrt{T}\left\|u\right\|$$
$$\leq M_0 + M_2 + CC_1 + \frac{C_1 C_\beta T^\beta}{\beta} + M_1 LT + MM_1\sqrt{T}\left\|u\right\| \qquad (3.5)$$

and

$$\left\|u\left(t\right)\right\| \leq \frac{1}{\alpha} MM_1\left(\left\|h\right\| + M_0 + M_2 + CC_1 + \int_0^T \frac{C_1 C_\beta}{\left(T - s\right)^{1-\beta}} ds + M_1 TL\right)$$
$$\leq \frac{1}{\alpha} MM_1\left(\left\|h\right\| + M_0 + M_2 + CC_1 + \frac{C_1 C_\beta T^\beta}{\beta} + M_1 TL\right). \qquad (3.6)$$

The last two inequalities (3.5) and (3.6) imply that for large enough $r > 0$ the following inequality holds

$$\|(\mathfrak{F}x)(t)\| \leq r.$$

That is $(\mathfrak{F}x) \in Y_r$. Therefore $\mathfrak{F}$ maps Y_r into itself.

Step 2. $V(t) = \{(\mathfrak{F}x)(t) : x \in Y_r\}$ is relatively compact for arbitrary $t \in [0, T]$.

Since $V(0) = \{x_0\}$, the case where $t = 0$ is trivial. So let t, $0 < t \leq T$, be a fixed point and let η be a given real number satisfying $0 < \eta < t$. Define

$$(\mathfrak{F}_1^\eta x)(t) = S(\eta)(\mathfrak{F}_1 x)(t - \eta),$$

where

$$(\mathfrak{F}_1 x)(t - \eta) = S(t - \eta)(x_0 + g(0, x_0))$$
$$+ \int_0^{t-\eta} S(t - s - \eta) BB^* S^*(T - s)\left(\alpha I + \Gamma_0^T\right)^{-1} p(x) ds$$
$$- \int_0^{t-\eta} (-A)^{1-\beta} S(t - s - \eta)(-A)^\beta g(s, x(s)) ds$$
$$+ \int_0^{t-\eta} S(t - s - \eta) f(s, x(s)) ds.$$

Since $S(t)$ is analytic and $\mathfrak{F}_1$ is bounded on Y_r, by assumption $(H4)$, the set

$$V_\eta(t) = \{(\mathfrak{F}_1^\eta x)(t) + g(t, x(t)) : x \in Y_r\}$$

is relatively compact set in X. That is, a finite set $\{y_i, 1 \leq i \leq m\}$ in X exists such that

$$V_\eta(t) \subset \bigcup_{i=0}^m N\left(y_i, \frac{\varepsilon}{2}\right),$$

where $N\left(y_i, \frac{\varepsilon}{2}\right)$ is an open ball in X with the center at y_i and radius $\frac{\varepsilon}{2}$.

On the other hand,

$$\|(\mathfrak{F}_1 x)(t) - (\mathfrak{F}_1^\eta x)(t)\| = \left\|\int_{t-\eta}^t S(t - s) BB^* S^*(T - s)\left(\alpha I + \Gamma_0^T\right)^{-1} p(x) ds\right.$$
$$- \int_{t-\eta}^t (-A)^{1-\beta} S(t - s)(-A)^\beta g(s, x(s)) ds$$
$$+ \left.\int_{t-\eta}^t S(t - s) f(s, x(s)) ds\right\|$$
$$\leq \frac{1}{\alpha} M^2 M_1^2 P\eta + \frac{C_1 C_\beta \eta^\beta}{\beta} + M_1 L\eta$$
$$\leq \frac{\varepsilon}{2}.$$

Consequently,

$$V(t) \subset \bigcup_{i=0}^m N(y_i, \varepsilon).$$

Hence, $V(t)$ is relatively compact in X for each $t \in [0, T]$.

Step 3. $V = \{\mathfrak{F}x : x \in Y_r\}$ is equicontinuous on $t \in [0, T]$.
For $0 < t_1 - \varepsilon < t_1 < t_2 < T$,

$$(\mathfrak{F}_1 x)(t_1) - (\mathfrak{F}_1 x)(t_2) = (S(t_1 - s) - S(t_2 - s))(x_0 + g(0, x_0))$$

$$+ \int_0^{t_1} [S(t_1 - s) - S(t_2 - s)] f(s, x(s)) ds - \int_{t_1}^{t_2} S(t_2 - s) f(s, x(s)) ds$$

$$+ \int_0^{t_1 - \varepsilon} (-A)^{1-\beta} S(t_1 - \varepsilon - s)(I - S(t_2 - t_1)) S(\varepsilon)(-A)^\beta g(s, x(s)) ds$$

$$+ \int_{t_1 - \varepsilon}^{t_1} (I - S(t_2 - t_1))(-A)^{1-\beta} S(t_1 - s)(-A)^\beta g(s, x(s)) ds$$

$$- \int_{t_1}^{t_2} (-A)^{1-\beta} S(t_2 - s)(-A)^\beta g(s, x(s)) ds$$

$$+ \int_0^{t_1} (S(t_1 - s) - S(t_2 - s)) Bu(s) ds - \int_{t_1}^{t_2} S(t_2 - s) Bu(s) ds. \tag{3.7}$$

Taking norm on (3.7) and using hypotheses $(H2), (H3)$ and Lemma 2.1,

$$\| (\mathfrak{F}_1 x)(t_1) - (\mathfrak{F}_1 x)(t_2) \| \le \| S(t_1 - s) - S(t_2 - s) \| (\| x_0 \| + M_3)$$

$$+ L \int_0^t \| S(t_1 - s) - S(t_2 - s) \| ds + (t_2 - t_1) M_1 L$$

$$+ \frac{(t_1 - \varepsilon)^\beta}{\beta} M_1 C_1 \| I - S(t_2 - t_1) \|$$

$$+ \frac{\varepsilon^\beta}{\beta} C_1 \| I - S(t_2 - t_1) \| + \frac{C_1 (t_2 - t_1)^\beta}{\beta}$$

$$+ M \int_0^t \| S(t_1 - s) - S(t_2 - s) \| \| u(s) \| ds$$

$$+ M M_1 \| u \| \sqrt{t_2 - t_1}$$

$$= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8. \tag{3.8}$$

The right hand side of (3.8) does not depend on particular choices of x. It is clear that $I_3 \to 0$, $I_6 \to 0$ and $I_8 \to 0$ as $t_2 - t_1 \to 0$. Since the semigroup $S(t), t > 0$ is analytic

$$\| S(t_1 - s) - S(t_2 - s) \| \to 0$$

as $t_1 - t_2 \to 0$ for arbitrary t, s such that $t - s > 0$. By Lebesque dominated convergence theorem $I_2 \to 0$ and $I_7 \to 0$ as $t_2 - t_1 \to 0$. It is obvious that, $I_4 \to 0$, $I_5 \to 0$ as $t_2 - t_1 \to 0$. Hence V is equicontinuous and the equicontinuity of g follows from $(H4)$.

Note that we only considered the case when $0 < t_1 < t_2$, since the other case $0 < t_2 < t_1$ is similar. Therefore $\mathfrak{F}[Y_r]$ is equicontinuous and bounded. By the Arzela - Ascoli theorem, $\mathfrak{F}[Y_r]$ is relatively compact in $C([0, T], X)$.

Step 4. $\mathfrak{F}$ is continuous on $C([0, T], X)$.

In order to apply the Schauder fixed point theorem it remains to show the continuity of $\mathfrak{F}$ on $C([0, T], X)$. Let $\{y_n\} \subset C([0, T], X)$ with $y_n \to y$ in $C([0, T], X)$. Since

$$f(s, y^n(s)) \to f(s, y(s)), \quad \text{for each } s \in [0, T]$$

$$(-A)^\beta g(s, y^n(s)) \to (-A)^\beta g(s, y(s)), \text{ for each } s \in [0, T],$$

and

$$\| (\mathfrak{F}y^n) (t) - (\mathfrak{F}y) (t) \| \leq C \left\| (-A)^\beta (g (t, y^n (t)) - g (t, y (t))) \right\|$$
$$+ M M_1 \sqrt{T} \left(\int_0^t \| u^n (s) - u (s) \|^2 \, ds \right)^{1/2}$$
$$+ \frac{t^\beta}{\beta} \left\| (-A)^\beta (g (t, y^n (t)) - g (t, y (t))) \right\|$$
$$+ M_1 \int_0^t \| f (s, y^n (s)) - f (s, y (s)) \| \, ds,$$

the Lebesque dominated convergence theorem implies that

$$\sup_{0 \leq t \leq T} \| (\mathfrak{F}y^n) (t) - (\mathfrak{F}y) (t) \| \to 0,$$

which shows the continuity of $\mathfrak{F}$. Thus all the conditions of the Schauder fixed point theorem are satisfied, and consequently the operator $\mathfrak{F}$ has a fixed point in $C ([0, T], X)$.

Theorem 3.2: *Assume hypotheses* $(H1), (H2), (H3)$ *and* $(BA1)$ *are satisfied. Then the system* (1.1) *is approximately controllable on* $[0, T]$.

Proof: Let $x^\alpha (\cdot)$ be a fixed point of $\mathfrak{F}$ in Y_r. By Theorem 2.1, any fixed point of $\mathfrak{F}$ is a mild solution of (1.1) on $[0, T]$ under the control

$$u^\alpha (t) = B^* S^* (T - t) R \left(\alpha, \Gamma_0^T \right) p (x^\alpha)$$

and satisfies

$$x^\alpha (T) = h - \alpha R \left(\alpha, \Gamma_0^T \right) \left\{ h - S (T) [x_0 + g (0, x_0)] + g (T, x (T)) \right.$$
$$\left. + \int_0^T A S (T - s) g (s, x^\alpha (s)) \, ds - \int_0^T S (T - s) f (s, x^\alpha (s)) \, ds \right\}. \qquad (3.9)$$

By $(H2)$ and $(H3)$,

$$\int_0^T \| f (s, x^\alpha (s)) \|^2 \, ds \leq L^2 T$$

$$\int_0^T \left\| (-A)^\beta g (s, x^\alpha (s)) \right\|^2 \, ds \leq C_1^2 T.$$

Consequently, the sequences $\{ f (s, x^\alpha (s)) \}$ and $\left\{ (-A)^\beta g (s, x^\alpha (s)) \right\}$ are bounded in $L_2 ([0, T], X)$. Thus there are subsequences, still denoted by $\{ f (s, x^\alpha (s)) \}$ and $\left\{ (-A)^\beta g (s, x^\alpha (s)) \right\}$, that converge weakly to say $f (s)$ and $g (s)$ in $L_2 ([0, T], X)$, respectively. Define

$$q = h - S (T) [x_0 + g (0, x_0)] + g (T, x (T))$$
$$+ \int_0^T (-A)^{-\beta} S (T - s) g (s) \, ds - \int_0^T S (T - s) f (s) \, ds.$$

It follows that

$$\|p(x^\alpha) - q\| \le \left\| \int_0^T S(T-s)(f(s, x^\alpha(s)) - f(s)) \, ds \right\|$$

$$+ \left\| \int_0^T (-A)^{-\beta} S(T-s) \left((-A)^\beta g(s, x^\alpha(s)) - g(s) \right) ds \right\|. \quad (3.10)$$

By the compactness of the operators $k(t) \to \int_0^t S(t-s) k(s) \, ds : L_2([0,T], X) \to C([0,T], X)$ and $k(t) \to \int_0^t (-A)^{-\beta} S(t-s) k(s) \, ds : L_2([0,T], X) \to C([0,T], X)$, the right hand side of (3.10) tends to zero as $\alpha \to 0^+$.

By (3.9),

$$\|x^\alpha(T) - h\| = \left\| \alpha R(\alpha, \Gamma_0^T) p(x^\alpha) \right\|$$
$$= \left\| \alpha R(\alpha, \Gamma_0^T) (p(x^\alpha) - q + q) \right\|$$
$$\le \left\| \alpha R(\alpha, \Gamma_0^T) q \right\| + \left\| \alpha R(\alpha, \Gamma_0^T) (p(x^\alpha) - q) \right\|$$
$$\le \left\| \alpha R(\alpha, \Gamma_0^T) q \right\| + \left\| \alpha R(\alpha, \Gamma_0^T) \right\| \|p(x^\alpha) - q\|$$
$$\le \left\| \alpha R(\alpha, \Gamma_0^T) q \right\| + \|p(x^\alpha) - q\|, \quad (3.11)$$

where (3.11) tends to zero as $\alpha \to 0^+$ by the estimation (3.10) and the assumption $(BA1)$. This proves the approximate controllability of system (1.1).

Note that compactness of the operators $k(\cdot) \to \int_0^{\cdot} S(\cdot - s) k(s) \, ds : L_2([0,T], X) \to C([0,T], X)$ and $k(\cdot) \to \int_0^{\cdot} (-A)^{-\beta} S(\cdot - s) k(s) \, ds : L_2([0,T], X) \to C([0,T], X)$ are essential in the above proof.

4 Applications

In this section, we illustrate the obtained result. Let $X = L_2[0, \pi]$ and A be defined as follows

$$Az = z''$$

with domain

$$D(A) = \{f(\cdot) \in L_2[0, \pi] : f'' \in L_2[0, \pi], \ f(0) = f(\pi) = 0\}.$$

Recall that A is the infinitesimal generator of a strongly continuous semigroup $S(t), t > 0$, on X which is analytic and self-adjoint, the eigenvalues are $-n^2$, $n \in N$, with corresponding normalized eigenvectors $z_n(\xi) := (2/\pi)^{1/2} \sin(n\xi)$. Moreover the following hold:

(a) $\{z_n : n \in N\}$ is an orthonormal basis of X.
(b) If $f \in D(A)$ then $A(f) = -\sum_{n=1}^\infty n^2 \langle f, z_n \rangle z_n$.
(c) For $f \in X$, $(-A)^{-1/2} f = \sum_{n=1}^\infty \frac{1}{n} \langle f, z_n \rangle z_n$.
(d) The operator $(-A)^{1/2}$ is given as $(-A)^{1/2} f = \sum_{n=1}^\infty n \langle f, z_n \rangle z_n$ on the space $D\left((-A)^{1/2}\right) = \{f \in X : \sum_{n=1}^\infty n \langle f, z_n \rangle z_n \in X\}$.

Consider the neutral system

$$\frac{d}{dt}\left[x\left(t,\xi\right)+q\left(t,x\left(t,\xi\right)\right)\right]=\frac{\partial^2}{\partial\xi^2}x\left(t,\xi\right)$$
$$+p\left(t,x\left(t,\xi\right)\right)+Bu\left(t,\xi\right), \tag{4.1}$$
$$x\left(t,0\right)=x\left(t,\pi\right)=0,\ t\geq0, \tag{4.2}$$
$$x\left(0,\xi\right)=\varphi\left(\xi\right),\ 0\leq\xi\leq\pi, \tag{4.3}$$

where $p:[0,T]\times R\to R$, $q:[0,T]\times R\to R$ are continuous functions, the control $u\in U=L_2\left([0,T]\times[0,\pi]\right)$ and B is a linear bounded operator from U to X.

To write the initial-boundary value problem $(4.1)-(4.3)$ in the abstract form we assume the following:

(i) The substitution operator $f:R\times X\to X$ defined by $f\left(t,x\left(t\right)\right)=p\left(t,x\left(t,\xi\right)\right)$ is continuous and there exists $L>0$ such that $\|f\left(t,x\right)\|\leq L$, $(t,x)\in R\times X$.

(ii) The substitution operator $g:R\times X\to X$ defined by $g\left(t,x\left(t\right)\right)=q\left(t,x\left(t,\xi\right)\right)$ is such that $(-A)^{1/2}g$ is completely continuous on $R\times X$ and there exists $L>0$ such that $\left\|(-A)^{1/2}g\left(t,x\right)\right\|\leq L$, $(t,x)\in R\times X$.

Thus the problem $(4.1)-(4.3)$ can be written in the abstract form

$$\frac{d}{dt}\left(x\left(t\right)+g\left(t,x\left(t\right)\right)\right)=Ax\left(t\right)+f\left(t,x\left(t\right)\right)+Bu\left(t\right)$$
$$x\left(0\right)=x_0,\quad t\in[0,T]$$

and all the conditions of the Theorem 3 are satisfied.This gives the approximate controllability of the neutral system $(4.1)-(4.3)$.

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