ESSENTIAL $\mathcal{U}_c^\kappa$-TYPE MAPS AND BIRKHOFF-KELLOGG THEOREMS

R. P. AGARWAL AND DONAL O'REGAN

We present a new continuation theorem for $\mathcal{U}_c^\kappa$-type maps. The analysis is elementary and relies on properties of retractions and fixed point theory for self-maps. Also we present some Birkhoff-Kellogg type theorems on invariant directions.

1. Introduction

This paper presents a new essential map approach, motivated in part by the paper of Granas [8], for $\mathcal{U}_c^\kappa$-type maps. The theory differs from that in [2]. In particular, we obtain new results for maps which are either

  (a) Kakutani;
  (b) acyclic;
  (c) O'Neill;
  (d) approximable;
  (e) admissible in the sense of Górniewicz; or
  (f) in $\mathcal{U}_c^\kappa$.

The maps considered will also satisfy various compactness criteria described in Section 2. Our analysis is elementary and combines properties of the Minkowski functional with fixed point theory for self-maps. Also using our new homotopy theorem, we will present an "invariant direction" result for particular classes of maps. The theory and results in this paper complement and extend previously known results in the literature (see [2, 7, 8, 10, 11, 12] and the references therein).

For the remainder of this section, we present some definitions and some known facts. Let X and Y be subsets of Hausdorff topological vector spaces E_1 and E_2, respectively. We will look at maps $F : X \to K(Y)$; here $K(Y)$ denotes the family of nonempty compact subsets of Y. We say that $F : X \to K(Y)$ is *Kakutani* if F is upper semicontinuous with convex values. A nonempty topological space is said to be acyclic if all its reduced Čech homology groups over the rationals are trivial. Now $F : X \to K(Y)$ is *acyclic* if F is upper semicontinuous with acyclic values. The map $F : X \to K(Y)$ is said to be an *O'Neill* map

if F is continuous and if the values of F consist of one or m acyclic components (here m is fixed).

Given two open neighborhoods U and V of the origins in E_1 and E_2, respectively, a (U, V)-approximate continuous selection [4] of $F : X \to K(Y)$ is a continuous function $s : X \to Y$ satisfying

$$s(x) \in (F[(x + U) \cap X] + V) \cap Y \quad \text{for every } x \in X. \tag{1.1}$$

We say that $F : X \to K(Y)$ is *approximable* if it is a closed map and if its restriction $F|_K$ to any compact subset K of X admits a (U, V)-approximate continuous selection for every open neighborhood U and V of the origins in E_1 and E_2, respectively.

For our next definition, let X and Y be metric spaces. A continuous single-valued map $p : Y \to X$ is called a Vietoris map if the following two conditions are satisfied:

(i) for each $x \in X$, the set $p^{-1}(x)$ is acyclic,
(ii) p is a proper map, that is, for every compact $A \subseteq X$, we have that $p^{-1}(A)$ is compact.

Definition 1.1. A multifunction $\phi : X \to K(Y)$ is *admissible* (strongly) in the sense of Górniewicz if $\phi : X \to K(Y)$ is upper semicontinuous and if there exists a metric space Z and two continuous maps $p : Z \to X$ and $q : Z \to Y$ such that

(i) p is a Vietoris map,
(ii) $\phi(x) = q(p^{-1}(x))$ for any $x \in X$.

Remark 1.2. It should be noted [7, page 179] that ϕ upper semicontinuous is redundant in Definition 1.1.

Suppose that X and Y are Hausdorff topological spaces. Given a class $\mathscr{X}$ of maps, $\mathscr{X}(X, Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathscr{X}$, and $\mathscr{X}_c$ the set of finite compositions of maps in $\mathscr{X}$. A class $\mathscr{U}$ of maps is defined by the following properties:

(i) $\mathscr{U}$ contains the class $\mathscr{C}$ of single-valued continuous functions;
(ii) each $F \in \mathscr{U}_c$ is upper semicontinuous and compact valued;
(iii) for any polytope P, $F \in \mathscr{U}_c(P, P)$ has a fixed point, where the intermediate spaces of composites are suitably chosen for each $\mathscr{U}$.

Definition 1.3. The map $F \in \mathscr{U}_c^\kappa(X, Y)$ if for any compact subset K of X, there is a $G \in \mathscr{U}_c(K, Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

Examples of $\mathscr{U}_c^\kappa$ maps are the Kakutani maps, the acyclic maps, the O'Neill maps, the approximable maps, and the maps admissible in the sense of Górniewicz.

Let (E, d) be a pseudometric space. For $S \subseteq E$, let $B(S, \epsilon) = \{x \in E : d(x, S) \le \epsilon\}$, $\epsilon > 0$, where $d(x, S) = \inf_{y \in Y} d(x, y)$. The measure of noncompactness [6] of the set $M \subseteq E$ is defined by $\alpha(M) = \inf Q(M)$, where

$$Q(M) = \{\epsilon > 0 : M \subseteq B(A, \epsilon) \text{ for some finite subset } A \text{ of } E\}. \tag{1.2}$$

Let E be a locally convex Hausdorff topological vector space, and let P be a defining system of seminorms on E. Suppose that $F : S \to 2^E$; here $S \subseteq E$. The map F is said to be

a countably P-concentrative mapping [6] if $F(S)$ is bounded, and for $p \in P$ for each countably bounded subset X of S, we have $\alpha_p(F(X)) \leq \alpha_p(X)$, and for $p \in P$ for each countably bounded non-p-precompact subset X of S (i.e., X is not precompact in the pseudonormed space (E, p)), we have $\alpha_p(F(X)) < \alpha_p(X)$; here $\alpha_p(\cdot)$ denotes the measure of noncompactness in the pseudonormed space (E, p). In this paper, when we consider countably P-concentrative maps, it is worth remarking here that in fact, the results hold if the maps are countably condensing in the sense of [13, pages 353, 356].

2. Essential maps and invariant directions

In this section, we begin by presenting a "homotopy"-type property for a general class of maps. Here E is a Hausdorff locally convex topological vector space, C is a closed convex subset of E, $U \subseteq C$ is convex, U is an open subset of E, and $0 \in U$. Notice that $\text{int}_C U = U$ since U is open in C. We will consider maps $F : \overline{U} \to K(C)$; here $\overline{U}$ denotes the closure of U in C. Throughout, our map F will satisfy *one* of the following conditions:

- (H1) F is compact;
- (H2) if $D \subseteq C$ and $D \subseteq \overline{\text{co}}(\{0\} \cup F(\text{co}(\{0\} \cup D) \cap \overline{U}))$, then $\overline{D}$ is compact;
- (H3) F is countably P-concentrative and E is Fréchet (here P is a defining system of seminorms);
- (H4) if $D \subseteq C$ and $D \subseteq \text{co}(\{0\} \cup F(\text{co}(\{0\} \cup D) \cap \overline{U}))$, then $\overline{D}$ is compact and in this case, we also assume that for any relatively compact subset A of $\overline{U}$, we have $F(\overline{A}) \subseteq \overline{F(A)}$; or
- (H5) if $D \subseteq C$, $D \subseteq \text{co}(\{0\} \cup F(\text{co}(\{0\} \cup D) \cap \overline{U}))$ with $K \subseteq D$ countable and $\overline{K} = \overline{D}$, then $\overline{D}$ is compact and in this case, we also assume that
  - (i) for any relatively compact subset A of $\overline{U}$, we have $F(\overline{A}) \subseteq \overline{F(A)}$,
  - (ii) F maps compact sets into relatively compact sets,
  - (iii) for any relatively compact convex set A of E, there exists a countable set $B \subseteq A$ with $\overline{B} = \overline{A}$,
  - (iv) if Q is a compact subset of E, then $\overline{\text{co}}(Q)$ is compact.

Remark 2.1. If F is a Kakutani map, then the condition "for any relatively compact subset A of $\overline{U}$, we have $F(\overline{A}) \subseteq \overline{F(A)}$" can be removed in (H4) and (H5).

Also in this paper, we will consider maps $F : C \to K(C)$ which satisfy (Hi)* for some $i \in \{1, 2, 3, 4, 5\}$. Now (H1)* = (H1), (H3)* = (H3), and the others are defined as follows:

(H2)* if $D \subseteq C$ and $D = \overline{\text{co}}(\{0\} \cup F(D))$, then D is compact;

(H4)* if $D \subseteq C$ and $D = \text{co}(\{0\} \cup F(D))$, then $\overline{D}$ is compact and in this case, we also assume for any relatively compact subset A of C, we have $F(\overline{A}) \subseteq \overline{F(A)}$;

(H5)* if $D \subseteq C$, $D = \text{co}(\{0\} \cup F(D))$ with $K \subseteq D$ countable and $\overline{K} = \overline{D}$, then $\overline{D}$ is compact and in this case, we also assume that
  - (i) for any relatively compact subset A of C, we have $F(\overline{A}) \subseteq \overline{F(A)}$,
  - (ii) F maps compact sets into relatively compact sets,
  - (iii) for any relatively compact convex set A of E, there exists a countable set $B \subseteq A$ with $\overline{B} = \overline{A}$,
  - (iv) if Q is a compact subset of E, then $\overline{\text{co}}(Q)$ is compact.

Remark 2.2. If F is a Kakutani map, then the condition "for any relatively compact subset A of C, we have $F(\overline{A}) \subseteq \overline{F(A)}$" can be removed in (H4)* and (H5)*.

Definition 2.3. The map $F \in LS(\overline{U}, C)$ if $F : \overline{U} \to K(C)$ satisfies condition (A).
    We assume condition (A) is such that

$$
\begin{array}{l}
\text{for any map } F \in LS(\overline{U}, C) \text{ and any continuous} \\
\text{single-valued map } r : E \longrightarrow \overline{U}, \text{ we have that } Fr \\
\text{satisfies condition (A).}
\end{array}
\tag{2.1}
$$

Example 2.4. If condition (A) means the map F is either

  (a) Kakutani;
  (b) acyclic;
  (c) O'Neill;
  (d) approximable;
  (e) admissible (strongly) with respect to Górniewicz; or
  (f) in $\mathcal{U}_c^\kappa(\overline{U}, C)$,

then clearly (2.1) holds.

    Fix $i \in \{1, 2, 3, 4, 5\}$.

Definition 2.5. The map $F \in LS^i(\overline{U}, C)$ if $F \in LS(\overline{U}, C)$ satisfies (Hi).

Definition 2.6. The map $F \in LS^i_{\partial U}(\overline{U}, C)$ if $F \in LS^i(\overline{U}, C)$ with $x \notin Fx$ for $x \in \partial U$; here ∂U denotes the boundary of U in C.

Definition 2.7. A map $F \in LS^i_{\partial U}(\overline{U}, C)$ is essential in $LS^i_{\partial U}(\overline{U}, C)$ if for every $G \in LS^i_{\partial U}(\overline{U}, C)$ with $G|_{\partial U} = F|_{\partial U}$, there exists $x \in U$ with $x \in G(x)$.

Definition 2.8. The map $F \in ELS^i(C, C)$ if $F \in LS(C, C)$ satisfies (Hi)*.

THEOREM 2.9. *Fix $i \in \{1, 2, 3, 4, 5\}$ and let E be a Hausdorff locally convex topological vector space, C a closed convex subset of E, $U \subseteq C$ convex, U an open subset of E, $0 \in U$, and $F \in LS^i(\overline{U}, C)$, and assume that (2.1) and the following conditions are satisfied:*

$$x \notin \lambda Fx \quad \text{for } x \in \partial U, \lambda \in (0, 1], \tag{2.2}$$

$$\text{any map } \Phi \in ELS^i(C, C) \text{ has a fixed point.} \tag{2.3}$$

Then F is essential in $LS^i_{\partial U}(\overline{U}, C)$ (in particular, F has a fixed point in U).

Proof. Let $H \in LS^i_{\partial U}(\overline{U}, C)$ with $H|_{\partial U} = F|_{\partial U}$. We must show that H has a fixed point in U. Let μ be the Minkowski functional on $\overline{U}$ and let $r : E \to \overline{U}$ be given by

$$r(x) = \frac{x}{\max\{1, \mu(x)\}} \quad \text{for } x \in E. \tag{2.4}$$

Let $G = Hr$. Now $G \in LS(C, C)$ from (2.1). Next we show that $G \in ELS^i(C, C)$. We will just consider the case $i = 4$ since the cases $i = 2$ and $i = 5$ are similar and the cases $i = 1$ and $i = 3$ are immediate. Let $i = 4$ and we show that G satisfies (H4)*. Since $H \in LS^4(\overline{U}, C)$

and r is continuous, we have

$$G(\overline{A}) = H(r(\overline{A})) \subseteq H(\overline{r(A)}) \subseteq \overline{H(r(A))} = \overline{G(A)} \tag{2.5}$$

for any relatively compact subset A of C (note that $\overline{r(A)}$ is compact since r is continuous). Now let $D \subseteq C$ and $D = \mathrm{co}(\{0\} \cup G(D))$. Then since $r(A) \subseteq \mathrm{co}(\{0\} \cup A)$ for any subset A of E, we have

$$D \subseteq \mathrm{co}\left(\{0\} \cup H\left(\mathrm{co}\left(\{0\} \cup D\right) \cap \overline{U}\right)\right). \tag{2.6}$$

Now since $H \in LS^4(\overline{U}, C)$, we have that $\overline{D}$ is compact, and as a result, $G \in ELS^4(C, C)$.

Now (2.3) guarantees that there exists $y \in C$ with $y \in G(y) = Hr(y)$. Also notice that (2.2) with $H|_{\partial U} = F|_{\partial U}$ guarantees that

$$x \notin \lambda H(x) \quad \text{for } x \in \partial U, \lambda \in (0, 1]. \tag{2.7}$$

Let $z = r(y)$. Then $z \in rH(z)$, that is, $z = r(w)$ for some $w \in H(z)$. Now either $w \in \overline{U}$ or $w \notin \overline{U}$. If $w \in \overline{U} = U \cup \partial U$ (note that $\mathrm{int}_C U = U$ since U is open in E), then $r(w) = w$, so $z = w \in H(z)$, and we are finished (note that (2.7) implies that $z = w \in U$). If $w \notin \overline{U}$, then $z = r(w) = w/\mu(w)$ with $\mu(w) > 1$. Thus $z = \lambda w$ (i.e., $z \in \lambda H(z)$) with $0 < \lambda = 1/\mu(w) < 1$. Note that $z \in \partial U$ since $\mu(z) = \mu(\lambda w) = 1$ (note that $\partial U = \partial_E U$ since $\mathrm{int}_C U = U$). As a result, $z \in \lambda H(z)$ with $\lambda = 1/\mu(w) \in (0, 1)$ and $z \in \partial U$. This of course contradicts (2.7). $\square$

Remark 2.10. In fact, Theorem 2.9 is a homotopy result since we will now show that the zero map is essential in $LS^i_{\partial U}(\overline{U}, C)$. Then the zero map is essential in $LS^i_{\partial U}(\overline{U}, C)$ with $F \cong 0$, and (2.1), (2.2), and (2.3) guarantee (Theorem 2.9) that F is essential in $LS^i_{\partial U}(\overline{U}, C)$.

To show that the zero map is essential in $LS^i_{\partial U}(\overline{U}, C)$, let $\theta \in LS^i_{\partial U}(\overline{U}, C)$ with $\theta|_{\partial U} = \{0\}$. Let μ and r be as in Theorem 2.9 and let $J = \theta r$. As in Theorem 2.9, there exists $y \in C$ with $y \in J(y) = \theta r(y)$. Let $z = r(y)$ and essentially the same argument as in Theorem 2.9 yields $x \in U$ with $z \in \theta(z)$.

Example 2.11. If condition (A) means the map F is either

   (a) Kakutani;
   (b) acyclic;
   (c) O'Neill;
   (d) approximable;
   (e) admissible (strongly) with respect to Górniewicz; or
   (f) in $\mathcal{U}^\kappa_c(\overline{U}, C)$,

then [1, 3, 9] guarantees that (2.3) holds. Note that if condition (A) means the map is Kakutani, then the condition "for any relatively compact subset A of $\overline{U}$, we have $F(\overline{A}) \subseteq \overline{F(A)}$" can be removed from (H4) and (H5).

Now from Theorem 2.9, we obtain the following Birkhoff-Kellogg type theorem.

THEOREM 2.12. *Let $i = 1$ and let E be a Hausdorff locally convex topological vector space, C a closed convex subset of E, $U \subseteq C$ convex, U an open subset of E, $0 \in U$, and $F \in LS^1(\overline{U}, C)$, and assume that (2.1) and (2.3) (with $i = 1$) hold. Also assume that condition (A) satisfies the following condition:*

$$\text{for any map } F \in LS(\overline{U}, C) \text{ and any } \lambda \in \mathbb{R}, \tag{2.8}$$
$$\text{we have that } \lambda F \text{ satisfies condition (A).}$$

Finally, suppose that the following condition holds:

$$\exists \mu \in \mathbb{R} \quad \text{with} \quad \mu F(\overline{U}) \cap \overline{U} = \varnothing. \tag{2.9}$$

Then there exist $\lambda \in (0,1)$ and $x \in \partial U$ with $(\lambda^{-1}\mu^{-1})x \in Fx$; here $\mu \neq 0$ is chosen as in (2.9).

Remark 2.13. Notice that $0 \in U$ guarantees that $\mu \neq 0$ in (2.9).

Remark 2.14. If $\mu = 1$ in (2.9), then assumption (2.8) is not needed in the statement of Theorem 2.12.

Example 2.15. If condition (A) means the map F is either
   (a) Kakutani;
   (b) acyclic;
   (c) O'Neill;
   (d) approximable; or
   (e) admissible (strongly) with respect to Górniewicz,
then clearly (2.8) (and of course (2.1) and (2.3) (with $i = 1$)) is true.

Proof of Theorem 2.12. Let $\mu \neq 0$ be chosen as in (2.9). Now (2.8) guarantees that $\mu F \in LS^1(\overline{U}, C)$. Also (2.9) guarantees that μF has no fixed points in $\overline{U}$. Theorem 2.9 (applied to μF) guarantees that there exist $\lambda \in (0,1)$ and $x \in \partial U$ with $x \in \lambda(\mu F)x$. As a result $(\lambda^{-1}\mu^{-1})x \in Fx$ and we are finished. $\square$

THEOREM 2.16. *Fix $i = 3$ and let E be a Hausdorff locally convex topological vector space, C a closed convex subset of E, $U \subseteq C$ convex, U an open subset of E, $0 \in U$, and $F \in LS^3(\overline{U}, C)$, and assume that (2.1) and (2.3) (with $i = 3$) hold. Also assume that condition (A) satisfies the following condition:*

$$\text{for any map } F \in LS(\overline{U}, C) \text{ and any } \lambda \in \mathbb{R} \text{ with } |\lambda| \leq 1, \tag{2.10}$$
$$\text{we have that } \lambda F \text{ satisfies condition (A).}$$

Finally, suppose that the following condition is satisfied:

$$\exists \mu \in \mathbb{R} \quad \text{with} \quad |\mu| \leq 1, \quad \mu F(\overline{U}) \cap \overline{U} = \varnothing. \tag{2.11}$$

Then there exists $\lambda \in (0,1)$ and $x \in \partial U$ with $(\lambda^{-1}\mu^{-1})x \in Fx$.

Remark 2.17. If $\mu = 1$ in (2.11), then assumption (2.10) is not needed in the statement of Theorem 2.16.

Proof. Let $\mu \neq 0$ be chosen as in (2.11). Now $\mu F \in LS(\overline{U}, C)$ from (2.10) and it is easy to check that $\mu F \in LS^3(\overline{U}, C)$ since $|\mu| \leq 1$. Apply Theorem 2.9 to μF. □

Theorem 2.18. *Fix $i = 2$ and let E be a Hausdorff locally convex topological vector space, C a closed convex subset of E, $U \subseteq C$ convex, U an open subset of E, $0 \in U$, and $F \in LS^2(\overline{U}, C)$, and assume that (2.1), (2.3) (with $i = 2$), (2.8), and (2.9) hold. Also assume that the following condition is satisfied:*

$$\text{if } D \subseteq C \text{ with } D \subseteq \overline{co}(\{0\} \cup \mu F(co(\{0\} \cup D) \cap \overline{U})), \\ \text{then } \overline{D} \text{ is compact; here } \mu \text{ is as in (2.9).} \tag{2.12}$$

Then there exists $\lambda \in (0,1)$ and $x \in \partial U$ with $(\lambda^{-1}\mu^{-1})x \in Fx$.

Remark 2.19. If $\mu = 1$ in (2.9), then assumptions (2.8) and (2.12) are not needed in the statement of Theorem 2.18.

Proof of Theorem 2.18. Let $\mu \neq 0$ be chosen as in (2.9). Now $\mu F \in LS^2(\overline{U}, C)$ from (2.8) and (2.12). Apply Theorem 2.9 to μF. □

Remark 2.20. One could also obtain an analogue of Theorem 2.18 for the cases $i = 4$ and $i = 5$. We leave the details to the reader.

In Theorem 2.12, if $\mu > 0$ in (2.9), we say that $F|_{\partial U}$ has an invariant direction. We complete this paper by presenting one invariant direction result.

Theorem 2.21. *Let $i = 1$, $E = (E, \|\cdot\|)$ an infinite-dimensional normed linear space, $C = E$, $U = B$, and $F \in LS^1(\overline{B}, E)$, and assume that (2.1), (2.3) (with $i = 1$), and (2.8) hold; here $B = \{x \in E : \|x\| < 1\}$. In addition, suppose that the following two conditions are satisfied:*

$$\text{for any continuous map } r : \overline{B} \longrightarrow S, \\ \text{we have that } Fr \text{ satisfies condition (A),} \tag{2.13}$$

$$0 \notin \overline{F(S)}; \tag{2.14}$$

here $S = \{x \in E : \|x\| = 1\}$. Then F has an invariant direction.

Example 2.22. If condition (A) means the map F is either
  (a) Kakutani;
  (b) acyclic;
  (c) O'Neill;
  (d) approximable;
  (e) admissible (strongly) with respect to Górniewicz; or
  (f) in $\mathcal{U}_c^\kappa(\overline{U}, C)$,
then (2.13) holds.

Remark 2.23. In Theorem 2.21, $F \in LS^1(\overline{B}, E)$ could be replaced by $F \in LS^1(S, E)$.

Remark 2.24. In Theorem 2.21, we could replace B by any open set U of E with $0 \in U$ (here E is any Hausdorff locally convex topological vector space), provided ∂U is a retract of $\overline{U}$, and in this case, (2.14) is replaced by $\exists \mu > 0$ with $\mu F(\partial U) \cap \overline{U} = \varnothing$ (note that if $\mu = 1$, then assumption (2.8) is not needed in the statement of Theorem 2.21).

Proof of Theorem 2.21. We know [5] that there exists a continuous retraction $r : \overline{B} \to S$. Let $G = Fr$ and notice that $G \in LS(\overline{B}, E)$ from (2.13). We now claim that there exists $\mu > 0$ with

$$\mu F(S) \cap \overline{B} = \varnothing. \tag{2.15}$$

If (2.15) is true, then

$$\mu G(\overline{B}) \cap \overline{B} = \varnothing, \tag{2.16}$$

and so Theorem 2.12 (applied to G with $U = B$ and $C = E$) guarantees that there exist $\lambda \in (0, 1)$ and $x \in \partial B = S$ with $\lambda^{-1}\mu^{-1}x \in Gx = Frx = Fx$, and we are finished. It remains to prove (2.15), but this is immediate since $0 \notin \overline{F(S)}$ (i.e., if (2.15) was false, then for each $n \in \{1, 2, \dots\}$, there exist $y_n \in F(S)$ and $w_n \in \overline{B}$ with $y_n = (1/n)w_n$).   □

References

[1]   R. P. Agarwal and D. O'Regan, *Collectively fixed point theorems*, Nonlinear Anal. Forum **7** (2002), no. 2, 167–179.

[2]   ———, *Homotopy and Leray-Schauder principles for multi maps*, Nonlinear Anal. Forum **7** (2002), no. 1, 103–111.

[3]   ———, *Fixed point theory of Mönch type for general classes of admissible maps*, Nonlinear Funct. Anal. Appl. **8** (2003), 609–622.

[4]   H. Ben-El-Mechaiekh and P. Deguire, *Approachability and fixed points for nonconvex set-valued maps*, J. Math. Anal. Appl. **170** (1992), no. 2, 477–500.

[5]   Y. Benyamini and Y. Sternfeld, *Spheres in infinite-dimensional normed spaces are Lipschitz contractible*, Proc. Amer. Math. Soc. **88** (1983), no. 3, 439–445.

[6]   J. Daneš, *Generalized concentrative mappings and their fixed points*, Comment. Math. Univ. Carolin. **11** (1970), 115–136.

[7]   L. Górniewicz and M. Slosarski, *Topological essentiality and differential inclusions*, Bull. Austral. Math. Soc. **45** (1992), no. 2, 177–193.

[8]   A. Granas, *On the Leray-Schauder alternative*, Topol. Methods Nonlinear Anal. **2** (1993), no. 2, 225–231.

[9]   D. O'Regan, *A unified fixed point theory for countably P-concentrative multimaps*, Appl. Anal. **81** (2002), no. 3, 565–574.

[10]   S. Park, *Generalized Leray-Schauder principles for condensing admissible multifunctions*, Ann. Mat. Pura Appl. (4) **172** (1997), 65–85.

[11]   ———, *A unified fixed point theory of multimaps on topological vector spaces*, J. Korean Math. Soc. **35** (1998), no. 4, 803–829.

[12]   S. Reich, *A remark on set-valued mappings that satisfy the Leray-Schauder condition*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **61** (1976), no. 3-4, 193–194.

[13]   M. Väth, *Fixed point theorems and fixed point index for countably condensing maps*, Topol. Methods Nonlinear Anal. **13** (1999), no. 2, 341–363.

R. P. Agarwal: Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, FL 32901-6975, USA
E-mail address: agarwal@fit.edu

Donal O'Regan: Department of Mathematics, National University of Ireland, Galway, Ireland
E-mail address: donal.oregan@nuigalway.ie