DOUBLE-BARRIERS-REFLECTED BSDEs WITH JUMPS AND VISCOSITY SOLUTIONS OF PARABOLIC INTEGRODIFFERENTIAL PDEs

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Received 30 April 2004 and in revised form 9 July 2004

We give a probabilistic interpretation of the viscosity solutions of parabolic integrodifferential partial equations with two obstacles via the solutions of forward-backward stochastic differential equations with jumps.

1. Introduction

We consider the following obstacle problem for a parabolic integrodifferential partial equation

$$\begin{gathered}\Big(-\frac{\partial u}{\partial t}-Lu-F\Big)(u-l)^{+}\leq 0,\\ \Big(-\frac{\partial u}{\partial t}-Lu-F\Big)(u-h)^{-}\geq 0,\\ l(t,x)\leq u(t,x)\leq h(t,x),\quad \forall (t,x)\in[0,T]\times\mathbb{R}^{d},\\ u(T,x)=g(x),\quad \forall x\in\mathbb{R}^{d},\end{gathered}\tag{1.1}$$

where

$$F=f\big(t,x,u(t,x),(\nabla u\sigma)(t,x),Bu(t,x)\big),\qquad L=A+K\tag{1.2}$$

with

$$Bu(t,x)=\int_{\mathbb{R}^{l}\setminus\{0\}}\big[u(t,x+\beta(x,e))-u(t,x)\big]\gamma(x,e)\lambda(de),\tag{1.3}$$

and A, K respectively, second-order differential operator and integrodifferential partial operator defined by

$$\begin{aligned}Au(t,x)&=\frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^{*})_{i,j}(t,x)\frac{\partial^{2}u}{\partial x_i\partial x_j}(t,x)+\sum_{i=1}^{d}b_i(t,x)\frac{\partial u}{\partial x_i}(t,x),\\ Ku(t,x)&=\int_{\mathbb{R}^{l}\setminus\{0\}}\big[u(t,x+\beta(x,e))-u(t,x)-\langle\nabla u(t,x),\beta(x,e)\rangle\big]\lambda(de).\end{aligned}\tag{1.4}$$

Journal of Applied Mathematics and Stochastic Analysis 2005:1 (2005) 37–53
DOI: 10.1155/JAMSA.2005.37

In this paper, we obtain a probabilistic interpretation for the viscosity solution of this parabolic integrodifferential variational inequality via the theory of two barriers reflected backward stochastic differential equations (BSDEs) with jumps.

As is well known, BSDEs provide probabilistic formulae for the viscosity solution of semilinear partial differential equations (PDE) (see, e.g., Pardoux and Peng [16]). These results have been next extended to integrodifferential partial equations by Barles et al. [1].

At the same time, El Karoui et al. have introduced in [5] the notion of one-barrier reflected BSDEs, which is a backward equation but the solution is forced to stay above a given continuous obstacle. The authors have established the existence and uniqueness of the solution via a penalization as well as a Picard's iteration method. Next, Hamadène and Ouknine [10] have generalized this result to one-barrier reflected BSDEs with jumps, that is, when the noise is driven by a Brownian motion and an independent Poisson random measure.

The notion of double barriers reflected BSDEs has been introduced by Cvitanić and Karatzas [3], where the solution is forced to remain between two prescribed upper and lower barriers L and H. Then Hamadène et al. [9] and Lepeltier and San Martin [12] have successively improved the result on the existence of a solution when the drift is only continuous and with linear growth.

The main aim of this work is to link the viscosity solution of the parabolic integrodifferential variational inequality (1.1) with the solution (Y,Z,U,K^+,K^-) of the following two barriers reflected BSDE with jumps: for all $0 \le t \le T$,

$$\begin{aligned} Y_t = \xi + \int_t^T F(s,Y_s,Z_s,U_s)\,ds \\ - \int_t^T Z_s\,dW_s + (K_T^+ - K_t^+) - (K_T^- - K_t^-) - \int_t^T \int_{\mathbb{R}^l\setminus\{0\}} U_s(e)\tilde{\mu}(de,ds), \\ L_t \le Y_t \le H_t. \end{aligned} \tag{1.5}$$

The key of the proofs is the existence and uniqueness of a solution for the above BSDE in [15], which is put in a Markovian framework.

The paper is organized as follows. In Section 2, we define the solutions of double reflected BSDE with jumps and present a result of the existence and uniqueness of the solution. The Markovian case is considered in Section 3 and we also give some properties of the corresponding solution. Finally, we deal in the last section with the connection between the solutions of the forward BSDE with jumps (3.10) and the variational inequality (1.1).

2. BSDEs with jumps: existence and uniqueness of a solution

2.1. Notations and assumptions. Let $(\Omega,\mathcal{F},\mathbb{P},\mathcal{F}_t,W_t,\tilde{\mu}_t,\ t\in[0,T])$ be a complete Wiener-Poisson space in $\mathbb{R}^d\times\mathbb{R}^l\setminus\{0\}$ with Lévy's measure λ, that is:

(i) $(\Omega,\mathcal{F},\mathbb{P})$ is a complete probability space with a filtration $(\mathcal{F}_t,\ t\in[0,T])$ that is a right continuous increasing family of complete sub σ-algebras of $\mathcal{F}$, on which are defined two mutually independent processes,

(ii) a d-dimensional standard Wiener process $(W_t,\ t \in [0,T])$ with respect to the filtration $(\mathcal{F}_t,\ t \in [0,T])$,

(iii) a Poisson random measure p on $\mathbb{R}^+ \times E$, where $E \triangleq \mathbb{R}^l \setminus \{0\}$ is equipped with its Borel σ-field $\mathcal{U}$, with compensator $q(dt,de) = dt \times \lambda(de)$, such that $\{\tilde{\mu}([0,t] \times A) = (p-q)([0,t] \times A)\}_{t\in[0,T]}$ is martingale for all $A \in \mathcal{U}$ satisfying $\lambda(A) < \infty$. λ is assumed to be a σ-finite measure on $(E,\mathcal{U})$ satisfying

$$\int_E (1 \wedge |e|^2)\lambda(de) < +\infty. \tag{2.1}$$

Readers are referred to Gīhman-Skorohod [8] or Jacod [11] for more precise definitions and properties of random measures.

We assume that

$$\mathcal{F}_t = \sigma\left[\int_{A\times(0,s]} p(ds,dx);\ s \le t,\ A \in \mathcal{U}\right] \otimes \sigma[W_s;\ s \le t] \otimes \mathcal{N}, \tag{2.2}$$

where $\mathcal{N}$ is the class of $\mathbb{P}$-null sets and $\sigma_1 \otimes \sigma_2$ denotes the σ-field generated by $\sigma_1 \cup \sigma_2$.

We introduce the following spaces:

(i) L^2 of $\mathcal{F}_T$-measurable random variables $\xi : \Omega \to \mathbb{R}$ such that $E|\xi|^2 < +\infty$,

(ii) S^2 of $\mathcal{F}_t$-adapted right continuous with left limit (rcll in short) processes $(Y_t)_{t\le T}$ with values in $\mathbb{R}$ and such that $E[\sup_{0\le t\le T}|Y_t|^2] < +\infty$,

(iii) $H^{2,k}$ of $\mathcal{F}_t$-progressively measurable processes with values in $\mathbb{R}^k$ such that

$$\|Z\|_{H^{2,k}} := E\left[\int_0^T |Z_s|^2 ds\right] < +\infty, \tag{2.3}$$

(iv) $\mathcal{L}^2(\tilde{\mu})$ of mappings $V : \Omega \times [0,T] \times E \to \mathbb{R}$ which are $\mathcal{P} \otimes \mathcal{U}$-measurable and satisfy

$$\|V\|^2_{L^2(\tilde{\mu})} := E\left[\int_0^T ds \int_E (V_s(e))^2 \lambda(de)\right] < +\infty; \tag{2.4}$$

$\mathcal{P}$ is the σ-algebra of predictable sets in $\Omega \times [0,T]$,

(v) $\mathcal{A}^2$ of continuous, increasing, $\mathcal{F}_t$-adapted processes $K : [0,T] \times \Omega \to [0,+\infty)$ with $K(0) = 0$ and $E[(K_T)^2] < +\infty$.

Finally, for a given rcll process $(w_t)_{t\le T}$, we define for any $t \in [0,T]$,

$$w_{t-} = \lim_{s\nearrow t} w_s, \quad (w_{0-} := w_0), \quad w_- = (w_{t-})_{0\le t\le T}. \tag{2.5}$$

Hereafter we have four objects.

(A1) A terminal value $\xi \in L^2$.

(A2) A coefficient "F" which is a map $F : \Omega \times [0,T] \times \mathbb{R}^{1+d} \times L^2(E,\mathcal{U},\lambda;\mathbb{R}) \to \mathbb{R}$, $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{1+d}) \otimes \mathcal{B}(L^2(E,\mathcal{U},\lambda;\mathbb{R}))$-measurable and satisfy

(i) $(F(t,0,0,0))_{t\le T} \in L^2(\Omega \times [0,T], d\mathbb{P} \otimes dt)$, that is,

$$E\left[\int_0^T (F(t,0,0,0))^2 dt\right] < +\infty, \tag{2.6}$$

(ii) F is uniformly Lipschitz with respect to (y,z,v), that is, there exists a constant $k \geq 0$ such that for any $y,y' \in \mathbb{R}$, $z,z' \in \mathbb{R}^d$ and $v,v' \in L^2(E,\mathcal{U},\lambda;\mathbb{R})$,

$$|F(\omega,t,y,z,v) - F(\omega,t,y',z',v')| \leq k(|y-y'| + |z-z'| + \|v-v'\|), \quad \mathbb{P}\text{-a.s.} \tag{2.7}$$

(A3) Two reflecting barriers L, H which are real valued and $\mathcal{P}$-measurable processes satisfying

(i) $E[\sup_{0\leq t\leq T}\{(H_t^-)^2 + (L_t^+)^2\}] < +\infty$, where $L_t^+ := \max\{L_t, 0\}$, $H_t^- := \max\{-H_t, 0\}$,

(ii) $L_t \leq H_t$, for all $0 \leq t \leq T$ and $L_T \leq \xi \leq H_T$, $\mathbb{P}$-a.s.,

(iii) $\{L_t,\ 0 \leq t \leq T\}$ is rcll and its jumping times are inaccessible stopping times (see, e.g., [4]).

2.2. Existence and uniqueness for a BSDE with jumps. The process $(Y_t, Z_t, K_t^+, K_t^-, U_t)_{t\leq T}$ with values in $\mathbb{R}^{(1+d)} \times \mathbb{R}^+ \times \mathbb{R}^+ \times L^2(E,\mathcal{U},\lambda;\mathbb{R})$ is called a solution for the double barriers reflected BSDE with jumps if

(i) $Y \in S^2$, $Z \in H^{2,d}$, $U \in \mathcal{L}^2(\tilde{\mu})$ and $K^\pm \in \mathcal{A}^2$,

(ii) for all $t \leq T$,

$$\begin{aligned} Y_t = \xi &+ \int_t^T F(s,Y_s,Z_s,U_s)\,ds \\ &- \int_t^T Z_s\,dW_s + (K_T^+ - K_t^+) - (K_T^- - K_t^-) - \int_t^T \int_E U_s(e)\tilde{\mu}(de,ds), \end{aligned} \tag{2.8}$$

(iii) for all $t \leq T$, $L_t \leq Y_t \leq H_t$, and $\int_0^T (Y_t - L_t)\,dK_t^+ = \int_0^T (H_t - Y_t)\,dK_t^- = 0$, $\mathbb{P}$-a.s.

The double barriers reflected BSDE (2.8) with jumps associated with (f,ξ,L,H) has a unique solution if the upper barrier satisfies the supplementary following assumption.

(A4) There exists a sequence of processes $(H^n)_{n\geq 0}$ such that

(i)

$$\forall t \leq T, \quad H_t^n \geq H_t^{n+1}, \quad \lim_{n\to+\infty} H_t^n = H_t, \quad \mathbb{P}\text{-a.s.}, \tag{2.9}$$

(ii)

$$\forall n \geq 0,\ \forall t \leq T, \qquad H_t^n = H_0^n + \int_0^T u_s^n\,ds + \int_0^T v_s^n\,dW_s + \int_0^T \int_E w_s^n(e)\tilde{\mu}(de,ds), \tag{2.10}$$

where the processes u^n, v^n, w^n are $\mathcal{F}$-adapted such that

$$\begin{aligned} &\sup_{n\geq 0}\sup_{t\in[0,T]} |u_t^n| \leq M, \qquad E\left[\int_0^T |v_s^n|^2 ds\right]^{1/2} < +\infty, \\ &E\left[\int_0^T \int_E |w_s^n|\lambda(de)\,ds\right]^{1/2} < +\infty, \qquad \forall n \geq 1. \end{aligned} \tag{2.11}$$

We can recall the following result which is proved in [6].

Proposition 2.1. *Assume that (A1)–(A4) hold, then the reflected BSDE with jumps (2.8) associated with* (f,ξ,L,H) *admits one and only one solution.*

From now, we consider the Markovian case in order to give a probabilistic representation of solution of (1.1) via the solution of (2.8).

3. A class of diffusion processes with jumps

We introduce a class of diffusion processes.

(A5) Let $b:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ and $\sigma:[0,T]\times\mathbb{R}^d\to\mathbb{R}^{d\times d}$ be functions such that for some positive constant C, for all $x,x'\in\mathbb{R}^d$ and for all $t\in[0,T]$,

$$\begin{gathered}|b(t,x)-b(t,x')|-|\sigma(t,x)-\sigma(t,x')|\le C|x-x'|,\\ |b(t,x)|-|\sigma(t,x)|\le C(1+|x|).\end{gathered}\tag{3.1}$$

(A6) Let $\beta:\mathbb{R}^d\times E\to\mathbb{R}^d$ be a measurable function such that for some constant K and for all $e\in E$,

$$\begin{gathered}|\beta(x,e)|\le K(1\wedge|e|),\quad \forall x\in\mathbb{R}^d,\\ |\beta(x,e)-\beta(x',e)|\le K|x-x'|(1\wedge|e|),\quad \forall x,x'\in\mathbb{R}^d.\end{gathered}\tag{3.2}$$

For each $(t,x)\in[0,T]\times\mathbb{R}^d$, we consider $\{(X_s^{t,x}),\ s\in[0,T]\}$ the unique solution of the stochastic differential equation

$$X_s^{tx}=x+\int_t^{t\vee s}b(r,X_r^{tx})\,dr+\int_t^{t\vee s}\sigma(r,X_r^{tx})\,dB_r+\int_t^{t\vee s}\int_E\beta(X_{r^-}^{tx},e)\tilde{\mu}(de,ds).\tag{3.3}$$

We state some properties of the process $\{(X_s^{tx}),\ s\in[0,T]\}$ which can be found in [7, Theorems 2.2 and 2.3].

Proposition 3.1. *For each* $t\ge 0$, *there exists a version of* $\{(X_s^{tx}),\ s\in[t,T]\}$ *such that* $s\to X_s^t$ *is a* C^2 $(\mathbb{R}^d)$*-valued rcll process. Moreover,*

(1) X_s^t *and* X_{s-t}^0 *have the same distribution,* $0\le t\le s$;
(2) $X_{t_1}^{t_0},X_{t_2}^{t_1},\dots,X_{t_n}^{t_{n-1}}$ *are independent for all* $n\in\mathbb{N}$, $0\le t_0<t_1<\cdots<t_n$;
(3) $X_r^t=X_r^s\circ X_s^t$, $0\le t<s<r$;
(4) *for all* $p\ge 2$, *there exists a real* C_p *such that for all* $0\le t<s$, *for any* $x,x'\in\mathbb{R}^d$,

$$\begin{gathered}E\Big(\sup_{t\le r\le s}|X_r^{t,x}-x|^p\Big)\le C_p(s-t)(1+|x|^p),\\ E\Big(\sup_{t\le r\le s}|X_r^{t,x}-X_r^{t,x'}-(x-x')|^p\Big)\le C_p(s-t)|x-x'|^p.\end{gathered}\tag{3.4}$$

In the rest of the section, we consider the RBSDE with data (ξ,F,L,H), where

$$
\begin{gathered}
\xi(\omega)=g\left(X_T^{t,x}(\omega)\right),\\
F(\omega,s,y,z,u)=f\left(s,X_s^{t,x}(\omega),y,z,\int_E u(e)\gamma\left(X_s^{t,x},e\right)\lambda(de)\right),\\
L_s(\omega)=l\left(s,X_s^{t,x}(\omega)\right),\qquad H_s(\omega)=h\left(s,X_s^{t,x}(\omega)\right)
\end{gathered}
\tag{3.5}
$$

with

(A7) $g:\mathbb{R}^d\to\mathbb{R}$ $f:[0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$, $l:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ and $h:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ some functions such that

(i) $g\in C(\mathbb{R}^d;\mathbb{R})$ and $|g(x)|\le C(1+|x|^p)$; $|f(t,x,0,0,0)|^2\le C(1+|x|^p)$ for some C, $p>0$,

(ii) f is globally Lipschitz in (y,z,u) uniformly in (t,x) and for each $(t,x,y,z)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d$ the function $u\to f(t,x,y,z,u)$ is nondecreasing,

(iii) l and h are Lipschitz in x uniformly with respect to $t\in[0,T]$, and for any $(s,x)\in[0,T]\times\mathbb{R}^d$

$$
\begin{gathered}
l(s,x)\le h(s,x),\qquad l(T,x)\le g(x)\le h(T,x),\\
l(s,x)\le C\left(1+|x|^p\right),\qquad -C\left(1+|x|^p\right)\le h(t,x).
\end{gathered}
\tag{3.6}
$$

(iv) there exists $C>0$ such that for any $x,x'\in\mathbb{R}^d$, $e\in E$,

$$
\begin{gathered}
0\le\gamma(x,e)\le C\left(1\wedge|e|\right),\\
\left|\gamma(x,e)-\gamma(x',e)\right|\le C\left|x-x'\right|\left(1\wedge|e|\right).
\end{gathered}
\tag{3.7}
$$

For each $t\ge 0$, we denote by $\{\mathcal{F}_s^{t,W}: s\in[t,T]\}$ the natural filtration of the Brownian motion $\{W_s-W_t: s\in[t,T]\}$ augmented with $\mathcal{N}$.

We put

$$
\mathcal{F}_s^t=\mathcal{F}_s^{t,W}\otimes\mathcal{F}_s^{t,\tilde\mu},
\tag{3.8}
$$

where

$$
\mathcal{F}_s^{t,\tilde\mu}=\sigma\left(\tilde\mu_s(A)-\tilde\mu_t(A): s\in[t,T],\ A\in\mathcal{U}\right)\otimes\mathcal{N}.
\tag{3.9}
$$

Under the assumptions (A1)–(A7), Proposition 2.1 implies that for each $(t,x)\in[0,T]\times\mathbb{R}^d$, there exists a unique $\mathcal{F}_s^t$-progressively measurable $(Y^{tx},Z^{tx},U^{tx},K^{tx+},K^{tx-})$ such that

$$
\begin{gathered}
Y_s^{tx}=g\left(X_T^{tx}\right)+\int_s^T F\left(r,Y_r^{tx},Z_r^{tx},U_r^{tx}\right)ds+\left(K_T^{tx+}-K_s^{tx+}\right)-\left(K_T^{tx-}-K_s^{tx-}\right)\\
-\int_s^T Z_r^{tx}dW_r-\int_s^T\int_E U_s^{tx}(e)\tilde\mu(de,dr),\quad \forall s\in[0,T],\ \mathbb{P}\text{-a.s.}\\
\int_0^T\left(Y_s^{tx}-l\left(s,X_s^{tx}\right)\right)dK_s^{tx+}=0=\int_0^T\left(h\left(s,X_s^{tx}\right)-Y_s^{tx}\right)dK_s^{tx-},\quad \mathbb{P}\text{-a.s.}
\end{gathered}
\tag{3.10}
$$

We have extended Y_s^{tx}, Z_s^{tx}, U_s^{tx}, K_s^{tx+}, K_s^{tx-} for $s \in [0,t]$ by putting

$$Y_s^{tx} = Y_t^{tx}, \quad Z^{tx} = U_s^{tx} = K_s^{tx+} = K_s^{tx-} = 0, \quad \text{for } s \in [0,t]. \tag{3.11}$$

It follows immediately that

$$\begin{aligned} Y_s^{tx} = g(X_T^{tx}) &- \int_s^T Z_r^{tx} dW_r - \int_s^T \int_E U_s^{tx}(e)\tilde{\mu}(de,dr) \\ &+ \int_s^T f\Big(r, X_r^{tx}, Y_r^{tx}, Z_r^{tx}, \int_E U_r^{tx}(e)\gamma(X_r^{tx},e)\lambda(de)\Big)ds + (K_T^{tx+} - K_s^{tx+}) - (K_T^{tx-} - K_s^{tx-}). \end{aligned} \tag{3.12}$$

The following proposition is classical and follows from Proposition 3.1, Itô formula, and Gronwall inequality.

Proposition 3.2. *The following holds:*

(i)

$$E\Big(\sup_{0\le s\le T} |Y_r^{t,x}|^2\Big) \le C(1+|x|^p), \tag{3.13}$$

(ii)

$$\begin{aligned} E\Big(\sup_{0\le s\le T} |Y_s^{tx} - Y_s^{t'x'}|\Big) &\le CE\Big(|g(X_T^{tx}) - g(X_T^{t'x'})|^2\Big) \\ &\quad + E\Big(\int_0^T |1_{[t',T]}(r)F(r, Y_r^{tx}, Z_r^{tx}, U_r^{tx}) \\ &\qquad - 1_{[t',T]}(r)F(r, Y_r^{t'x'}, Z_r^{t'x'}, U_r^{t'x'})|^2 dr\Big) \end{aligned} \tag{3.14}$$

for all $t,t' \in [0,T]$, $x,x' \in \mathbb{R}^d$ ($C>0$ and $p\ge 2$ are constants independent of t,t', x,x').

Proof. By virtue of Itô formula, we have

$$\begin{aligned} &|Y_s^{tx}|^2 + \int_s^T |Z_r^{tx}|^2 dr + \int_s^T \int_E |U_r^{tx}(e)|^2 \mu(dr,de) + \sum_{s<r\le T} (\Delta_r Y_r^{tx})^2 \\ &\quad = |\xi|^2 - 2\int_s^T Y_r^{tx} Z_r^{tx} dW_r + 2\int_s^T Y_r^{tx} F(r, Y_r^{tx}, Z_r^{tx}, U_r^{tx}) dr \\ &\qquad - 2\int_s^T \int_E Y_{r-}^{tx} U_r^{tx}(e)\tilde{\mu}(dr,de) + 2\int_s^T Y_r^{tx} dK_r^{tx+} - 2\int_s^T Y_r^{tx} dK_r^{tx-}. \end{aligned} \tag{3.15}$$

It follows that

$$
\begin{aligned}
&|Y_s^{tx}|^2+\int_s^T|Z_r^{tx}|^2dr+\int_s^T\int_E|U_r^{tx}(e)|^2\mu(dr,de)\\
&\quad\leq|\xi|^2+2\int_s^T Y_r^{tx}F(r,Y_r^{tx},Z_r^{tx},U_r^{tx})dr-2\int_s^T Y_r^{tx}Z_r^{tx}dW_r\\
&\qquad-2\int_s^T\int_E Y_{r-}^{tx}U_r^{tx}(e)\tilde{\mu}(dr,de)+2\int_s^T Y_r^{tx}dK_r^{tx+}-2\int_s^T Y_r^{tx}dK_r^{tx-}.
\end{aligned}
\tag{3.16}
$$

Using the facts

$$
\begin{aligned}
\Big[E\sup_{0\leq t\leq T}(L_t^+)^2\Big]^{1/2}\Big[E(K_T^{tx+})^2\Big]^{1/2}<+\infty,\\
\Big[E\sup_{0\leq t\leq T}(H_t^-)^2\Big]^{1/2}\Big[E(K_T^{tx-})^2\Big]^{1/2}<+\infty,
\end{aligned}
\tag{3.17}
$$

we deduce that

$$
\begin{aligned}
&E\bigg(|Y_s^{tx}|^2+\int_s^T|Z_r^{tx}|^2dr+\int_s^T\int_E|U_r^{tx}(e)|^2\mu(dr,de)\bigg)\\
&\quad\leq C+E\bigg(\int_s^T|F(r,0,0,0)|\,|Y_r^{tx}|dr\bigg)+E(|\xi|^2)\\
&\qquad+E\bigg(2k\int_s^T|Y_r^{tx}|(|Y_r^{tx}|+|Z_r^{tx}|+\|U_r^{tx}\|)dr\bigg).
\end{aligned}
\tag{3.18}
$$

Therefore, from Gronwall lemma, we obtain

$$
E\big(|Y_s^{tx}|^2\big)\leq CE\bigg(1+|g(X_T^{tx})|^2+\int_0^T|F(r,0,0,0)|^2dr\bigg).
\tag{3.19}
$$

Since $F(r,0,0,0)=f(r,X_r^{tx},0,0,0)$, by virtue of assumptions (A7) on f and g and Proposition 3.1, we deduce (i),

$$
E\big(|Y_s^{tx}|^2\big)\leq CE\bigg(1+|g(X_T^{tx})|^2+\int_0^T\big(1+|X_r^{tx}|^p\big)dr\bigg)\leq C(1+|x|^p).
\tag{3.20}
$$

Now, (ii) is a straightforward consequence of Proposition 3.1. □

Now, we deal with the connection between the RBSDE studied in the Markovian framework and the parabolic integrodifferential partial equation.

4. Viscosity solutions of integrodifferential partial equation with two obstacles

We introduce the notion of viscosity solution for the following parabolic integral-differential variational inequality (1.1), where F and L are defined in (1.2).

4.1. Preliminaries.

We define

$$
u(t,x):=Y_t^{tx},\quad\forall(t,x)\in[0,T]\times\mathbb{R}^d.
\tag{4.1}
$$

Since Y_t^{tx} is $\mathcal{F}_t^t$-measurable and $\mathcal{F}_t^t$ is a trivial σ-algebra, u is a deterministic function, which verifies the following properties of regularity.

Lemma 4.1. *For any* $(t,x) \in [0,T] \times \mathbb{R}^d$,

(1) $l(t,x) \le u(t,x) \le h(t,x)$ *and* $u(T,x) = g(x)$,

(2) u *grows at most polynomially at infinity, that is, for some real* C *and* $p \ge 2$,

$$|u(t,x)| \le C(1+|x|^p), \quad \forall (t,x) \in [0,T] \times \mathbb{R}^d, \tag{4.2}$$

(3) $u \in C([0,T] \times \mathbb{R}^d)$.

Proof. (1) is a direct consequence of the relation $u(t,x) = Y_t^{tx}$ and we deduce (2) from Proposition 3.2(i).

In order to prove (3), we define Y_s^{tx} for all $s \in [0,T]$ by choosing $Y_s^{tx} = Y_t^{tx}$ for $0 \le s \le t$.

Let $\{(t_n,x_n) : n \in \mathbb{N}\}$ be a sequence of $[0,T] \times \mathbb{R}^d$ converging to (t,x). Using Proposition 3.2, we have

$$\begin{aligned} |u(t_n,x_n) - u(t,x)|^2 &= |Y_{t_n}^{t_nx_n} - Y_t^{t,x}|^2 \\ &\le E\Big(\sup_{s\in[0,T]} |Y_s^{t_nx_n} - Y_s^{t,x}|^2\Big) \\ &\le CE\Big(|g(X_T^{t_nx_n}) - g(X_T^{t,x})|^2\Big) \\ &\quad + E\int_0^T \Big(|1_{[t,T]}(r)\big(F(r,Y_r^{t_nx_n},Z_r^{t_nx_n},U_r^{t_nx_n}) - F(r,Y_r^{tx},Z_r^{tx},U_r^{tx})\big)|^2\Big)dr. \end{aligned} \tag{4.3}$$

Then, Proposition 3.1 induces that $u(t_n,x_n) \to u(t,x)$ as $(t_n,x_n) \to (t,x)$, which gives (3). □

Now, we consider, for each $(t,x) \in [0,T] \times \mathbb{R}^d$, $(Y_{n,s}^{tx}, Z_{n,s}^{tx}, U_{n,s}^{tx})$ the solution of the following BSDE:

$$\begin{aligned} Y_{n,s}^{tx} &= g(X_T^{tx}) - \int_s^T Z_{n,r}^{tx}\,dW_r - \int_s^T\int_E U_{n,s}^{tx}(e)\tilde{\mu}(de,dr) \\ &\quad + \int_s^T f\Big(r, X_r^{tx}, Y_{n,r}^{tx}, Z_{n,r}^{tx}, \int_E U_{n,r}^{tx}(e)\gamma(X_r^{tx},e)\lambda(de)\Big)ds \\ &\quad - n\int_s^T (Y_{n,r}^{tx} - h(X_r^{tx}))^+ dr + n\int_s^T (Y_{n,r}^{tx} - l(X_r^{tx}))^- dr. \end{aligned} \tag{4.4}$$

If we define the deterministic function $u_n(t,x) = Y_{n,t}^{tx}$, then we have the following lemma.

Lemma 4.2. (1) *(See* [1, Theorem 3.4]*):* u_n *is a viscosity solution of the parabolic integral PDE*

$$\begin{gathered} -\frac{\partial u_n}{\partial t} - Lu_n - F + n(u_n(t,x) - h(t,x))^+ - n(u_n(t,x) - l(t,x))^- = 0, \quad \forall (t,x) \in [0,T] \times \mathbb{R}^d, \\ u(T,x) = g(x), \quad \forall x \in \mathbb{R}^d. \end{gathered} \tag{4.5}$$

(2) *(See* [15]*)*:

$$|u_n(t,x) - u(t,x)|^2 \le E\Big(\sup_{s\in[t,T]} |Y_{n,s}^{tx} - Y_s^{t,x}|^2\Big) \le \frac{1}{n}(1+|x|^p). \tag{4.6}$$

In particular, u_n converges to u uniformly on compact sets.

We now show that u is a viscosity solution of (1.1).

4.2. Definitions. As the function u defined in (4.1) is not smooth, (1.1) should be interpreted in a weak sense. Let $C([0,T]\times\mathbb{R}^d)$ denote the set of real-valued continuous functions on $[0,T]\times\mathbb{R}^d$. Adapting the notion of viscosity solution introduced by Crandall and Lions and then by Soner [18], Sayah [17], and Barles et al. [1], we define the following.

Definition 4.3. Let $u \in C([0,T]\times\mathbb{R}^d)$ satisfying $u(T) = g$.

(a) u is a viscosity subsolution of (1.1) if the following holds:
   (i) $u(t,x) \le h(t,x)$, for all $(t,x) \in [0,T]\times\mathbb{R}^d$,
   (ii) for any $\varphi \in C^2([0,T]\times\mathbb{R}^d)$, whenever $(t,x)\in[0,T)\times\mathbb{R}^d$ is a global maximum point of $u-\varphi$,

$$\begin{aligned}&\Big[-\frac{\partial\varphi}{\partial t} - A\varphi(t,x) - K^\delta(u,\varphi)(t,x)\\ &\quad - f\big(t,x,u(t,x),(\nabla\varphi\sigma)(t,x),B^\delta(u,\varphi)(t,x)\big)\Big](u-l)^+(t,x) \le 0\end{aligned} \tag{4.7}$$

   for any $0<\delta<1$, where

$$\begin{aligned}K^\delta(u,\varphi)(t,x) &= \int_{E_\delta}\big(\varphi(t,x+\beta(x,e)) - \varphi(t,x) - \langle\nabla\varphi(t,x),\beta(x,e)\rangle\big)\lambda(de)\\ &\quad + \int_{E_\delta^c}\big(u(t,x+\beta(x,e)) - u(t,x) - \langle\nabla\varphi(t,x),\beta(x,e)\rangle\big)\lambda(de)\\ &:= K_1^\delta(t,x,\varphi) + K_2^\delta(t,x,u,\nabla\varphi),\\ B^\delta(u,\varphi)(t,x) &= \int_{E_\delta}\big(\varphi(t,x+\beta(x,e)) - \varphi(t,x)\big)\gamma(x,e)\lambda(de)\\ &\quad + \int_{E_\delta^c}\big(u(t,x+\beta(x,e)) - u(t,x)\big)\gamma(x,e)\lambda(de)\\ &:= B_1^\delta(t,x,\varphi) + B_2^\delta(t,x,u)\end{aligned} \tag{4.8}$$

   with $E_\delta = \{e\in E;\ |e|<\delta\}$.

(b) u is a viscosity supersolution of (1.1) if the following holds:
   (i) $l(t,x) \le u(t,x)$, for all $(t,x)\in[0,T]\times\mathbb{R}^d$,

(ii) for any $\varphi \in C^2([0,T] \times \mathbb{R}^d)$, whenever $(t,x) \in [0,T) \times \mathbb{R}^d$ is a global minimum point of $u - \varphi$,

$$\begin{aligned}\Big[-\frac{\partial \varphi}{\partial t} - A\varphi(t,x) - K^{\delta}(u,\varphi)(t,x) \\ - f\big(t,x,u(t,x),(\nabla\varphi\sigma)(t,x),B^{\delta}(u,\varphi)(t,x)\big)\Big](u-h)^{-}(t,x) \geq 0\end{aligned} \tag{4.9}$$

for any $0 < \delta < 1$.

(c) u is a viscosity solution of (1.1) if it is both a viscosity subsolution and supersolution.

Remark 4.4. (1) We have introduced the operators K_1^{δ} and B_1^{δ} because of the singularity of $\lambda(de)$ at 0 and since u is only continuous in x. The operators K_2^{δ} and B_2^{δ} make sense thanks to Lemma 4.1(2).

(2) We can clearly replace "global maximum point" or "global minimum point" by "strict global maximum point" or "strict global minimum point."

To prove the uniqueness result for viscosity solutions of second-order equations, it is convenient to give an intrinsic characterization of viscosity solutions. So, we recall the notion of parabolic semijets as introduced in [13, 14].

Let $S(d)$ stand for the set of $d \times d$ symmetric nonnegative matrices.

Definition 4.5. Let $u \in C([0,T] \times \mathbb{R}^d)$ and $(t,x) \in [0,T] \times \mathbb{R}^d$.

Denote by $\wp^{2+}u(t,x)$, (the parabolic superjet of u at (t,x)), the set of triple $(p,q,X) \in \mathbb{R} \times \mathbb{R}^d \times S(d)$, which are such that

$$u(s,y) \leq u(t,x) + p(s-t) + \langle q, y-x\rangle + \frac{1}{2}\langle X(y-x), y-x\rangle + \circ\big(|t-s| + |y-x|^2\big) \tag{4.10}$$

and its closure

$$\begin{aligned}\bar{P}^{2+}u(t,x) = \Big\{(p,q,X) = \lim_{n\to+\infty}(p_n,q_n,X_n) \text{ with } (p_n,q_n,X_n) \in \wp^{2+}u(t_n,x_n), \\ \lim_{n\to+\infty}\big(t_n,x,u(t_n,x_n)\big) = \big(t,x,u(t,x)\big)\Big\}.\end{aligned} \tag{4.11}$$

Similarly, we consider the parabolic subjet of u at (t,x), $\wp^{2-}u(t,x) = -\wp^{2+}(-u)(t,x)$.

4.3. Main results

Theorem 4.6. *u, defined in (4.1), is a viscosity solution of (1.1).*

Proof. We already know from Lemma 4.1 that $u(T) = g$ and

$$l(t,x) \leq u(t,x) \leq h(t,x), \quad \forall (t,x) \in [0,T] \times \mathbb{R}^d. \tag{4.12}$$

(1) We show that u is a subsolution.

Let $\varphi \in C^2([0,T] \times \mathbb{R}^d)$ and let $(t_0,x_0) \in (0,T) \times \mathbb{R}^d$ be a strict global maximum point of $u - \varphi$.

If $u(t_0,x_0) = l(t_0,x_0)$, then (4.9) is trivially verified.

Assume that $u(t_0,x_0) > l(t_0,x_0)$, we have to show that for any $\delta > 0$,

$$-\frac{\partial\varphi}{\partial t} - A\varphi(t,x) - K^\delta(u,\varphi)(t,x) - f\big(t,x,u(t,x),(\nabla\varphi\sigma)(t,x),B^\delta(u,\varphi)(t,x)\big) \le 0. \tag{4.13}$$

Since $u_n \to u$ uniformly on compact sets of $[0,T]\times\mathbb{R}^d$ by Lemma 4.2, there exists n_0 such that for all $n \ge n_0$, $u_n(t_0,x_0) \ge l(t_0,x_0)$, $u_n(t_0,x_0) \le h(t_0,x_0)$ and (t_0,x_0) is a maximum point of $u_n - \varphi$ in a compact $[0,T]\times\bar{B}_R$.

Modifying if necessary the test function, we can suppose that (t_0,x_0) is a global maximum point of $u_n - \varphi$ in $[0,T]\times\mathbb{R}^d$. Thus, we have

$$\begin{aligned}
-\frac{\partial\varphi}{\partial t} &- A\varphi(t_0,x_0) - K^\delta(u_n,\varphi)(t_0,x_0) - f\big(t_0,x_0,u_n(t_0,x_0),(\nabla\varphi\sigma)(t_0,x_0),B^\delta(u,\varphi)(t_0,x_0)\big)\\
&+ n\big(u_n(t_0,x_0) - h(t_0,x_0)\big)^+ - n\big(u_n(t_0,x_0) - l(t_0,x_0)\big)^-\\
&\le 0.
\end{aligned} \tag{4.14}$$

Passing to the limit when $n \to 0$ in the above inequality, we obtain (4.9).

(2) A similar argument leads to the supersolution counterpart. □

In order to establish a uniqueness result, we need an additional assumption on F and γ.

(A8) For each $R > 0$, there exists a continuous function $m : \mathbb{R}^+ \to \mathbb{R}^+$, $m(0) = 0$ such that

$$\|F(t,x,y,z,u) - F(t,x',y,z,u)\| \le m\big(|x - x'|(1 + |z|)\big), \tag{4.15}$$

$\forall (t,u) \in [0,T]\times\mathbb{R}^d$, $\forall x,x' \in \mathbb{R}^d$, $\forall y \in \mathbb{R}/|x| < R$, $|x'| < R$, $|y| < R$.

(A9) There exists $C > 0$ such that for all $x,x' \in \mathbb{R}^d$, for all $e \in E$,

$$|\gamma(x,e) - \gamma(x',e)| \le C|x - x'|(1 \wedge |e|^2). \tag{4.16}$$

Theorem 4.7. *Under the assumptions (A1)–(A9), there exists a unique viscosity solution of (1.1) in the class of functions satisfying*

$$\lim_{|x|\to+\infty} |u(t,x)|e^{-\tilde{A}[\log(|x|)]^2} = 0 \tag{4.17}$$

uniformly for $t \in [0,T]$, *for some* $\tilde{A} > 0$.

Remark 4.8. (1) Barles et al. have shown that the assumption (4.17) is optimal to get such a uniqueness result for (1.1).

(2) By Lemma 4.1(2), $u(t,x) = Y_t^{t,x}$ satisfies (4.17).

Proof. Let u and v be two viscosity solutions of (1.1). The proof consists in several steps. It follows from adaptation to standard techniques and proof of [1, Theorem 3.5]. For completeness we will give the first part and sketch the rest.

We set $w := u - v$. Let $\varphi \in C^2([0,T]\times\mathbb{R}^d)$ and let $(t_0,x_0)\in(0,T)\times\mathbb{R}^d$ be a strict global maximum point of $w-\varphi$. We introduce the function

$$\psi_{\varepsilon,\alpha}(t,x,s,y) = u(t,x) - v(s,y) - \frac{|x-y|^2}{\varepsilon^2} - \frac{(t-s)^2}{\alpha^2} - \varphi(t,x), \tag{4.18}$$

where ε, α are positive parameters which are devoted to tend to zero.

Since (t_0,x_0) is a strict global maximum point of $u-v-\varphi$, by a classical argument in the theory of viscosity, there exists a sequence $(\bar t,\bar x,\bar s,\bar y)$ such that the following holds:

(i) $(\bar t,\bar x,\bar s,\bar y)$ is a global maximum point of $\psi_{\varepsilon,\alpha}$ in $([0,T]\times\bar B_R)^2$, where B_R is a ball with a large radius R,
(ii) $(\bar t,\bar x),(\bar s,\bar y)\to(t_0,x_0)$ as $(\varepsilon,\alpha)\to 0$,
(iii) $|\bar x-\bar y|^2/\varepsilon^2$, $(\bar t-\bar s)^2/\alpha^2$ are bounded and tend to zero when $(\varepsilon,\alpha)\to 0$.

We omit the dependence of $\bar t$, $\bar x$, $\bar s$, $\bar y$ in ε and α to alleviate notations.

Furthermore, it follows from [2, Theorem 8.3] that there exist $X,Y\in S^d$ such that

$$\left(\bar p+\frac{\partial\varphi}{\partial t}(\bar t,\bar x),\bar q+D\varphi(\bar t,\bar x),X\right)\in\bar P^{2+}u(\bar t,\bar x), \tag{4.19}$$

$$(\bar p,\bar q,Y)\in\bar P^{2-}v(\bar s,\bar y), \tag{4.20}$$

$$\begin{pmatrix} X & 0\\ 0 & -Y\end{pmatrix}\le\frac{4}{\varepsilon^2}\begin{pmatrix} I & -I\\ -I & I\end{pmatrix}+\begin{pmatrix} D^2\varphi(\bar t,\bar x) & 0\\ 0 & 0\end{pmatrix}, \tag{4.21}$$

where

$$\bar p=\frac{2(\bar t-\bar s)}{\alpha^2},\qquad \bar q=\frac{2(\bar x-\bar y)}{\varepsilon^2}. \tag{4.22}$$

Modifying if necessary $\psi_{\varepsilon,\alpha}$, we may assume that $(\bar t,\bar x,\bar s,\bar y)$ is a global maximum point of $\psi_{\varepsilon,\alpha}$ in $([0,T]\times\mathbb{R}^d)^2$.

First case. We assume that $u(\bar t,\bar x)\le l(\bar t,\bar x)$, but v being a supersolution of (1.1), we have $v(\bar t,\bar y)\le l(\bar t,\bar y)$. Then, by (A7)(iii), we deduce that

$$\limsup_{\varepsilon\searrow 0}\,(u(\bar t,\bar x)-v(\bar t,\bar y))=0 \tag{4.23}$$

and finally

$$u(t,x)\le v(t,x),\quad \forall\,(t,x)\in[0,T]\times\mathbb{R}^d. \tag{4.24}$$

As u and v play symmetric roles, we conclude that $u=v$.

Second case. We assume that $v(\bar s,\bar y)\ge h(\bar s,\bar y)$, but we know that $u(\bar s,\bar x)\le h(\bar s,\bar x)$. A similar argument as above shows that $u=v$.

Third case. We suppose that $u(\bar t,\bar x)>l(\bar t,\bar x)$ and $v(\bar s,\bar y)<h(\bar s,\bar y)$.

First step. We are going to show that $u-v$ and $v-u$ are viscosity subsolution of an integrodifferential partial equation

$$-\frac{\partial w}{\partial t}-Lw-\tilde{k}\left[|w|+|\nabla w\sigma|+(Bw)^{+}\right]=0,\quad \text{in } [0,T]\times\mathbb{R}^{d}. \tag{4.25}$$

Since u and v are, respectively, subsolution and supersolution of (1.1), we have for δ small enough

$$\begin{aligned}\bar{p}&-\frac{\partial\varphi}{\partial t}(\bar{t},\bar{x})-\frac{1}{2}Tr\big(a(\bar{t},\bar{x})X\big)-\big\langle b(\bar{t},\bar{x}),\bar{q}+D\varphi(\bar{t},\bar{x})\big\rangle\\&-\int_{E_\delta}\frac{|\beta(\bar{x},e)|^{2}}{\varepsilon^{2}}\lambda(de)-K_1^{\delta}(\bar{t},\bar{x},\varphi)-K_2^{\delta}\big(\bar{t},\bar{x},u,\bar{q}+D\varphi(\bar{t},\bar{x})\big)\\&-f\big(\bar{t},\bar{x},u(\bar{t},\bar{x}),(\bar{q}+D\varphi(\bar{t},\bar{x}))\sigma(\bar{t},\bar{x}),\hat{B}^{\delta}\big)\\&\leq 0,\end{aligned} \tag{4.26}$$

where

$$\hat{B}^{\delta}=\int_{E^{\delta}}\left(\langle\bar{q},\beta(\bar{x},e)\rangle+\frac{|\beta(\bar{x},e)|^{2}}{\varepsilon^{2}}\right)\gamma(\bar{x},e)\lambda(de)+B_2^{\delta}(\bar{t},\bar{x},u), \tag{4.27}$$

$$\begin{aligned}\bar{p}&-\frac{1}{2}Tr\big(a(\bar{s},\bar{y})Y\big)-\langle b(\bar{s},\bar{y}),\bar{q}\rangle\\&+\int_{E_\delta}\frac{|\beta(\bar{y},e)|^{2}}{\varepsilon^{2}}\lambda(de)-K_2^{\delta}(\bar{s},\bar{y},v,\bar{q})-f\left(\bar{s},\bar{y},v(\bar{s},\bar{y}),\bar{q}\sigma(\bar{s},\bar{y}),\hat{\hat{B}}^{\delta}\right)\\&\geq 0,\end{aligned} \tag{4.28}$$

where

$$\hat{\hat{B}}^{\delta}=\int_{E^{\delta}}\left(-\langle\bar{q},\beta(\bar{y},e)\rangle-\frac{|\beta(\bar{y},e)|^{2}}{\varepsilon^{2}}\right)\gamma(\bar{y},e)\lambda(de)+B_2^{\delta}(\bar{s},\bar{y},v,\bar{q}). \tag{4.29}$$

Before subtracting these inequalities, we need to estimate differences between terms of the same type.

First, there exists $C>0$ such that

$$Tr\big(a(\bar{t},\bar{x})X\big)-Tr\big(a(\bar{s},\bar{y})Y\big)\leq C\frac{|x-y|^{2}}{\varepsilon^{2}}+Tr\big(a(\bar{t},\bar{x})D^{2}\varphi(\bar{t},\bar{x})\big) \tag{4.30}$$

because of (4.21) and the Lipschitz continuity of σ in x.

Then,

$$\big|\langle b(\bar{t},\bar{x})-b(\bar{s},\bar{y}),\bar{q}\rangle\big|\leq C\frac{|x-y|^{2}}{\varepsilon^{2}}. \tag{4.31}$$

The fact that $(\bar{t},\bar{x},\bar{s},\bar{y})$ is a global maximum point of $\psi_{\varepsilon,\alpha}$ in $([0,T]\times\bar{B}_R)^2$ is the key to estimate the differences of the integrodifferential term. From the inequality

$$\psi_{\varepsilon,\alpha}\big(\bar{t},\bar{x}+\beta(\bar{x},e),\bar{s},\bar{y}+\beta(\bar{y},e)\big)\leq\psi_{\varepsilon,\alpha}(\bar{t},\bar{x},\bar{s},\bar{y}), \tag{4.32}$$

we deduce

$$
\begin{aligned}
&[u(\bar{t},\bar{x}+\beta(\bar{x},e)) - u(\bar{t},\bar{x})] - [v(\bar{s},\bar{y}+\beta(\bar{y},e)) - v(\bar{s},\bar{y})] \\
&\qquad - \langle \bar{q},\beta(\bar{x},e)-\beta(\bar{y},e)\rangle - \frac{1}{\varepsilon^2}|\beta(\bar{x},e)-\beta(\bar{y},e)|^2 \\
&\quad \le \varphi(\bar{t},\bar{x}+\beta(\bar{x},e)) - \varphi(\bar{t},\bar{x}).
\end{aligned}
\tag{4.33}
$$

Therefore,

$$
\begin{aligned}
&\int_{E_\delta^c}[u(\bar{t},\bar{x}+\beta(\bar{x},e)) - u(\bar{t},\bar{x}) - \langle \bar{q}+D\varphi(\bar{t},\bar{x}),\beta(\bar{x},e)\rangle]\lambda(de) \\
&\qquad - \int_{E_\delta^c}[v(\bar{s},\bar{y}+\beta(\bar{y},e)) - v(\bar{s},\bar{y}) - \langle \bar{q},\beta(\bar{y},e)\rangle]\lambda(de) \\
&\quad \le K_2^\delta(\bar{t},\bar{x},\varphi,D\varphi) + \int_{E_\delta^c}\frac{1}{\varepsilon^2}|\beta(\bar{x},e)-\beta(\bar{y},e)|^2\lambda(de),
\end{aligned}
\tag{4.34}
$$

and we note that the last term is estimated by $C(|x-y|^2/\varepsilon^2)$ with C independent of δ.

In the same way, setting $M_\varepsilon(x,e) = \langle \bar{q},\beta(x,e)\rangle + |\beta(x,e)|^2/\varepsilon^2$, we get

$$
\begin{aligned}
\hat{B}^\delta - \hat{\hat{B}}^\delta \le &\int_{E_\delta}(M_\varepsilon(\bar{x},e)\gamma(\bar{x},e) + M_\varepsilon(\bar{y},e)\gamma(\bar{y},e))\lambda(de) + B\varphi(\bar{t},\bar{x}) \\
&+ \int_{E_\delta^c}[v(\bar{s},\bar{y}+\beta(\bar{y},e)) - v(\bar{s},\bar{y})](\gamma(\bar{x},e)-\gamma(\bar{y},e))\lambda(de) \\
&+ \int_{E_\delta^c}\Big[\langle \bar{q},\beta(\bar{x},e)-\beta(\bar{y},e)\rangle - \frac{1}{\varepsilon^2}|\beta(\bar{x},e)-\beta(\bar{y},e)|^2\Big]\gamma(\bar{x},e)\lambda(de) \\
\le &\int_{E_\delta}(M_\varepsilon(\bar{x},e)\gamma(\bar{x},e) + M_\varepsilon(\bar{y},e)\gamma(\bar{y},e))\lambda(de) + B\varphi(\bar{t},\bar{x}) + C\frac{|x-y|^2}{\varepsilon^2} + C|x-y|,
\end{aligned}
\tag{4.35}
$$

the last inequality following from the assumption on β, γ, and the continuity of v with (A9).

Finally, using the continuity of f in t, (A8), and (A7)(ii), we obtain for the nonlinear terms:

$$
\begin{aligned}
&f(\bar{t},\bar{x},u(\bar{t},\bar{x}),(\bar{q}+D\varphi(\bar{t},\bar{x}))\sigma(\bar{t},\bar{x}),\hat{B}^\delta) - f\Big(\bar{s},\bar{y},v(\bar{s},\bar{y}),\bar{q}\sigma(\bar{s},\bar{y}),\hat{\hat{B}}^\delta\Big) \\
&\le \rho_{\varepsilon,\delta}(|\bar{t}-\bar{s}|) + m_R(|\bar{x}-\bar{y}|(1+|\bar{q}\sigma(\bar{s},\bar{y})|)) + \tilde{k}|u(\bar{t},\bar{x})-v(\bar{s},\bar{y})| \\
&\quad + \tilde{k}|\bar{q}(\sigma(\bar{t},\bar{x})-\sigma(\bar{s},\bar{y})) + D\varphi(\bar{t},\bar{x})\sigma(\bar{t},\bar{x})| + \tilde{k}\Big(\hat{B}^\delta - \hat{\hat{B}}^\delta\Big)^+
\end{aligned}
\tag{4.36}
$$

for R large enough and $\rho_{\varepsilon,\delta}(s) \to 0$ when $s \to 0^+$ for fixed ε and δ.

Now subtracting (4.28) from (4.26), we can write

$$
\begin{aligned}
&-\frac{\partial\varphi}{\partial t}(\bar{t},\bar{x}) - A\varphi(\bar{t},\bar{x}) - K\varphi(\bar{t},\bar{x}) - \tilde{k}|u(\bar{t},\bar{x})-v(\bar{s},\bar{y})| \\
&\qquad - \tilde{k}|D\varphi(\bar{t},\bar{x})\sigma(\bar{t},\bar{x})| - \tilde{k}(B\varphi(\bar{t},\bar{x}))^+ \\
&\quad \le \rho_{\varepsilon,\delta}(|\bar{t}-\bar{s}|) + \omega_1(\varepsilon,\alpha) + \omega_2^\varepsilon(\delta),
\end{aligned}
\tag{4.37}
$$

where we have gathered in the $\omega_1(\varepsilon,\alpha)$ term all the terms of the form $|\bar{x}-\bar{y}|^2/\varepsilon^2$ and $|\bar{x}-\bar{y}|$; $\omega_1(\varepsilon,\alpha)\to 0$ when (α,δ) tends to 0. The term $\omega_2^{\varepsilon}(\delta)$ contains all the remaining integrals on E_δ. At last, we first let α go to 0 (since $(\bar{t}-\bar{s})/\alpha^2$ is bounded, $|\bar{t}-\bar{s}|\to 0$), then we let δ go to zero keeping ε fixed, and finally we let $\varepsilon\to 0$ to get

$$\begin{aligned} -\frac{\partial\varphi}{\partial t}(t_0,x_0) - A\varphi(t_0,x_0) - K\varphi(t_0,x_0) - \tilde{k}\,|\,w(t_0,x_0)\,| \\ - \tilde{k}\,|\,D\varphi(t_0,x_0)\sigma(t_0,x_0)\,| - \tilde{k}\big(B\varphi(t_0,x_0)\big)^+ \\ \le 0, \end{aligned} \tag{4.38}$$

that is, $w=u-v$ is a subsolution of (4.25).

Second step. We build a suitable sequence of smooth supersolution of this equation to show that $|u-v|=0$ in $[0,T]\times\mathbb{R}^d$. We need the following lemma which is proved in [1, Lemma 3.8].

Lemma 4.9. *For any $\tilde{A}>0$, there exists $C_1>0$ such that the function*

$$\chi(t,x)=\exp\big[\big(C_1(T-t)+\tilde{A}\big)\psi(x)\big], \tag{4.39}$$

where

$$\psi(x)=\Big[\log\Big(\big(|x|^2+1\big)^{1/2}\Big)+1\Big]^2 \tag{4.40}$$

satisfies

$$-\frac{\partial\chi}{\partial t}-L\chi-\tilde{k}\chi-\tilde{k}|D\chi\sigma|-\tilde{k}(B\chi)^+>0 \quad \text{in } [t_1,T]\times\mathbb{R}^d \tag{4.41}$$

with $t_1=T-\tilde{A}/C_1$.

The end of the demonstration consists in showing that $|w(t,x)|\le\alpha\chi(t,x)$ in $[0,T]\times\mathbb{R}^d$, for any $\alpha>0$. The conclusion is then immediate. □

Acknowledgment

The authors are grateful to the referee for a careful reading that improved the first version of the paper.

References

[1] G. Barles, R. Buckdahn, and E. Pardoux, *Backward differential equations and integral-partial differential equations*, Stochastics Stochastics Rep. **60** (1997), no. 1-2, 57–83.
[2] M. G. Crandall, H. Ishii, and P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.) **27** (1992), no. 1, 1–67.
[3] J. Cvitanić and I. Karatzas, *Backward stochastic differential equations with reflection and Dynkin games*, Ann. Probab. **24** (1996), no. 4, 2024–2056.
[4] C. Dellacherie and P.-A. Meyer, *Probabilités et Potentiel. Chapitres V à VIII*, Actualités Scientifiques et Industrielles, vol. 1385, Hermann, Paris, 1980.

[5] N. El Karoui, C. Kapoudjian, E. Pardoux, S. Peng, and M. C. Quenez, *Reflected solutions of backward SDE's, and related obstacle problems for PDE's*, Ann. Probab. **25** (1997), no. 2, 702–737.

[6] E. H. Essaki, N. Harraj, and Y. Ouknine, *Backward stochastic differential equation with two reflecting barriers and jumps*, submitted to Stochastic Anal. Appl.

[7] T. Fujiwara and H. Kunita, *Stochastic differential equations of jump type and Lévy processes in diffeomorphisms group*, J. Math. Kyoto Univ. **25** (1985), no. 1, 71–106.

[8] Ĭ. Gīhman and A. V. Skorohod, *Stochastic Differential Equations*, Springer, New York, 1972.

[9] S. Hamadène, J.-P. Lepeltier, and A. Matoussi, *Double barrier backward SDEs with continuous coefficient*, Backward Stochastic Differential Equations (Paris, 1995–1996), Pitman Res. Notes Math. Ser., vol. 364, Longman, Harlow, 1997, pp. 161–175.

[10] S. Hamadène and Y. Ouknine, *Reflected backward stochastic differential equation with jumps and random obstacle*, Electron. J. Probab. **8** (2003), no. 2, 1–20.

[11] J. Jacod, *Calcul Stochastique et Problèmes de Martingales*, Lecture Notes in Mathematics, vol. 714, Springer, Berlin, 1979.

[12] J.-P. Lepeltier and J. San Martín, *Backward SDEs with two barriers and continuous coefficient: an existence result*, J. Appl. Probab. **41** (2004), no. 1, 162–175.

[13] P.-L. Lions, *Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. I. The dynamic programming principle and applications*, Comm. Partial Differential Equations **8** (1983), no. 10, 1101–1174.

[14] ______, *Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. II. Viscosity solutions and uniqueness*, Comm. Partial Differential Equations **8** (1983), no. 11, 1229–1276.

[15] M. N'zi and Y. Ouknine, *Probabilistic interpretation for integral-partial differential equations with subdifferential operator*, Random Oper. Stochastic Equations **9** (2001), no. 1, 87–101.

[16] E. Pardoux and S. Peng, *Backward stochastic differential equations and quasilinear parabolic partial differential equations*, Stochastic Partial Differential Equations and Their Applications (Charlotte, NC, 1991), Lecture Notes in Control and Inform. Sci., vol. 176, Springer, Berlin, 1992, pp. 200–217.

[17] A. Sayah, *Équations d'Hamilton-Jacobi du premier ordre avec termes intégro-différentiels. I. Unicité des solutions de viscosité*, Comm. Partial Differential Equations **16** (1991), no. 6-7, 1057–1074 (French).

[18] H. M. Soner, *Optimal control of jump-Markov processes and viscosity solutions*, Stochastic Differential Systems, Stochastic Control Theory and Applications (Minneapolis, Minn, 1986), IMA Vol. Math. Appl., vol. 10, Springer, New York, 1988, pp. 501–511.

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