$\mathscr{BC}$-TOTAL STABILITY AND ALMOST PERIODICITY FOR SOME PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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In this paper, we study the existence of an almost periodic solution for some partial functional differential equation with infinite delay. We assume that the linear part is nondensely defined and satisfies the known Hille-Yosida condition. We prove if the null solution of the homogeneous equation is $\mathscr{BC}$-total stable, then the nonhomogeneous equation has an almost periodic solution.

1. Introduction

In this work, we study the existence of an almost periodic solution of the following partial functional differential equation with infinite delay:

$$\frac{du(t)}{dt} = Au(t) + L(t,u_t) + f(t), \quad t \in \mathbb{R}, \tag{1.1}$$

where $A : D(A) \subset X \to X$ is a nondensely defined linear operator on a Banach space X, we assume that A satisfies the Hille-Yosida condition: there exist $\bar{M} \geq 0$, $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$, and

$$\sup\left\{(\lambda-\omega)^n \left| R(\lambda,A)^n \right|,\, n \in \mathbb{N},\, \lambda > \omega\right\} \leq \bar{M}, \tag{1.2}$$

where $\rho(A)$ is the resolvent set of A and $R(\lambda,A) = (\lambda - A)^{-1}$. The phase space $\mathscr{B}$ satisfies the axioms which have been introduced at first by Hale and Kato [9]. For every $t \geq 0$, the history function $u_t \in \mathscr{B}$ is defined by

$$u_t(\theta) = u(t+\theta), \quad \text{for } \theta \in (-\infty, 0]. \tag{1.3}$$

L is a continuous function from $\mathbb{R} \times \mathscr{B}$ into X and f is a continuous function from $\mathbb{R}$ to X.

The theory of functional differential equations with infinite delay has been developed by several authors. The fundamental task of that kind of equations is the choice of phase space $\mathscr{B}$. Since the history function $t \to u_t$ is not continuous in general if $u : (-\infty, T] \to X$ is continuous (where $T > 0$). The paper [9] contains the fundamental theory of functional

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differential equations with infinite delay in finite-dimensional space. When the delay is finite, the phase space is the space of continuous functions from $[-r,0]$ to X, for more details about this topics, we refer to [10, 17, 18]. In [11], the author proved the existence of periodic solution of (1.1), when A generates a strongly continuous semigroup on X which is equivalent by Hille-Yosida theorem that A satisfies the Hille-Yosida condition and $\overline{D(A)} = X$. In [1, 2], by using the integrated semigroup theory, the authors proved the existence and regularity of solutions of (1.1), where A is nondensely defined and satisfies the Hille-Yosida condition. In [4], the authors proved the existence of ω-periodic solutions of (1.1), where L and f are ω-periodic in t, since the existence of periodic solution is equivalent to the existence of a fixed point for the Poincaré map, for this subject the authors used Chow and Hale's fixed point theorem for affine map [5].

The problem of existence of almost periodic solutions of partial functional differential equations with infinite delay has been studied by several authors. Recently in [13], the authors proved the existence of almost periodic solutions of (1.1), where A generates a compact strongly continuous semigroup on X and $\mathcal{B}$ is a fading memory space. Using the $\mathcal{BC}$-total stability of the null solution of the following homogeneous equation:

$$\frac{du(t)}{dt} = Au(t) + L(t,u_t), \quad \text{for } t \in \mathbb{R}, \tag{1.4}$$

they proved the existence of an asymptotic almost periodic solution of (1.1) and consequently, they obtained the existence of an almost periodic solution, where L is almost periodic in t uniformly with respect to the second argument and f is almost periodic. In this work, our goal is to obtain the same result as in [13] if we assume that A is nondensely defined and satisfies the Hille-Yosida condition. We will prove if the null solution of (1.4) is $\mathcal{BC}$-total stable and X is a separable Banach space, then (1.1) has an asymptotic almost periodic solution and consequently it has an almost periodic solution. This work is an extension of [13].

The organization of this work is as follows: in Section 2, we recall some results about the phase space and the $\mathcal{BC}$-total stability. In Section 3, we prove the existence of almost periodic solution of (1.1). Finally, we propose an application to a reaction diffusion equation with infinite delay.

2. Boundedness and $\mathcal{BC}$-total stability

Here and hereafter, we assume that $\mathcal{B}$ is a normed linear space of functions mapping $(-\infty,0]$ into $(X,|\cdot|)$, endowed with a norm $|\cdot|_{\mathcal{B}}$ and satisfying the following fundamental axioms introduced by Hale and Kato [9].

($\mathbf{A_1}$) There exist a positive constant H and functions $K(\cdot),M(\cdot) : [0,+\infty) \to [0,+\infty)$, with K continuous and M locally bounded, such that for any $\sigma \in \mathbb{R}$ and $a > 0$, if $x : (-\infty,\sigma+a] \to X$, $x_\sigma \in \mathcal{B}$, and $x(\cdot)$ is continuous on $[\sigma,\sigma+a]$, then for all t in $[\sigma,\sigma+a]$, the following conditions hold:

(i) $x_t \in \mathcal{B}$,

(ii) $|x(t)| \le H|x_t|_{\mathcal{B}}$,

(iii) $|x_t|_{\mathcal{B}} \le K(t-\sigma)\sup_{\sigma\le s\le t}|x(s)| + M(t-\sigma)|x_\sigma|_{\mathcal{B}}$.

($\mathbf{A_2}$) For the function $x(\cdot)$ in ($\mathbf{A_1}$), $t \mapsto x_t$ is a $\mathcal{B}$-valued continuous function for t in $[\sigma, \sigma + a]$.

($\mathbf{B_1}$) The space $\mathcal{B}$ is complete.

In [1, 2], it has been proved that axioms ($\mathbf{A_1}$), ($\mathbf{A_2}$), and ($\mathbf{B_1}$) are enough to deal with the quantitative analysis of (1.1). Let $\mathcal{BC}$ be the space of bounded continuous functions mapping $(-\infty, 0]$ into X, provided with the uniform norm topology, then we assume that $\mathcal{B}$ satisfies the following:

($\mathbf{C}$) if $(\phi_n)_{n\geq 0}$ is a sequence in $\mathcal{B} \cap \mathcal{BC}$ converging compactly to ϕ on $(-\infty, 0]$ and $\sup_{n\in\mathbb{N}} |\phi_n|_{\mathcal{BC}} < +\infty$, then ϕ is in $\mathcal{B}$ and $|\phi_n - \phi|_{\mathcal{B}} \to 0$.

Lemma 2.1 [14, Proposition 7.1.1, page 187]. *If $\mathcal{B}$ satisfies axiom* ($\mathbf{C}$), *then $\mathcal{BC} \subset \mathcal{B}$ and there exists a positive constant J such that $|\phi|_{\mathcal{B}} \leq J|\phi|_{\mathcal{BC}}$.*

For $\phi \in \mathcal{B}$, $t \geq 0$, and $\theta \leq 0$, we define the linear operator $W(t)$ by

$$[W(t)\phi](\theta) = \begin{cases} \phi(0), & \text{if } t+\theta \geq 0, \\ \phi(t+\theta), & \text{if } t+\theta < 0. \end{cases} \tag{2.1}$$

$(W(t))_{t\geq 0}$ is exactly the solution semigroup associated to the following trivial equation:

$$\frac{du(t)}{dt} = 0, \qquad u_0 = \varphi. \tag{2.2}$$

Let

$$W_0(t) = W(t)_{/\mathcal{B}_0}, \quad \text{where } \mathcal{B}_0 := \{\phi \in \mathcal{B} : \phi(0) = 0\}. \tag{2.3}$$

$\mathcal{B}$ is called a fading memory space if it satisfies the axiom ($\mathbf{C}$) and $|W_0(t)\phi|_{\mathcal{B}} \to 0$ as $t \to +\infty$, for all $\phi \in \mathcal{B}_0$.

Lemma 2.2 [14, Proposition 7.1.5, page 190]. *If $\mathcal{B}$ is a fading memory space, then the functions $K(\cdot)$ and $M(\cdot)$ in* ($\mathbf{A_1}$) *can be taken as constants.*

In the sequel, we suppose the following:

($\mathbf{B_2}$) $\mathcal{B}$ is a fading memory space,

($\mathbf{H_1}$) A satisfies the Hille-Yosida condition on X.

We consider the following definition of integral solutions of (1.1), which are taken from [1].

Definition 2.3 (see [1]). Let $(\sigma, \varphi) \in \mathbb{R} \times \mathcal{B}$, a function $u : (-\infty, +\infty) \to X$ is an integral solution of (1.1) if the following conditions hold:

(i) $u : [\sigma, +\infty) \to X$ is continuous,

(ii) $\int_\sigma^t u(s)ds \in D(A)$, for $t \geq \sigma$,

(iii) $u(t) = \varphi(0) + A\int_\sigma^t u(s)ds + \int_\sigma^t [L(s, u_s) + f(s)]ds$ for $t \geq \sigma$,

(iv) $u_\sigma = \varphi$.

If $D(A)$ is dense, then the integral solutions coincide with the mild solutions given in [13]. Let A_0 be the part of A in $\overline{D(A)}$ given by

$$\begin{aligned} D(A_0) &= \{x \in D(A) : Ax \in \overline{D(A)}\}, \\ A_0 x &= Ax, \quad \text{for } x \in D(A_0). \end{aligned} \tag{2.4}$$

Lemma 2.4 [3, Lemma 3.3.12, page 140]. *A_0 generates a strongly continuous semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$.*

For the existence of the integral solutions, one has the following theorem.

Theorem 2.5 [1, Theorem 19]. *Assume that $(\mathbf{H_1})$ holds and $\mathcal{B}$ satisfies axioms $(\mathbf{A_1})$, $(\mathbf{A_2})$, and $(\mathbf{B_1})$. If $L : \mathbb{R} \times \mathcal{B} \to X$ is continuous and Lipschitzian with respect to the second argument, then for any $(\sigma, \varphi) \in \mathbb{R} \times \mathcal{B}$ such that $\varphi(0) \in \overline{D(A)}$, (1.1) has a unique integral solution u on $(-\infty, +\infty)$. Moreover, u is given by*

$$u(t) = T_0(t-\sigma)\varphi(0) + \lim_{\lambda \to +\infty} \int_\sigma^t T_0(t-s)\lambda R(\lambda, A)[L(s, u_s) + f(s)]ds, \quad \textit{for } t \geq \sigma. \tag{2.5}$$

Throughout this work, we call integral solutions as solutions. The solution of (1.1) will be denoted by $u(\cdot, \sigma, \varphi, L + f)$.

We recall some properties about almost periodic functions. Let $\mathcal{C}(\mathbb{R} \times \mathcal{B}, X)$ be the set of continuous functions from $\mathbb{R} \times \mathcal{B}$ to X.

Definition 2.6 (see [8]). Let $g \in \mathcal{C}(\mathbb{R} \times \mathcal{B}, X)$. g is an almost periodic function in t uniformly for $\varphi \in \mathcal{B}$, if for any $\varepsilon > 0$ and compact set W in $\mathcal{B}$, there is a positive number $l(\varepsilon, W)$, such that any interval of length $l(\varepsilon, W)$ contains a τ for which

$$|g(t+\tau, \varphi) - g(t, \varphi)| < \varepsilon, \quad \text{for } (t, \varphi) \in \mathbb{R} \times W. \tag{2.6}$$

Theorem 2.7 (see [8]). *Let $g \in \mathcal{C}(\mathbb{R} \times \mathcal{B}, X)$. If g is an almost periodic function in t uniformly for $\varphi \in \mathcal{B}$, then there exists a sequence $(t_n)_n \subset \mathbb{R}$, $t_n \to +\infty$ as $n \to +\infty$ such that*

$$g(t+t_n, \varphi) \longrightarrow g(t, \varphi) \quad \textit{as } n \longrightarrow +\infty \textit{ uniformly on } \mathbb{R} \times W, \textit{ for any compact set } W \subset \mathcal{B}. \tag{2.7}$$

Definition 2.8. Let $g : \mathbb{R} \times \mathcal{B} \to X$ be an almost periodic function in t uniformly for $\varphi \in \mathcal{B}$, define $H_\infty(g)$ by

$$\begin{aligned} H_\infty(g) = \{\tilde{g} \in \mathcal{C}(\mathbb{R} \times \mathcal{B}, X) :\ & \text{there exists a sequence } (t_n)_{n\geq 0} \text{ such that } t_n \longrightarrow +\infty \\ & \text{as } n \longrightarrow +\infty \text{ and } g(t+t_n, \varphi) \longrightarrow \tilde{g}(t, \varphi) \text{ as } n \longrightarrow +\infty \text{ uniformly on} \\ & \mathbb{R} \times W, \text{ for any compact set } W \subset \mathcal{B}\}. \end{aligned} \tag{2.8}$$

Remark 2.9. If $\tilde{g} \in H_\infty(g)$, then $\tilde{g}$ is almost periodic in t uniformly with respect to the second argument.

Theorem 2.10 [8, Theorem 1.17, page 12]. *The following are equivalent:*

(i) *g is almost periodic in t uniformly with respect to the second argument,*

(ii) *for any real sequences $(\alpha'_n)_n$, $(\beta'_n)_n$ there exist subsequences $(\alpha_n)_n \subset (\alpha'_n)_n$, $(\beta_n)_n \subset (\beta'_n)_n$ such that*

$$\lim_{n\to+\infty} g(t+\alpha_n+\beta_n,\varphi) = \lim_{n\to+\infty}\left\{\lim_{l\to+\infty} g(t+\alpha_n+\beta_l,\varphi)\right\}, \quad \textit{pointwise on } \mathbb{R}\times\mathcal{B}. \tag{2.9}$$

Definition 2.11. A continuous function $g:[a,+\infty)\to X$ is said to be asymptotically almost periodic if $g=g_1+g_2$, where g_1 is an almost periodic function and $g_2(t)\to 0$ as $t\to+\infty$.

The following notions and definitions are taken from [13].

Definition 2.12 (see [13]). The null solution of (1.4) is said to be $\mathcal{BC}$-totally stable ($\mathcal{BC}$-TS) if for any $\varepsilon>0$, there exists a $\delta=\delta(\varepsilon)>0$ such that if $\sigma\in\mathbb{R}$, $\varphi\in\mathcal{BC}$ with $|\varphi|_{\mathcal{BC}}<\delta$, $\varphi(0)\in\overline{D(A)}$, and $h\in\mathcal{BC}([\sigma,+\infty),X)$ with $\sup\{|h(t)|,t\geq\sigma\}<\delta$, then

$$|u(t,\sigma,\varphi,L+h)|<\varepsilon, \quad \text{for } t\geq\sigma, \tag{2.10}$$

where $\mathcal{BC}([\sigma,+\infty),X)$ denotes the space of bounded continuous functions from $[\sigma,+\infty)$ to X.

Definition 2.13 (see [13]). The null solution of (1.4) is said to be $\mathcal{BC}$-uniformly asymptotically stable ($\mathcal{BC}$-UAS), if for any $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon)>0$ such that if $\sigma\in\mathbb{R}$, $\varphi\in\mathcal{BC}$ with $|\varphi|_{\mathcal{BC}}<\delta$ and $\varphi(0)\in\overline{D(A)}$, then

$$|u(t,\sigma,\varphi,L)|<\varepsilon, \quad \text{for } t\geq\sigma. \tag{2.11}$$

In addition, there exists a $\delta_0>0$ with the propriety that for any $\varepsilon>0$ there exists a $t_0(\varepsilon)>0$ such that for $\sigma\in\mathbb{R}$, $\varphi\in\mathcal{BC}$ with $|\varphi|_{\mathcal{BC}}<\delta_0$ and $\varphi(0)\in\overline{D(A)}$ there exists

$$|u(t,\sigma,\varphi,L)|<\varepsilon, \quad \text{for } t\geq\sigma+t_0(\varepsilon). \tag{2.12}$$

Remark 2.14. As a consequence of the $\mathcal{BC}$-total stability of the null solution of (1.4), we have the existence of a bounded solution of (1.1) on $\mathbb{R}^+$ if so is f. In fact, let

$$\alpha=\sup_{t\in\mathbb{R}^+}|f(t)|, \qquad g(t)=\frac{\delta(1)}{2\alpha}f(t), \tag{2.13}$$

where $\delta(1)$ is given by definition of $\mathcal{BC}$-TS. Since $g\in\mathcal{BC}([0,+\infty),X)$ with $\sup_{t\in\mathbb{R}^+}|g(t)|<\delta(1)$, then for all $\varphi\in\mathcal{BC}$ such that $|\varphi|_{\mathcal{BC}}<\delta(1)$, we have

$$|u(t,0,\varphi,L+g)|<1, \quad \forall t\geq 0. \tag{2.14}$$

Let $u(t)=(2\alpha/\delta(1))u(t,0,\varphi,L+g)$. Then,

$$|u(t)|\leq\frac{2\alpha}{\delta(1)}<+\infty, \quad \text{for } t\geq 0, \tag{2.15}$$

which implies that (1.1) has a bounded solution on $\mathbb{R}^+$.

Proposition 2.15. *If the null solution of (1.4) is $\mathscr{BC}$-UAS, then zero is the only bounded solution on $\mathbb{R}$ of (1.4).*

Proof. We proceed by contradiction and suppose that there exists a bounded solution y of (1.4) on $\mathbb{R}$, $\varepsilon_0 > 0$ and $t_1 \in \mathbb{R}$ such that

$$|y(t_1)| > \varepsilon_0. \tag{2.16}$$

Let $(\delta_0, t_0(\cdot))$ be given by the $\mathscr{BC}$-UAS of the null solution of (1.4) and choose $\beta > 0$ such that

$$\sup_{t\in\mathbb{R}} |y(t)| < \frac{\delta_0}{\beta}. \tag{2.17}$$

Let $\vartheta(t) = \beta y(t)$ and $\sigma = t_1 - t_0(\beta\varepsilon_0/2)$, $\varphi = \vartheta_\sigma$. Then ϑ is a bounded solution on $\mathbb{R}$ of (1.4) such that $|\varphi|_{\mathscr{BC}} \leq \delta_0$, and we get that

$$|\vartheta(t)| < \frac{\beta\varepsilon_0}{2}, \quad \text{for } t \geq t_1, \tag{2.18}$$

which implies that $|y(t)| < \varepsilon_0/2$, for $t \geq t_1$. Hence $|y(t_1)| < \varepsilon_0/2$, which gives a contradiction with (2.16). □

In the following we assume that

($\mathbf{H_2}$) L is linear with respect to the second argument, almost periodic in t uniformly with respect to the second argument, and f is almost periodic,

($\mathbf{H_3}$) the semigroup $(T_0(t))_{t\geq 0}$ is compact in $\overline{D(A)}$.

Recall that for linear case, it has been proved in [12, Theorem 3] the equivalence between the $\mathscr{BC}$-totally stable and $\mathscr{BC}$-uniformly asymptotically stable under the following assumptions:

(i) $\mathscr{B}$ is separable,

(ii) $\sup_{t\geq 0} \|L(t,\cdot)\| < +\infty$,

(iii) for each $\varphi \in \mathscr{B}$, $L(t,\varphi)$ is uniformly continuous in $t \in \mathbb{R}^+$, and the set $\{L(t,\varphi) : t \in \mathbb{R}^+\}$ is relatively compact in X,

(iv) A generates a compact strongly continuous semigroup.

Assumptions (i), (ii), and (iii) are used in the proof in [12, Theorem 3]. Through the proof, the authors used the following argument: for any sequence $(t'_n)_n$, $t'_n \to +\infty$ as $n \to +\infty$, there exists a subsequence $(t_n)_n$ of $(t'_n)_n$ such that $(L(\cdot + t_n, \varphi))_{n\geq 0}$ converges compactly on $\mathbb{R} \times \mathscr{B}$. Conditions (i), (ii), and (iii) are true if ($\mathbf{H_2}$) holds. Then the same result holds where assumptions ($\mathbf{H_1}$), ($\mathbf{H_2}$), and ($\mathbf{H_3}$) hold. The proof is omitted here.

Proposition 2.16. *Assume that* ($\mathbf{H_1}$), ($\mathbf{H_2}$), *and* ($\mathbf{H_3}$) *hold. Then the null solution of (1.4) is $\mathscr{BC}$-totally stable if and only if it is $\mathscr{BC}$-uniformly asymptotically stable.*

Let u be a bounded solution of (1.1) on $\mathbb{R}^+$.

Lemma 2.17. *(i) $\overline{\{u(t),\, t \geq 0\}}$ is compact in X.*

(ii) u is uniformly continuous on $\mathbb{R}^+$.

Proof. (i) Let $\eta \geq 0$. Then

$$\{u(t),\ t \geq 0\} = O_\eta \cup \widetilde{O}_\eta, \tag{2.19}$$

with $O_\eta = \{u(t), t \geq \eta\}$ and $\widetilde{O}_\eta = \{u(t), 0 \leq t \leq \eta\}$. Let $\alpha(\cdot)$ be the Kuratowski measure of noncompactness of sets in X. Then,

$$\alpha(\{u(t),\ t \geq 0\}) = \alpha(O_\eta \cup \widetilde{O}_\eta) = \max(\alpha(O_\eta), \alpha(\widetilde{O}_\eta)). \tag{2.20}$$

Since the set $\widetilde{O}_\eta$ is compact in X, it follows that $\alpha(\widetilde{O}_\eta) = 0$ and

$$\alpha(\{u(t),\ t \geq 0\}) = \alpha(O_\eta). \tag{2.21}$$

Putting $F(s,\varphi) = L(s,\varphi) + f(s)$, let $0 < \nu < \min(\eta, 1)$, if $t \geq \eta$, we have

$$u(t) = T_0(\nu)u(t-\nu) + \lim_{\lambda \to +\infty} \int_{t-\nu}^{t} T_0(t-s)F(s,u_s)\,ds. \tag{2.22}$$

From the compactness of the semigroup $(T_0(t))_{t \geq 0}$, we get that

$$\{T_0(\nu)u(t-\nu),\ t \geq \nu\} \text{ is relatively compact in } X. \tag{2.23}$$

Let $\varphi \in \mathcal{B}$. Then by $(\mathbf{H_2})$ we have $\sup_{t \in \mathbb{R}} |L(t,\varphi)| < \infty$ and by Banach-Stainhauss theorem, we get that $\sup_{t \in \mathbb{R}, |\varphi|_{\mathcal{B}} \leq 1} |L(t,\varphi)| < \infty$. Since $\mathcal{B}$ is a fading memory space, by axiom $(\mathbf{A_1})$-(**iii**), we get that $\{u_s,\ s \in \mathbb{R}^+\}$ is a bounded set in $\mathcal{B}$, which implies that $\sup_{s \geq 0} |F(s,u_s)| < \infty$. Consequently,

$$\left| \int_{t-\nu}^{t} T_0(t-s)\lambda R(\lambda,A)F(s,u_s)\,ds \right| \leq m_0 \frac{\lambda}{\lambda - \omega}\nu, \tag{2.24}$$

for some positive constant m_0. It follows that

$$\lim_{\lambda \to +\infty} \left| \int_{t-\nu}^{t} T_0(t-s)\lambda R(\lambda,A)F(s,u_s)\,ds \right| \leq m_0 \nu, \tag{2.25}$$

and $\alpha(O_\eta) \leq m_0\nu$, letting $\nu \to 0$ in the above, we get $\alpha(O_\eta) = 0$, which proves that $\{u(t),\ t \geq 0\}$ is relatively compact in X.

(ii) Let $0 \leq t_0 \leq t \leq t_0 + 1$. Then,

$$\begin{aligned} u(t) - u(t_0) &= T_0(t-t_0)u(t_0) + \lim_{\lambda \to +\infty} \int_{t_0}^{t} T_0(t-s)\lambda R(\lambda,A)F(s,u_s)\,ds - u(t_0), \\ |u(t) - u(t_0)| &\leq |T_0(t-t_0)u(t_0) - u(t_0)| + \left| \lim_{\lambda \to +\infty} \int_{t_0}^{t} T_0(t-s)\lambda R(\lambda,A)F(s,u_s)\,ds \right|, \end{aligned} \tag{2.26}$$

which gives that

$$|u(t) - u(t_0)| \leq |T_0(t-t_0)u(t_0) - u(t_0)| + m_1(t - t_0), \tag{2.27}$$

for some positive constant m_1 and

$$|u(t)-u(t_0)| \leq \sup_{x\in W} |(T_0(t-t_0)-I)x| + m_1(t-t_0), \tag{2.28}$$

where $W=\overline{\{u(t),\ t\in\mathbb{R}^+\}}$ is a compact set of X, by Banach-Stainhauss theorem, we get that

$$\sup_{x\in W} |(T_0(t-t_0)-I)x| \longrightarrow 0 \quad \text{as } t-t_0 \longrightarrow 0. \tag{2.29}$$

Consequently,

$$\sup_{0\leq t_0\leq t\leq t_0+1} |u(t)-u(t_0)| \longrightarrow 0 \quad \text{as } t-t_0 \longrightarrow 0, \tag{2.30}$$

which implies the uniform continuity of u over $\mathbb{R}^+$. □

Lemma 2.18. *Let u be a bounded solution of (1.1) defined on $\mathbb{R}^+$. Then for any sequence $(t'_n)_n$, $t'_n \to +\infty$ as $n\to+\infty$, there exists a subsequence $(t_n)_n$ of $(t'_n)_n$ and functions v, $\widetilde{L}$, and $\widetilde{f}$ such that*

$$L(t+t_n,\varphi)+f(t+t_n) \longrightarrow \widetilde{L}(t,\varphi)+\widetilde{f}(t) \quad \textit{as } n \longrightarrow +\infty \textit{ uniformly on } \mathbb{R}\times W, \tag{2.31}$$

for any compact set $W\subset\mathcal{B}$, and

$$u(t+t_n) \longrightarrow v(t) \quad \textit{as } n \longrightarrow +\infty \textit{ uniformly on any compact in } \mathbb{R}. \tag{2.32}$$

Moreover, v is a bounded solution on $\mathbb{R}$ of equation

$$\frac{dv(t)}{dt} = Av(t)+\widetilde{L}(t,\varphi)+\widetilde{f}(t), \quad t\in\mathbb{R}. \tag{2.33}$$

Proof. By virtue of $(\mathbf{H_2})$, it follows that for any sequence $(t'_n)_n$, $t'_n \to +\infty$ as $n\to+\infty$, there exists a subsequence $(t_n)_n$ of $(t'_n)_n$ and almost periodic functions $\widetilde{L}$ and $\widetilde{f}$ such that

$$L(t+t_n,\varphi)+f(t+t_n) \longrightarrow \widetilde{L}(t,\varphi)+\widetilde{f}(t) \quad \text{as } n \longrightarrow +\infty, \text{ uniformly on } \mathbb{R}\times W, \tag{2.34}$$

for any compact set $W\subset\mathcal{B}$.

Let $u^n(t)=u(t+t_n)$ for $t\in[-1,+\infty)$ and $n_1>0$ such that $t+t_n\geq 0$, for any $n\geq n_1$. Then the sequence $(u^n(t))_{n\geq 0}$ satisfies

$$\begin{aligned} u^n(t) &= T_0(t)u(t_n) \\ &\quad + \lim_{\lambda\to+\infty}\int_0^t T_0(t-s)\lambda R(\lambda,A)\left[L(s+t_n,u_{s+t_n})+f(s+t_n)\right]ds, \end{aligned} \tag{2.35}$$

for $n\geq n_1$ and $t\in[-1,+\infty)$. Using the same argument as in the proof of Lemma 2.17, we can see that $(u^n)_{n\geq 0}$ has a subsequence $(u_1^n)_{n\geq 0}$ which is compactly convergent on

$[-1,+\infty)$ to some function v_1 which is a solution of (2.33) on $[-1,+\infty)$. Consider $(u_1^n)_{n\geq 0}$ on $[-1,+\infty)$, then there exists a subsequence $(u_2^n)_{n\geq 0}$ of $(u_1^n)_{n\geq 0}$ which is compactly convergent on $[-2,+\infty)$ to some function v_2 such that $v_1 = v_2$ on $[-1,+\infty)$ and v_2 is a solution of (2.33) on $[-2,+\infty)$. By induction, for every $k \geq 1$, there is a subsequence $(u_k^n)_{n\geq 0}$ of $(u_{k-1}^n)_{n\geq 0}$ such that $(u_k^n)_{n\geq 0}$ converges compactly on $[-k,+\infty)$ to some function v_k with $v_k = v_{k-1}$ on $[-(k-1),+\infty)$ and v_k is a solution of (2.33) on $[-k,+\infty)$. We take the diagonal sequence $(u_n^n)_{n\geq 0}$, then $(u_n^n)_{n\geq 0}$ is a subsequence of $(u^n)_{n\geq 0}$ which converges compactly on $\mathbb{R}$ to v which is defined by $v = v_n$ on $[-n,+\infty)$. Consequently, v is a bounded solution of (2.33) on $\mathbb{R}$. □

3. Almost periodic solutions

Theorem 3.1. *If (1.1) has an asymptotically almost periodic solution, then it has an almost periodic solution.*

Proof. The proof is a consequence of Lemma 2.18. In fact, let u be an asymptotically almost periodic solution of (1.1). Then $u(t) = p(t) + q(t)$, where p is the almost periodic component and q is the continuous component with $q(t) \to 0$ as $t \to +\infty$. By Theorem 2.7, there exists a sequence $(t_n)_n \subset \mathbb{R}$, with $t_n \to +\infty$ as $n \to +\infty$ such that

$$L(t+t_n,\varphi) + f(t+t_n) \longrightarrow L(t,\varphi) + f(t) \quad \text{as } n \longrightarrow +\infty, \text{ uniformly on } \mathbb{R}\times W, \tag{3.1}$$

for any compact set $W \subset \mathcal{B}$, and

$$p(t+t_n) \longrightarrow \widetilde{p}(t), \quad \text{uniformly on } \mathbb{R}. \tag{3.2}$$

By Lemma 2.18, $(u(\cdot + t_n))_{n\geq 0}$ converges to $\widetilde{p}$ uniformly on any compact set in $\mathbb{R}$ and $\widetilde{p}$ is a solution of (1.1) which is almost periodic. □

In order to prove the existence of asymptotic almost periodic solution, we need the following characterization.

Lemma 3.2 [13, Proposition 4]. *Let Y be any separable Banach space and let $g : [a,+\infty) \to Y$ be a continuous function. Then the following are equivalent:*

(i) *g is asymptotically almost periodic,*

(ii) *for any sequence $(t'_n)_n$ such that $t'_n \to +\infty$ as $n \to +\infty$, there exists a subsequence $(t_n)_n$ of $(t'_n)_n$ for which $(g(\cdot + t_n))_n$ converges uniformly on $[a,+\infty)$.*

In the sequel, we assume that

($\mathbf{H_4}$) X is a separable Banach space.

Theorem 3.3. *Assume that* ($\mathbf{H_1}$), ($\mathbf{H_2}$), ($\mathbf{H_3}$), *and* ($\mathbf{H_4}$) *hold. If the null solution of (1.4) is $\mathcal{BC}$-TS, then (1.1) has an asymptotically almost periodic solution.*

Recall that the above theorem has been established in [13] with the same context when A is densely defined.

Proof of Theorem 3.3. We use the same approach as in [13]. Let u be a bounded solution of (1.1) on $\mathbb{R}^+$. We claim that u is asymptotically almost periodic. We proceed by contradiction. Suppose that there exist $(t_n)_n$, $t_n \to +\infty$ as $n \to +\infty$, $\varepsilon > 0$ and sequences $(r_j)_j$, (k_j), $(m_j)_j$ such that $k_j, m_j \to +\infty$ as $j \to +\infty$ and

$$\left| u(r_j + t_{k_j}) - u(r_j + t_{m_j}) \right| \geq \varepsilon. \tag{3.3}$$

The sequence $(t_n)_n$, $t_n \to +\infty$ as $n \to +\infty$, can be chosen such that $(L(\cdot + t_n, \varphi))_n$ converges uniformly on $\mathbb{R} \times W$ for any compact set W of $\mathcal{B}$ and $(f(\cdot + t_n))_n$ converges uniformly on $\mathbb{R}$. By Theorem 2.10, for the sequences $(r_j)_j$, $(t_{k_j})_j$ there exist subsequences, will be denoted by $(r_j)_j$, (t_{k_j}) for simplicity, such that

$$\begin{aligned} \lim_{j\to+\infty} L(t + r_j + t_{k_j}, \varphi) &= \lim_{j\to+\infty} \left\{ \lim_{l\to+\infty} L(t + r_j + t_{k_l}, \varphi) \right\}, \quad \text{pointwise on } \mathbb{R}\times\mathcal{B}, \\ \lim_{j\to+\infty} f(t + r_j + t_{k_j}) &= \lim_{j\to+\infty} \left\{ \lim_{l\to+\infty} f(t + r_j + t_{k_l}) \right\}, \quad \text{pointwise on } \mathbb{R}. \end{aligned} \tag{3.4}$$

By the same argument, for sequences $(r_j)_j$, (t_{m_j}) there exist subsequences, will be denoted by $(r_j)_j$, (t_{m_j}) such that

$$\begin{aligned} \lim_{j\to+\infty} L(t + r_j + t_{m_j}, \varphi) &= \lim_{j\to+\infty} \left\{ \lim_{l\to+\infty} L(t + r_j + t_{m_l}, \varphi) \right\} \quad \text{on } \mathbb{R}\times\mathcal{B}, \\ \lim_{j\to+\infty} f(t + r_j + t_{m_j}) &= \lim_{j\to+\infty} \left\{ \lim_{l\to+\infty} f(t + r_j + t_{m_l}) \right\}, \quad \text{pointwise on } \mathbb{R}. \end{aligned} \tag{3.5}$$

Since L is almost periodic in t uniformly with respect to the second argument and f is almost periodic, the pointwise mode of convergence could be replaced by the uniform mode of convergence and we have

$$\lim_{j\to+\infty} L(t + r_j + t_{k_j}, \varphi) = \lim_{j\to+\infty} L(t + r_j + t_{m_j}, \varphi) = \widetilde{L}(t, \varphi), \quad \text{uniformly on } \mathbb{R} \times W, \tag{3.6}$$

for any compact set W of $\mathcal{B}$, and

$$\lim_{j\to+\infty} f(t + r_j + t_{k_j}) = \lim_{j\to+\infty} f(t + r_j + t_{m_j}) = \widetilde{f}(t), \tag{3.7}$$

uniformly on $\mathbb{R}$. Moreover, by Lemma 2.18, for $(r_j + t_{k_j})_{j\geq 0}$ and $(r_j + t_{m_j})_{j\geq 0}$, there exist subsequences, will be denoted, respectively, by $(r_j + t_{k_j})_{j\geq 0}$ and $(r_j + t_{m_j})_{j\geq 0}$, such that $u(t + r_j + t_{k_j}) \to p(t)$ and $u(t + r_j + t_{m_j}) \to q(t)$ uniformly on any compact set in $\mathbb{R}$ as $j \to +\infty$, where p and q are bounded solutions of equation

$$\frac{dw(t)}{dt} = Aw(t) + \widetilde{L}(t, w_t) + \widetilde{f}(t), \quad \widetilde{L} + \widetilde{f} \in H_\infty(L + f), \tag{3.8}$$

then $p - q$ is a bounded solution on $\mathbb{R}$ of the following equation:

$$\frac{dw(t)}{dt} = Aw(t) + \widetilde{L}(t, w_t). \tag{3.9}$$

To complete the proof, we need the following lemma and the proof is similar to the one given in [12, Theorem 2].

Lemma 3.4. *Let $\widetilde{L} \in H_\infty(L)$, if the null solution of (1.4) is $\mathcal{BC}$-UAS, then it is $\mathcal{BC}$-UAS for*

$$\frac{du(t)}{dt} = Au(t) + \widetilde{L}(t,\varphi), \quad t \in \mathbb{R}. \tag{3.10}$$

Since the null solution of (1.4) is $\mathcal{BC}$-TS, then by Proposition 2.15 and Lemma 3.4, we obtain that

$$p = q. \tag{3.11}$$

Letting $j \to +\infty$ in (3.3), then $|p(0) - q(0)| \geq \varepsilon > 0$, which gives a contradiction with (3.11). Then $(u(\cdot + t_n))_{n\geq 0}$ converges uniformly on $\mathbb{R}^+$ and u is asymptotically almost periodic. □

Consequently, by Theorem 3.1, we have the following result.

Corollary 3.5. *Assume that* $(\mathbf{H_1})$, $(\mathbf{H_2})$, $(\mathbf{H_3})$, *and* $(\mathbf{H_4})$ *hold. If the null solution of (1.4) is $\mathcal{BC}$-TS, then (1.1) has an almost periodic solution.*

4. Example

To illustrate the previous abstract results, we consider the following partial differential equations of the form:

$$\begin{gathered}\frac{\partial u(t,x)}{\partial t} = a\triangle u(t,x) - bu(t,x) + K(t)\int_{-\infty}^{0} G(\theta)u(t+\theta,x)d\theta + \Gamma(t,x), \quad \text{for } t \geq \sigma,\ x \in \overline{\Omega}, \\ u(t,x) = 0, \quad t \geq \sigma,\ x \in \partial\Omega, \\ u(\theta+\sigma,x) = \varphi(\theta)(x), \quad \theta \in \mathbb{R}^-,\ x \in \overline{\Omega},\end{gathered} \tag{4.1}$$

where a, b are positive constants, Ω is a bounded domain in $\mathbb{R}^n$ with regular boundary $\partial\Omega$, $\Delta = \sum_{i=1}^{n}(\partial^2/\partial x_i^2)$ is the Laplacian operator in $\mathbb{R}^n$, $K : \mathbb{R} \to \mathbb{R}$, $G : (-\infty,0] \to \mathbb{R}$ and $\Gamma : \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$ are continuous functions.

Let X be the separable Banach space $C(\overline{\Omega})$, endowed with the uniform norm topology, and define the operator $A : D(A) \subset X \to X$ by

$$D(A) = \{y \in C^1(\overline{\Omega},\mathbb{R}) \cap C^2(\Omega,\mathbb{R}) y_{/\partial\Omega} = 0\}, \qquad Ay = a\Delta y. \tag{4.2}$$

By [6, Example 14.5 and Proposition 14.6, pages 319–320], one has

$$(0,+\infty) \subset \rho(A), \qquad |(\lambda I - A)^{-1}| \leq \frac{1}{\lambda}, \quad \text{for } \lambda > 0. \tag{4.3}$$

Moreover,

$$\overline{D(A)} = \{y \in X : y_{/\partial\Omega} = 0\} \neq X. \tag{4.4}$$

This implies that A satisfies the Hille-Yosida condition.

The part A_0 of A in $\overline{D(A)} = C_0(\overline{\Omega}) = \{y \in X : y_{/\partial\Omega} = 0\}$, is given by

$$D(A_0) = \{y \in C^1(\overline{\Omega}) \cap C^2(\Omega) : y_{/\partial\Omega} = 0,\ \Delta y \in \overline{D(A)}\},$$
$$A_0 y = Ay, \quad \text{for } y \in D(A_0). \tag{4.5}$$

From [7], A_0 generates a strongly continuous compact semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$, which implies that $(\mathbf{H_3})$ holds.

Let $\gamma > 0$. We introduce the following phase space:

$$\mathcal{B} = C_\gamma^0(X) := \Big\{\varphi \in C((-\infty,0];X) : \lim_{\theta\to-\infty} \exp(\gamma\theta)\varphi(\theta) = 0\Big\}, \tag{4.6}$$

provided with the norm

$$|\varphi|_\gamma := \sup_{-\infty<\theta\leq 0} \exp(\gamma\theta)|\varphi(\theta)|. \tag{4.7}$$

Lemma 4.1 [14, Proposition 1.4.2, page 22]. *The space $C_\gamma^0(X)$ satisfies axioms* $(\mathbf{A_1})$, $(\mathbf{A_2})$, $(\mathbf{B_1})$, *and* $(\mathbf{B_2})$.

We assume the following:
(a) $\int_{-\infty}^0 e^{-\gamma\theta}|G(\theta)|d\theta < \infty$,
(b) K is almost periodic and $\Gamma(t,x)$ is almost periodic in t uniformly for $x \in \overline{\Omega}$.
Let $f : \mathbb{R} \to X$ and $L : \mathbb{R}\times\mathcal{B} \to X$ be defined by

$$f(t)(x) = \Gamma(t,x), \quad \text{for } t \in \mathbb{R},\ x \in \overline{\Omega},$$
$$L(t,\varphi)(x) = -b\varphi(0)(x) + K(t)\int_{-\infty}^0 G(\theta)\varphi(\theta)(x)d\theta, \quad \text{for } t \in \mathbb{R},\ x \in \overline{\Omega}, \tag{4.8}$$

Then (4.1) takes the abstract form

$$\frac{du(t)}{dt} = Au(t) + L(t,u_t) + f(t), \quad t \geq \sigma,$$
$$u_\sigma = \varphi \in \mathcal{B}. \tag{4.9}$$

From the continuity of Γ, we get that f is a continuous function from $\mathbb{R}$ to X and L is continuous from $\mathbb{R}\times\mathcal{B}$ to X. In fact, given $t \in \mathbb{R}$, $\varphi \in \mathcal{B}$ and sequences $(t_n)_{n\geq 0}$ of $\mathbb{R}$ and $(\varphi_n)_{n\geq 0}$ of $\mathcal{B}$ such that $t_n \to t$ and $\varphi_n \to \varphi$ in $\mathcal{B}$, we have

$$\begin{aligned} &|L(t_n,\varphi_n) - L(t,\varphi)| \\ &\quad = \sup_{x\in\overline{\Omega}} |L(t_n,\varphi_n)(x) - L(t,\varphi)(x)|, \\ &\quad \leq \Big(b + |K(t_n)|\int_{-\infty}^0 e^{-\gamma\theta}|G(\theta)|d\theta\Big)|\varphi_n - \varphi|_{\mathcal{B}} \\ &\qquad + |K(t) - K(t_n)|\int_{-\infty}^0 e^{-\gamma\theta}|G(\theta)|d\theta|\varphi|_{\mathcal{B}}. \end{aligned} \tag{4.10}$$

Letting $n \to +\infty$, by continuity of K under condition $(\mathbf{a})$, we get the continuity of L at (t,φ). Moreover, by condition $(\mathbf{b})$, we get that $(\mathbf{H_2})$ is true.

By virtue of Theorem 2.5, for any $\sigma \in \mathbb{R}$ and $\varphi \in \mathcal{B}$ such that $\varphi(0) \in \{y \in X : y_{/\partial\Omega} = 0\}$, there exists a unique solution u of (4.9) such that $u_\sigma = \varphi$. Now, our goal is to study the $\mathcal{BC}$-TS of the null solution of the nonhomogeneous equation (1.4).

Let

$$c = \left(\sup_{t\in\mathbb{R}} |K(t)|\right) \int_{-\infty}^{0} |G(\theta)|\, d\theta. \tag{4.11}$$

Theorem 4.2. *Assume that (a) and (b) are satisfied with $b > c$. Then the null solution of (1.4) is $\mathcal{BC}$-TS.*

Proof. We use the same argument as in [13], however for the regularity we will use the theory of sectorial operators [15]. It is enough to prove that for $\varepsilon > 0$, $\sigma \in \mathbb{R}$ and for any $\varphi \in \mathcal{BC}$, with $|\varphi|_{\mathcal{BC}} < \varepsilon$, $\varphi(0) \in \overline{D(A)}$, and $h \in \mathcal{BC}([\sigma,+\infty),X)$ such that $\sup\{|h(t)|, t \ge \sigma\} < (b-c)\varepsilon$, we have

$$|u(t,\sigma,\varphi,L+h)| < \varepsilon, \quad \text{for } t \ge \sigma. \tag{4.12}$$

We proceed by contradiction and suppose that there exists $\tau > \sigma$ such that

$$|u(\tau,\sigma,\varphi,L+h)| = \varepsilon, \qquad |u(t,\sigma,\varphi,L+h)| < \varepsilon \quad \text{for } t < \tau. \tag{4.13}$$

Let $q(t) = L(t,v_t) + bv(t) + h(t)$, for all $t \in [\sigma,\tau]$, where

$$v(t) = u(t,\sigma,\varphi,L+h). \tag{4.14}$$

Then the function $q : [\sigma,\tau] \to X$ is continuous and $|q(t)| < b\varepsilon$, for all $t \in [\sigma,\tau]$. Let $(Q_n)_n$ be a sequence in $C^1([\sigma,\tau]\times\overline{\Omega},\mathbb{R})$ such that

$$\begin{gathered} Q = \sup\{|Q_n(t,x)|,\ t \in [\sigma,\tau],\ x \in \overline{\Omega},\ n \ge 1\} < b\varepsilon, \\ \sup\{|q_n(t) - q(t)|,\ \sigma \le t \le \tau\} \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty, \end{gathered} \tag{4.15}$$

with

$$q_n(t)(x) = Q_n(t,x), \quad x \in \overline{\Omega}. \tag{4.16}$$

Now, we consider the equation

$$\begin{gathered} \frac{\partial v(t,x)}{\partial t} = a\triangle v(t,x) - bv(t,x) + Q_n(t,x) \quad \text{in } (\sigma,\tau]\times\Omega, \\ v(t,x) = 0 \quad \text{in } (\sigma,\tau]\times\partial\Omega, \\ v(\sigma,x) = \varphi(0)(x), \quad x \in \overline{\Omega}. \end{gathered} \tag{4.17}$$

For the regularity of the solutions of (4.17), we have

Lemma 4.3 [15, Theorem 4.3.1, page 134]. *If $(Q_n)_{n\ge 0} \subset C^1([\sigma,\tau]\times\overline{\Omega},\mathbb{R})$ and $\varphi(0) \in \overline{D(A)}$, then (4.17) has a C^1-solution on $(\sigma,\tau]$.*

Let v_n be the C^1-solution on $(\sigma,\tau]$ of (4.17) obtained by Lemma 4.3, let $\vartheta_n(t)(x) = v_n(t,x)$, for $(t,x) \in (\sigma,\tau] \times \overline{\Omega}$, then for $t \in [\sigma,\tau]$ one has

$$\vartheta_n(t) = T_0(t-\sigma)\varphi(0) + \lim_{\lambda\to+\infty}\int_\sigma^t T_0(t-s)\lambda R(\lambda,A)\big[-b\vartheta_n(s)+q_n(s)\big]ds. \tag{4.18}$$

Then $(\vartheta_n)_{n\geq 0}$ converges to v uniformly on $[\sigma,\tau]$ as $n \to +\infty$. In fact,

$$v(t) = T_0(t-\sigma)\varphi(0) + \lim_{\lambda\to+\infty}\int_\sigma^t T_0(t-s)\lambda R(\lambda,A)\big[-bv(s)+q(s)\big]ds, \tag{4.19}$$

then

$$\vartheta_n(t) - v(t) = \lim_{\lambda\to+\infty}\int_\sigma^t T_0(t-s)\lambda R(\lambda,A)\big[b\big(v(s)-\vartheta_n(s)\big)+q_n(s)-q(s)\big]ds. \tag{4.20}$$

Since $\sup_{s\in[\sigma,\tau]}|q_n(s)-q(s)| \to 0$ as $n \to +\infty$, then by Gronwall's lemma we obtain that

$$\sup_{s\in[\sigma,\tau]}\big|\vartheta_n(t)-v(t)\big| \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty. \tag{4.21}$$

Let $k = \max(|\varphi(0)|, Q/b)$ and $n_0 \in \mathbb{N}$ such that $k + 1/n_0 < \varepsilon$, then we claim that

$$\big|v_n(t,x)\big| < k + \frac{1}{n} \quad \text{for } (t,x) \in [\sigma,\tau) \times \overline{\Omega},\ n \geq n_0. \tag{4.22}$$

Assume that there exist $n \geq n_0$ and $(t_0,x_0) \in (\sigma,\tau) \times \overline{\Omega}$ such that

$$\big|v_n(t_0,x_0)\big| = k + \frac{1}{n}, \qquad \big|v_n(t,x)\big| < k + \frac{1}{n} \quad \text{for } (t,x) \in [\sigma,t_0) \times \overline{\Omega}. \tag{4.23}$$

If $v_n(t_0,x_0) = k + 1/n$, then $x_0 \in \Omega$. Putting $w(t,x) = v_n(t,x) - k - 1/n$ for any $(t,x) \in [\sigma,\tau) \times \overline{\Omega}$, then $w(t_0,x_0) = 0$ and $w(t,x) < 0$ for $(t,x) \in [\sigma,t_0) \times \overline{\Omega}$. In addition, one has

$$\frac{\partial w(t,x)}{\partial t} = a\triangle w(t,x) - b\left[w(t,x) + k + \frac{1}{n_0}\right] + Q_n(t,x), \tag{4.24}$$

then in $(\sigma,t_0] \times \Omega$, we have

$$a\triangle w(t,x) - \frac{\partial w(t,x)}{\partial t} - bw(t,x) = b\left[k + \frac{1}{n_0}\right] - Q_n(t,x) \geq Q - Q_n(t,x), \tag{4.25}$$

hence,

$$a\triangle w(t,x) - \frac{\partial w(t,x)}{\partial t} - bw(t,x) \geq 0 \quad \text{for } (t,x) \in (\sigma,t_0] \times \Omega. \tag{4.26}$$

Since $w(t_0,x_0) = 0$ and $w(t,x) < 0$ for all $(t,x) \in [\sigma,t_0) \times \overline{\Omega}$, we obtain that $w(t,x) = 0$ for $(t,x) \in [\sigma,t_0] \times \overline{\Omega}$ by the strong maximum principle [16, Theorem 3.7] we have $w(\sigma,x) = 0$, that is, $v_n(\sigma,x) = k + 1/n$ on $\overline{\Omega}$, which is a contradiction, since $|v_n(\sigma,x)| < k + 1/n$ on $\overline{\Omega}$.

If $v_n(t_0,x_0) = -k - 1/n$, then $x_0 \in \Omega$. Putting $w(t,x) = -v_n(t,x) - k - 1/n$ for any $(t,x) \in [\sigma,\tau) \times \overline{\Omega}$, then $w(t_0,x_0) = 0$ and $w(t,x) < 0$, for $(t,x) \in [\sigma,t_0) \times \overline{\Omega}$. In addition, one has

$$a\triangle w(t,x) - \frac{\partial w(t,x)}{\partial t} - bw(t,x) \geq 0 \quad \text{for } (t,x) \in (\sigma,t_0] \times \Omega. \tag{4.27}$$

Since $w(t_0,x_0) = 0$ and $w(t,x) < 0$ for $(t,x) \in [\sigma,t_0) \times \overline{\Omega}$ and by the strong maximum principle, we obtain that $w(t,x) = 0$ for all $(t,x) \in [\sigma,t_0] \times \overline{\Omega}$ and hence $w(\sigma,x) = 0$, that is, $v_n(\sigma,x) = -k - 1/n$ on $\overline{\Omega}$, which is a contradiction because $|v_n(\sigma,x)| < k + 1/n$ on $\overline{\Omega}$. Consequently, (4.22) must be true, letting $n \to +\infty$ in (4.22), we obtain that

$$|v(t)| \leq k < \varepsilon \quad \text{for } t \in [\sigma,\tau], \tag{4.28}$$

which implies $|v(\tau)| < \varepsilon$, which is a contradiction because of $|v(\tau)| = \varepsilon$. Consequently, the null solution of (1.4) is $\mathcal{BC}$-TS. □

As an immediate consequence of Corollary 3.5 and Theorem 4.2, we obtain the main result of this part.

Theorem 4.4. *Assume that* (**b**) *is satisfied with $b > c$. Then there exists an almost periodic solution of (4.9).*

Remark 4.5. The above result has been established in [13], where the authors worked on $C_0(\overline{\Omega})$ instead of $C(\overline{\Omega})$.

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