HEMIEQUILIBRIUM PROBLEMS

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We consider a new class of equilibrium problems, known as hemiequilibrium problems. Using the auxiliary principle technique, we suggest and analyze a class of iterative algorithms for solving hemiequilibrium problems, the convergence of which requires either pseudomonotonicity or partially relaxed strong monotonicity. As a special case, we obtain a new method for hemivariational inequalities. Since hemiequilibrium problems include hemivariational inequalities and equilibrium problems as special cases, the results proved in this paper still hold for these problems.

1. Introduction

Variational inequalities theory, introduced in 1964, has emerged as a powerful tool to investigate and study a wide class of unrelated problems arising in industrial, regional, physical, pure, and applied sciences in a unified and general framework. The ideas and techniques of the variational inequalities are being applied in a variety of diverse areas and prove to be productive and innovative. Variational inequalities have been extended and generalized in several directions using novel and new techniques; see [1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]. There are significant developments of variational inequalities related to multivalued, nonmonotone, nonconvex optimization and structural analysis. An important and useful generalization of variational inequalities is a class of variational inequalities, which is known as hemivariational inequalities. The hemivariational inequalities were introduced and investigated by Panagiotopoulos [26] by using the concept of the generalized directional derivatives of nonconvex and nondifferentiable functions. This class has important applications in structural analysis and nonconvex optimization. In particular, it has been shown [3] that if a nonsmooth and nonconvex superpotential of a structure is quasidifferentiable, then these problems can be studied via hemivariational inequalities. The solution of the hemivariational inequalities gives the position of the state equilibrium of the structure. It is worth mentioning that hemivariational inequalities can be viewed as

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a special case of mildly nonlinear variational inequalities, considered and introduced by Noor [17]. However, numerical techniques considered for solving mildly nonlinear variational inequalities cannot be extended to hemivariational inequalities due to the presence of nonlinear and nondifferentiable terms. For the applications and formulation of the hemivariational inequalities, see [3, 15, 26, 27] and the references therein.

Equally important is the field of equilibrium problems. Equilibrium problems related to variational inequalities were introduced by Blum and Oettli [1] and Noor and Oettli [25]. It has been shown that variational inequalities, fixed-point problems, Nash equilibrium problems, and saddle-point problems can be studied in the framework of equilibrium problems. For recent applications and numerical methods for solving equilibrium problems, see [1, 6, 7, 8, 13, 14, 16, 22, 23, 24, 25] and the references therein.

Thus it is clear that hemivariational inequalities and equilibrium problems are different generalizations of variational inequalities. It is natural to consider the unification of these two generalized problems. Motivated and inspired by this fact, we consider here another class of equilibrium problems, which is called the *hemiequilibrium problems.* This class includes the hemivariational inequalities and equilibrium problems as special cases.

Variational inequalities and equilibrium problems have witnessed an explosive growth in theoretical advances and algorithmic developments and applications across almost all disciplines of engineering, pure, and applied sciences. As a result of interaction between different branches of mathematical and engineering sciences, we now have a variety of techniques to suggest and analyze various iterative algorithms for solving hemivariational inequalities and equilibrium problems. Analysis of these problems requires a blend of techniques and ideas from convex analysis, functional analysis, numerical analysis, and nonsmooth analysis. There are several methods for solving variational inequalities and equilibrium problems. Due to the nature of the hemiequilibrium problems, projection and resolvent methods cannot be applied for solving them. In recent years, the auxiliary principle technique is being used to suggest and analyze some iterative methods for solving variational inequalities and equilibrium problems. This technique is basically due to Lions and Stampacchia [11] and was used by Noor [17] to obtain the existence results for the mildly (strongly) nonlinear variational inequalities. However, Glowinski et al. [9] used this technique to study the existence problem for mixed variational inequalities. The main idea involving this technique is to first consider an auxiliary problem and then to show that the solution of the auxiliary problem is the solution of the original problem by using the fixed-point approach. Noor [16, 21, 22, 23, 24] has used this approach to suggest and analyze some iterative methods for solving various classes of variational inequalities and equilibrium problems. To the best of our knowledge, the auxiliary principle technique has not been applied for hemivariational inequalities and hemiequilibrium problems. In this paper, we show that this technique can be used to suggest some iterative schemes for hemiequilibrium problems. We also prove that the convergence of these methods requires either pseudomonotonicity or partially relaxed strong monotonicity. These are weaker conditions than monotonicity. As a special case, we obtain new iterative schemes for solving hemivariational inequalities. The comparison of these methods with other methods is a subject of future research.

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let K be a nonempty closed convex set in H. Let $f : H \to \mathbb{R}$ be a locally Lipschitz continuous function. Let Ω be an open bounded subset of $\mathbb{R}^n$.

First of all, we recall the following concepts and results from nonsmooth analysis; see Clarke [2].

Definition 2.1. Let f be locally Lipschitz continuous at a given point $x \in H$ and let v be any other vector in H. Clarke's generalized directional derivative of f at x in the direction v, denoted by $f^0(x,v)$, is defined as

$$f^0(x;v) = \lim_{t\to 0^+} \sup_{h\to 0} \frac{f(x+h+tv)-f(x+h)}{t}. \tag{2.1}$$

The generalized gradient of f at x, denoted by $\partial f(x)$, is defined to be a subdifferential of the function $f^0(x;v)$ at 0, that is,

$$\partial f(x) = \{w \in H : \langle w,v\rangle \le f^0(x;v),\ \forall v \in H\}. \tag{2.2}$$

Lemma 2.2. *Let f be locally Lipschitz continuous at a given point $x \in H$ with a constant L. Then*

(i) *$\partial f(x)$ is a noneempty compact subset of H and $\|\xi\| \le L$ for each $\xi \in \partial f(x)$;*
(ii) *for every $v \in H$, $f^0(x;v) = \max\{\langle \xi, c\rangle : \xi \in \partial f(x)\}$;*
(iii) *the function $v \to f^0(x;v)$ is finite, positively homogeneous, subadditive, convex, and continuous;*
(iv) *$f^0(x;-v) = (-f)^0(x,v)$;*
(v) *$f^0(x;v)$ is upper semicontinuous as a function of $(x;v)$;*
(vi) *for all $x \in H$, there exists a constant $\alpha > 0$ such that*

$$|f^0(x,v)| \le \alpha\|v\|, \quad \forall v \in H. \tag{2.3}$$

If f is convex on K and locally Lipschitz continuous at $x \in K$, then $\partial f(x)$ coincides with the subdifferential $f'(x)$ of f at x in the sense of convex analysis, and $f^0(x;v)$ coincides with the directional derivative $f'(x;v)$ for each $v \in H$.

For a given nonlinear function $F(\cdot,\cdot) : K \times K \to \mathbb{R}$, consider the problem of finding $u \in K$ such that

$$F(u,v) + \int_\Omega f^0(x,u;v-u)\,d\Omega \ge 0, \quad \forall v \in K. \tag{2.4}$$

Here, $f^0(x,u;v-u) := f^0(x,u(x);v(x)-u(x))$ denotes the generalized directional derivative of the function $f(x,\cdot)$ at $u(x)$ in the direction $v(x)-u(x)$. Problems of type (2.4) are called the *hemiequilibrium problems.*

If $F(u,v) = \langle Tu, v-u\rangle$, then problem (2.4) is equivalent to finding $u \in K$ such that

$$\langle Tu, v-u\rangle + \int_\Omega f^0(x,u;v-u)\,d\Omega \ge 0, \quad \forall v \in K, \tag{2.5}$$

which are known as the hemivariational inequalities introduced and studied by Panagiotopoulos [26] in order to formulate variational principles connected to energy functions which are neither convex nor smooth. It has been shown that the technique of hemivariational inequalities is very efficient to describe the behavior of complex structure arising in engineering and industrial sciences; see [3, 15, 26, 27] and the references therein.

If $f = 0$, then problem (2.4) is equivalent to finding $u \in K$ such that

$$F(u,v) \geq 0, \quad \forall v \in K. \tag{2.6}$$

Problem (2.6) is called the equilibrium problem, which is due to Blum and Oettli [1] and Noor and Oettli [25]. For recent applications and numerical methods for solving equilibrium problems, see [1, 6, 13, 16, 22, 23, 24].

Definition 2.3. The function $F(\cdot,\cdot) : K \times K \to H$ is said to be

(a) *monotone* if

$$F(u,v) + F(v,u) \leq 0, \quad \forall u,v \in K; \tag{2.7}$$

(b) *pseudomonotone* with respect to $\int_\Omega f^0(x,u;v-u)d\Omega$ if

$$\begin{aligned} &F(u,v) + \int_\Omega f^0(x,u;v-u)d\Omega \geq 0 \\ &\Longrightarrow -F(v,u) + \int_\Omega f^0(x,u;v-u)d\Omega \geq 0, \quad \forall u,v \in K; \end{aligned} \tag{2.8}$$

(c) *partially relaxed strongly monotone* if there exists a constant $\gamma > 0$ such that

$$F(u,v) + F(v,z) \leq \gamma \|u-z\|^2, \quad \forall u,v,z \in K; \tag{2.9}$$

(d) *hemicontinuous* if the mapping $t \in [0,1]$ implies that $F(u+t(v-u),v)$ is continuous for all $u,v \in K$.

Note that for $z = u$, partially relaxed strong monotonicity reduces to monotonicity. This shows that partially relaxed strong monotonicity implies monotonicity, but the converse is not true.

Definition 2.4. The function $\int_\Omega f^0(x,u;v-u)d\Omega$ is said to be partially relaxed strongly monotone if there exists a constant $\alpha > 0$ such that

$$\int_\Omega f^0(x,u;v-u)d\Omega + \int_\Omega f^0(x,z;u-v)d\Omega \leq \alpha \|z-v\|^2, \quad \forall u,v,z \in H. \tag{2.10}$$

Note that for $z = v$, partially relaxed strong monotonicity reduces to monotonicity.

Lemma 2.5. *Let the function $F(\cdot,\cdot)$ be hemicontinuous, pseudomonotone with respect to the function $\int_\Omega f^0(x,u;v-u)d\Omega$, and convex in the second argument. Then problem (2.4) is equivalent to finding $u \in K$ such that*

$$-F(v,u) + \int_\Omega f^0(x,u;v-u)d\Omega \geq 0, \quad \forall v \in K. \tag{2.11}$$

Proof. Let $u \in K$ be a solution of (2.4). Then

$$F(u,v) + \int_{\Omega} f^0(x,u;v-u)d\Omega \geq 0, \quad \forall v \in K, \tag{2.12}$$

which implies that

$$F(v,u) \leq \int_{\Omega} f^0(x,u;v-u)d\Omega, \quad \forall v \in K, \tag{2.13}$$

since $F(\cdot,\cdot)$ is pseudomonotone with respect to $\int_{\Omega} f^0(x,u;v-u)d\Omega$.

For $u,v \in K$, $t \in [0,1]$, $v_t = u + t(v-u) \in K$, since K is a convex set. Taking $v = v_t$ in (2.11), we have

$$F(v_t,u) \leq t\left\{\int_{\Omega} f^0(x,u;v-u)d\Omega\right\}. \tag{2.14}$$

Now, using (2.14), we have

$$\begin{aligned} 0 &\leq F(v_t,v_t) \\ &\leq tF(v_t,v) + (1-t)F(v_t,u) \\ &\leq tF(v_t,v) + t(1-t)\left\{\int_{\Omega} f^0(x,u;v-u)d\Omega\right\}. \end{aligned} \tag{2.15}$$

Dividing inequality (2.15) by t and taking the limit as $t \to 0$, since $F(\cdot,\cdot)$ is hemicontinuous, we have $F(u,v) + \int_{\Omega} f^0(x,u;v-u)d\Omega \geq 0$ for all $v \in K$, the required result (2.4). □

Remark 2.6. From Lemma 2.5, we see that problems (2.4) and (2.11) are equivalent. Problem (2.11) is called the *dual hemiequilibrium problem*. One can easily show that the solution set of problem (2.11) is a closed convex set. Lemma 2.2 can be viewed as a natural generalization of Minty's result.

Definition 2.7. A function $f : K \to H$ is said to be strongly convex if there exists a constant $\beta > 0$ such that

$$f(u + t(v-u)) \leq (1-t)f(u) + tf(v) - t(1-t)\beta\|v-u\|^2, \quad \forall u,v \in K,\ t \in [0,1]. \tag{2.16}$$

If the strongly convex function is differentiable, then

$$f(v) - f(u) \geq \langle f'(u), v-u\rangle + \beta\|v-u\|^2, \quad \forall u,v \in K, \tag{2.17}$$

and conversely.

3. Main results

In this section, we suggest and analyze some iterative methods for hemiequilibrium problems (2.4) using the auxiliary principle technique of Glowinski et al. [9] as developed by Noor [16, 21, 22, 23, 24].

For a given $u \in K$, consider the auxiliary problem of finding a unique $w \in K$ such that

$$\rho F(w,v) + \langle E'(w) - E'(u), v - w\rangle + \rho \int_\Omega f^0(x,u;v-w)\,d\Omega \geq 0, \quad \forall v \in K, \tag{3.1}$$

where $\rho > 0$ is a constant and $E'(u)$ is the differential of a strongly convex function $E(u)$ at $u \in K$. Since $E(u)$ is a strongly convex function, problem (3.1) has a unique solution. We note that if $w = u$, then clearly w is a solution of the hemiequilibrium problem (2.4). This observation enables us to suggest and analyze the following iterative method for solving (2.4).

Algorithm 3.1. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\rho F(u_{n+1},v) + \langle E'(u_{n+1}) - E'(u_n), v - u_{n+1}\rangle + \rho \int_\Omega f^0(x,u_n;v-u_{n+1})\,d\Omega \geq 0, \quad \forall v \in K, \tag{3.2}$$

where $\rho > 0$ is a constant.

Algorithm 3.1 is called the proximal method for solving the hemiequilibrium problem (2.4). In passing, we remark that the proximal point method was suggested by Martinet [12] in the context of convex programming problems as a regularization technique. For the recent developments and applications of the proximal point algorithms, see [4, 5, 24, 29].

If $F(u,v) = \langle Tu, v-u\rangle$, then Algorithm 3.1 reduces to the following.

Algorithm 3.2. For a given $u_0 \in H$, calculate the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho T u_{n+1} + E'(u_{n+1}) - E'(u_n), v - u_{n+1}\rangle + \rho \int_\Omega f^0(x,u_n;v-u_{n+1})\,d\Omega \geq 0, \quad \forall v \in K. \tag{3.3}$$

Algorithm 3.2 is called the proximal point method for solving hemivariational inequalities (2.5) and appears to be a new one.

If $f(x,u) = 0$, then Algorithm 3.1 collapses to the following.

Algorithm 3.3. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\rho F(u_n,v) + \langle E'(u_{n+1}) - E'(u_n), v - u_{n+1}\rangle \geq 0, \quad \forall v \in K. \tag{3.4}$$

Algorithm 3.3 is due to Noor [16, 23, 24], for solving the equilibrium problems (2.6).

In brief, for a suitable and appropriate choice of the operators and the spaces, one can obtain a number of known and new algorithms for solving variational-like inequalities and related problems.

Theorem 3.4. *Let $F(\cdot,\cdot)$ be pseudomonotone with respect to $\int_\Omega f^0(x,u;v-u)d\Omega$. Let $\int_\Omega f^0(\cdot;\cdot)d\Omega$ be partially relaxed strongly monotone with constant $\alpha > 0$ and let E be a* *differentiable strongly convex function with module $\beta > 0$. If $0 < \rho < \beta/\alpha$, then the approximate solution u_{n+1} obtained from Algorithm 3.1 converges to the exact solution $u \in K$ satisfying (2.4).*

Proof. Let $u \in K$ be a solution of (2.4). Then $F(u,v) + \int_\Omega f^0(x,u;v-u)d\Omega \geq 0$, for all $v \in K$, implies that

$$-F(v,u) + \int_\Omega f^0(x,u;v-u)d\Omega \geq 0, \quad \forall v \in K, \tag{3.5}$$

since $F(\cdot,\cdot)$ is pseudomonotone with respect to $\int_\Omega f^0(x,u;v-u)d\Omega$.

Taking $v = u$ in (3.2) and $v = u_{n+1}$ in (3.5), we have

$$\begin{aligned} \rho F(u_{n+1},u) + \langle E'(u_{n+1}) - E'(u_n), u - u_{n+1}\rangle &\geq -\rho\int_\Omega f^0(x,u_n;u-u_{n+1})d\Omega, \\ -F(u_{n+1},u) + \int_\Omega f^0(x,u;u_{n+1}-u)d\Omega &\geq 0. \end{aligned} \tag{3.6}$$

We now consider the function

$$\begin{aligned} B(u,w) &= E(u) - E(w) - \langle E'(w), u-w\rangle \\ &\geq \beta\|u-w\|^2 \quad \text{(using strong convexity of } E\text{)}. \end{aligned} \tag{3.7}$$

Now, combining (3.6) and (3.7), we have

$$\begin{aligned} B(u,u_n) - B(u,u_{n+1}) &= E(u_{n+1}) - E(u_n) - \langle E'(u_{n+1}), u_{n+1} - u_n\rangle \\ &\quad + \langle E'(u_{n+1}) - E'(u_n), u - u_{n+1}\rangle \\ &\geq \beta\|u_{n+1}-u_n\|^2 + \langle E'(u_{n+1}) - E'(u_n), u - u_{n+1}\rangle \\ &\geq \beta\|u_{n+1}-u_n\|^2 - \rho F(u_{n+1},u) - \rho\int_\Omega f^0(x,u_n;u-u_{n+1})d\Omega \\ &\geq \beta\|u_{n+1}-u_n\|^2 - \rho\int_\Omega f^0(x,u;u_{n+1}-u)d\Omega \\ &\quad - \rho\int_\Omega f^0(x,u_n;u-u_{n+1})d\Omega \\ &\geq \beta\|u_{n+1}-u_n\|^2 - \rho\alpha\|u_{n+1}-u_n\|^2, \end{aligned} \tag{3.8}$$

where we have used the fact that $\int_\Omega f^0(x,u;v-u)d\Omega$ is partially relaxed strongly monotone with constant $\alpha > 0$. If $u_{n+1} = u_n$, then clearly u_n is a solution of the hemiequilibrium problem (2.4). Otherwise, for $0 < \rho < \beta/\alpha$, it follows that $B(u,u_n) - B(u,u_{n+1})$ is nonnegative, and we must have

$$\lim_{n\to\infty}\|u_{n+1}-u_n\| = 0. \tag{3.9}$$

Now, using the technique of Zhu and Marcotte [31], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point u satisfying the hemiequilibrium problem (2.4). □

It is well known that to implement the proximal point methods, one has to find the approximate solution implicitly, which is itself a difficult problem. To overcome this drawback, we now consider another method for solving (2.4) using the auxiliary principle technique.

For a given $u \in K$, find a unique $w \in K$ such that

$$\rho F(u,v) + \langle E'(w) - E'(u), v - w\rangle + \rho \int_{\Omega} f^{0}(x,u;v-w)\,d\Omega, \quad \forall v \in K, \tag{3.10}$$

where $E'(u)$ is the differential of a strongly convex function $E(u)$ at $u \in K$. Problem (3.10) has a unique solution, since E is a strongly convex function. Note that problems (3.1) and (3.10) are quite different. It is clear that for $w = u$, w is a solution of (2.4). This fact allows us to suggest and analyze another iterative method for solving the hemiequilibrium problem (2.4).

Algorithm 3.5. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\rho F(u_n,v) + \langle E'(u_{n+1}) - E'(u_n), v - u_{n+1}\rangle \geq -\rho \int_{\Omega} (x,u_n,v-u_{n+1})\,d\Omega, \quad \forall v \in K. \tag{3.11}$$

Note that for $F(u,v) = \langle Tu, v-u\rangle$, Algorithm 3.5 reduces to the following.

Algorithm 3.6. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho Tu_n + E'(u_{n+1}) - E'(u_n), v - u_{n+1}\rangle \geq -\rho \int_{\Omega} (x,u_n,v-u_{n+1})\,d\Omega, \quad \forall v \in K. \tag{3.12}$$

Algorithm 3.6 is for solving the hemivariational inequalities (2.5) and appears to be a new one. Similarly, for a suitable and appropriate choice of the operators and the spaces, one can obtain various known and new algorithms for solving equilibrium problems and variational inequalities.

We now consider the convergence analysis of Algorithm 3.5 essentially using the technique of Theorem 3.4. For the sake of completeness and to convey an idea of the technique, we sketch the main points.

THEOREM 3.7. *Let $F(\cdot,\cdot)$ and $\int_{\Omega} f^{0}(x,u;v-u)\,d\Omega$ be partially relaxed strongly monotone with constants $\gamma > 0$ and $\alpha > 0$, respectively. If E is a strongly convex function with modulus $\beta > 0$ and $0 < \rho < \beta/(\alpha+\gamma)$, then the approximate solution u_{n+1} obtained from Algorithm 3.5 converges to a solution of (2.4).*

Proof. Let $u \in K$ be a solution of (2.4). Setting $v = u_{n+1}$ in (2.4) and $v = u$ in (3.11), we have

$$\begin{gathered} F(u,u_{n+1}) + \int_{\Omega} f^{0}(x,u;u_{n+1}-u)\,d\Omega \geq 0, \\ \rho F(u_n,u) + \langle E'(u_{n+1}) - E'(u_n), u - u_{n+1}\rangle \geq -\rho \int_{\Omega} f^{0}(x,u_n;u-u_{n+1})\,d\Omega. \end{gathered} \tag{3.13}$$

As in Theorem 3.4 and from (3.13), we have

$$\begin{aligned}
B(u,u_n)-B(u,u_{n+1}) &= E(u_{n+1})-E(u_n)-\langle E'(u_{n+1}),u_{n+1}-u_n\rangle \\
&\quad +\langle E'(u_{n+1})-E'(u_n),u-u_{n+1}\rangle \\
&\geq \beta\|u_{n+1}-u_n\|^2+\langle E'(u_{n+1})-E'(u_n),u-u_{n+1}\rangle \\
&\geq \beta\|u_{n+1}-u_n\|^2-\rho F(u_n,u)-\rho\int_\Omega f^0(x,u_n;u-u_{n+1})\,d\Omega \\
&\geq \beta\|u_{n+1}-u_n\|^2-\rho\{F(u_n,u)+F(u,u_{n+1})\} \\
&\quad -\rho\Big\{\int_\Omega f^0(x,u;u_{n+1}-u)\,d\Omega+\int_\Omega f^0(x,u_n;u-u_{n+1})\,d\Omega\Big\} \\
&\geq \beta\|u_{n+1}-u_n\|^2-\rho(\alpha+\gamma)\|u_{n+1}-u_n\|^2,
\end{aligned} \tag{3.14}$$

where we have used the fact that $F(\cdot,\cdot)$ and $\int_\Omega f^0(x,\cdot;\cdot)d\Omega$ are partially relaxed strongly monotone with constants $\alpha>0$ and $\gamma>0$, respectively.

If $u_{n+1}=u_n$, then clearly u_n is a solution of the hemiequilibrium problem (2.4). Otherwise, for $0<\rho<\beta/(\alpha+\gamma)$, it follows that $B(u,u_n)-B(u,u_{n+1})$ is nonnegative, and we must have

$$\lim_{n\to\infty}\|u_{n+1}-u_n\|=0. \tag{3.15}$$

Now, using the technique of Zhu and Marcotte [31], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point u satisfying the hemiequilibrium problem (2.4). □

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