EXISTENCE OF SOLUTIONS OF GENERAL NONLINEAR FUZZY VOLTERRA-FREDHOLM INTEGRAL EQUATIONS

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We study the problem of existence and uniqueness of solutions of a class of nonlinear fuzzy Volterra-Fredholm integral equations.

1. Introduction

Fuzzy differential and integral equations have been studied by many authors [1, 2, 5, 6, 7, 14]. Kaleva [5] discussed the properties of differentiable fuzzy set-valued mappings by means of the concept of H-differentiability introduced by Puri and Ralescu [9]. Seikkala [11] defined the fuzzy derivative which is a generalization of the Hukuhara derivative [9] and the fuzzy integral which is the same as that proposed by Dubois and Prade [3, 4]. Balachandran and Dauer [1] established the existence of solutions of perturbed fuzzy integral equations. Subrahmanyam and Sudarsanam [13] studied fuzzy Volterra integral equations. Park and Jeong [8] proved the existence and uniqueness of solutions of fuzzy Volterra-Fredholm integral equations of the form

$$x(t) = F\left(t, x(t), \int_0^t f(t,s,x(s))\,ds \int_0^T g(t,s,x(s))\,ds\right), \tag{1.1}$$

and Balachandran and Prakash [2] studied the same problem for the nonlinear fuzzy Volterra-Fredholm integral equations of the form

$$x(t) = f(t,x(t)) + F\left(t, x(t), \int_0^t g(t,s,x(s))\,ds, \int_0^T h(t,s,x(s))\,ds\right). \tag{1.2}$$

The purpose of this paper is to prove the existence and uniqueness of solutions of general nonlinear fuzzy Volterra-Fredholm integral equations of the form

$$\begin{aligned} x(t) = F\Bigg(t, x(t), &\int_0^t f_1(t,s,x(s))\,ds, \ldots, \int_0^t f_m(t,s,x(s))\,ds, \\ &\int_0^T g_1(t,s,x(s))\,ds, \ldots, \int_0^T g_m(t,s,x(s))\,ds\Bigg), \quad 0 \le t \le T. \end{aligned} \tag{1.3}$$

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2. Preliminaries

Let $P_K(R^n)$ denote the family of all nonempty, compact, convex subsets of R^n. Addition and scalar multiplication in $P_K(R^n)$ are defined as usual. Let A and B be two nonempty bounded subsets of R^n. The distance between A and B is defined by the Hausdorff metric $d(A,B) = \max\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\}$, where $\|\cdot\|$ denote the usual Euclidean norm in R^n. Then it is clear that $(P_K(R^n),d)$ becomes a metric space.

Let $I = [0,1] \subseteq R$ be a compact interval and denote

$$E^n = \{u : R^n \to I : u \text{ satisfies (i)–(iv) below}\}, \tag{2.1}$$

where

(i) u is normal, that is, there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,

(ii) u is fuzzy convex,

(iii) u is upper semicontinuous,

(iv) $[u]^0 = \text{cl}\{x \in R^n : u(x) > 0\}$ is compact.

For $0 < \alpha \le 1$ denote $[u]^\alpha = \{x \in R^n : u(x) \ge \alpha\}$. Then from (i)–(iv) it follows that the α-level set $[u]^\alpha \in P_K(R^n)$ for all $0 \le \alpha \le 1$.

If $g : R^n \times R^n \to R^n$ is a function, then using Zadeh's extension principle we can extend g to $E^n \times E^n \to E^n$ by the equation

$$\tilde{g}(u,v)(z) = \sup_{z=g(x,y)} \min\{u(x), v(y)\}. \tag{2.2}$$

It is well known that $[\tilde{g}(u,v)]^\alpha = g([u]^\alpha,[v]^\alpha)$ for all $u,v \in E^n$, $0 \le \alpha \le 1$, and continuous function g. Further, we have $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$, $[ku]^\alpha = k[u]^\alpha$, where $k \in R$. The real numbers can be embedded in E^n by the rule $c \to \hat{c}(t)$ where

$$\hat{c}(t) = \begin{cases} 1 & \text{for } t = c, \\ 0 & \text{elsewhere.} \end{cases} \tag{2.3}$$

Let $D : E^n \times E^n \to R^+$ be defined by $D(u,v) = \sup_{0\le\alpha\le1} d([u]^\alpha,[v]^\alpha)$, where d is the Hausdorff metric defined in $P_K(R^n)$. Then D is a metric in E^n and (E^n,D) is a complete metric space [5, 10]. Further $D(u+w,v+w) = D(u,v)$ and $D(\lambda u,\lambda v) = |\lambda| D(u,v)$ for every $u,v,w \in E^n$ and $\lambda \in R$.

It can be proved that $D(u+v,w+z) \le D(u,w) + D(v,z)$ for u, v, w, and $z \in E^n$.

Definition 2.1 [5]. A mapping $F : I \to E^n$ is strongly measurable if for all $\alpha \in [0,1]$, the set-valued map $F_\alpha : I \to P_K(R^n)$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is Lebesgue measurable when $P_K(R^n)$ has the topology induced by the Hausdorff metric d.

Definition 2.2 [5]. A mapping $F : I \to E^n$ is said to be integrably bounded if there is an integrable function $h(t)$ such that $\|x(t)\| \le h(t)$ for every $x \in F_0(t)$.

Definition 2.3 [10]. The integral of a fuzzy mapping $F : I \to E^n$ is defined level-wise by

$$\begin{aligned}\left[\int_I F(t)dt\right]^\alpha &= \int_I F_\alpha(t)dt \\ &= \left\{\int_I f(t)dt \mid f : I \to R^n \text{ is a measurable selection for } F_\alpha\right\}\end{aligned} \tag{2.4}$$

for all $\alpha \in [0,1]$.

It has been proved by Puri and Ralescu [10] that a strongly measurable and integrably bounded mapping $F : I \to E^n$ is integrable (i.e., $\int_I F(t)dt \in E^n$).

Theorem 2.4. *If $F : I \to E^n$ is continuous, then it is integrable.*

Theorem 2.5. *Let $F, G : I \to E^n$ be integrable and $\lambda \in R$. Then*
(i) $\int_I (F(t) + G(t))dt = \int_I F(t)dt + \int_I G(t)dt$,
(ii) $\int_I \lambda F(t)dt = \lambda \int_I F(t)dt$,
(iii) $D(F, G)$ *is integrable,*
(iv) $D(\int_I F(t)dt, \int_I G(t)dt) \le \int_I D(F(t), G(t))dt$.

Now we make the following assumptions.

(A_1) Let $J = [0,T]$, $\Delta = \{(t,s) : 0 \le s \le t \le T\}$. If $f_i, g_i \in C(\Delta \times E^n, E^n)$, $i = 1,2,\ldots,m$, $F \in C(J \times E^{(2m+1)n}, E^n)$ and if $x \in C(J,E^n)$ and

$$\begin{aligned}z(t) = F\bigg(&t, x(t), \int_0^t f_1(t,s,x(s))ds, \ldots, \int_0^t f_m(t,s,x(s))ds, \\ &\int_0^T g_1(t,s,x(s))ds, \ldots, \int_0^T g_m(t,s,x(s))ds\bigg),\end{aligned} \tag{2.5}$$

then $z \in C(J,E^n)$.

(A_2) There exist functions $\omega_i(t,s,p)$, $\hat{\omega}_i(t,s,p)$ such that $\omega_i, \hat{\omega}_i \in C(\Delta \times R^+, R^+)$, $R^+ = [0,\infty)$, which are nondecreasing in p and fulfil the conditions

$$\begin{aligned}D\big(f_i(t,s,x(s)), f_i(t,s,\overline{x}(s))\big) &\le \omega_i\big(t,s,D(x(s),\overline{x}(s))\big), \\ D\big(g_i(t,s,x(s)), g_i(t,s,\overline{x}(s))\big) &\le \hat{\omega}_i\big(t,s,D(x(s),\overline{x}(s))\big) \\ &\qquad \text{for } x, \overline{x} \in C(J,E^n),\ i = 1,2,\ldots,m.\end{aligned} \tag{2.6}$$

(A_3) There exists a function $H(t, p_1, p_2, \ldots, p_{2m+1})$ defined for $t \in J$ and $0 \le p_1 \le p_2 \le \cdots \le p_{2m+1} < \infty$ such that
(i) if $u \in C(J,J)$ and

$$\begin{aligned}v(t) = H\bigg(&t, u(t), \int_0^t \omega_1(t,s,u(s))ds, \ldots, \int_0^t \omega_m(t,s,u(s))ds, \\ &\int_0^T \hat{\omega}_1(t,s,u(s))ds, \ldots, \int_0^T \hat{\omega}_m(t,s,u(s))ds\bigg),\end{aligned} \tag{2.7}$$

then $v \in C(J,J)$;

(ii) if $u,\overline{u} \in C(J,J)$ and $u(t) \le \overline{u}(t)$ for $t \in J$, then

$$\begin{aligned} &H\bigg(t,u(t),\int_0^t \omega_1(t,s,u(s))\,ds,\ldots,\int_0^t \omega_m(t,s,u(s))\,ds,\\ &\quad\int_0^T \hat{\omega}_1(t,s,u(s))\,ds,\ldots,\int_0^T \hat{\omega}_m(t,s,u(s))\,ds\bigg)\\ &\quad\le H\bigg(t,\overline{u}(t),\int_0^t \omega_1(t,s,\overline{u}(s))\,ds,\ldots,\int_0^t \omega_m(t,s,\overline{u}(s))\,ds,\\ &\qquad\int_0^T \hat{\omega}_1(t,s,\overline{u}(s))\,ds,\ldots,\int_0^T \hat{\omega}_m(t,s,\overline{u}(s))\,ds\bigg)\quad \text{for } t\in J; \end{aligned} \tag{2.8}$$

(iii) if $u_n \in C(J,J)$, $u_{n+1}(t) \le u_n(t)$, $t \in J$, $n = 0,1,2,\ldots$, and $\lim_{n\to\infty} u_n(t) = u(t)$, then

$$\begin{aligned} &\lim_{n\to\infty} H\bigg(t,u_n(t),\int_0^t \omega_1(t,s,u_n(s))\,ds,\ldots,\int_0^t \omega_m(t,s,u_n(s))\,ds,\\ &\qquad\int_0^T \hat{\omega}_1(t,s,u_n(s))\,ds,\ldots,\int_0^T \hat{\omega}_m(t,s,u_n(s))\,ds\bigg)\\ &\quad= H\bigg(t,u(t),\int_0^t \omega_1(t,s,u(s))\,ds,\ldots,\int_0^t \omega_m(t,s,u(s))\,ds,\\ &\qquad\int_0^T \hat{\omega}_1(t,s,u(s))\,ds,\ldots,\int_0^T \hat{\omega}_m(t,s,u(s))\,ds\bigg). \end{aligned} \tag{2.9}$$

(A_4)

$$\begin{aligned} &D\big(F(t,x_1(t),x_2(t),\ldots,x_{2m+1}(t)),F(t,\overline{x}_1(t),\overline{x}_2(t),\ldots,\overline{x}_{2m+1}(t))\big)\\ &\quad\le H\big(t,D(x_1(t),\overline{x}_1(t)),D(x_2(t),\overline{x}_2(t)),\ldots,D(x_{2m+1}(t),\overline{x}_{2m+1}(t))\big) \end{aligned} \tag{2.10}$$

holds for $x_i,\overline{x}_i \in C(J,E^n)$, $t \in J$, $i = 1,2,\ldots,(2m+1)$.

(A_5) There exists a nonnegative continuous function $\overline{u} : J \to R^+$ being the solution of the inequality,

$$\begin{aligned} &H\bigg(t,u(t),\int_0^t w_1(t,s,u(s))\,ds,\ldots,\int_0^t w_m(t,s,u(s))\,ds,\\ &\qquad\int_0^T \hat{w}_1(t,s,u(s))\,ds,\ldots,\int_0^T \hat{w}_m(t,s,u(s))\,ds\bigg)+q(t)\le u(t), \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} q(t) = \sup_{t\in J} D\bigg(F\bigg(&t,\hat{0},\int_0^t f_1(t,s,\hat{0})ds,\ldots,\int_0^t f_m(t,s,\hat{0})ds,\\ &\int_0^T g_1(t,s,\hat{0})ds,\ldots,\int_0^T g_m(t,s,\hat{0})ds\bigg),\hat{0}\bigg). \end{aligned} \tag{2.12}$$

(A_6) In the class of functions satisfying the condition $0 \le u(t) \le \overline{u}(t)$, $t \in J$, the function $u(t) \equiv 0$, $t \in J$, is the only solution of the equation

$$\begin{aligned} u(t) = H\bigg(t, u(t), \int_0^t w_1(t,s,u(s))\,ds, \ldots, \int_0^t w_m(t,s,u(s))\,ds, \\ \int_0^T \hat{w}_1(t,s,u(s))\,ds, \ldots, \int_0^T \hat{w}_m(t,s,u(s))\,ds\bigg). \end{aligned} \tag{2.13}$$

In order to prove the existence of a solution of (1.3), we define the sequence

$$\begin{aligned} x_0(t) &\equiv \hat{0}, \\ x_{n+1}(t) &= F\bigg(t, x_n(t), \int_0^t f_1(t,s,x_n(s))\,ds, \ldots, \int_0^t f_m(t,s,x_n(s))\,ds, \\ &\qquad \int_0^T g_1(t,s,x_n(s))\,ds, \ldots, \int_0^T g_m(t,s,x_n(s))\,ds\bigg) \end{aligned} \tag{2.14}$$

for $n = 0, 1, 2, \ldots$.

To prove the convergence of the sequence $\{x_n\}$ to the solution $\overline{x}$ of (1.3), we define the sequence $\{u_n\}$ by the relations

$$\begin{aligned} u_0(t) &= \overline{u}(t), \\ u_{n+1}(t) &= H\bigg(t, u_n(t), \int_0^t w_1(t,s,u_n(s))\,ds, \ldots, \int_0^t w_m(t,s,u_n(s))\,ds, \\ &\qquad \int_0^T \hat{w}_1(t,s,u_n(s))\,ds, \ldots, \int_0^T \hat{w}_m(t,s,u_n(s))\,ds\bigg) \end{aligned} \tag{2.15}$$

for $n = 0, 1, 2, \ldots$, where the function $\overline{u}(t)$ is from the assumptions (A_5) and (A_6).

LEMMA 2.6. *If the conditions* (A_3), (A_5), *and* (A_6) *are satisfied, then*

$$\begin{aligned} 0 \le u_{n+1}(t) \le u_n(t) \le \overline{u}(t), \quad t \in J,\ n = 0, 1, 2, \ldots, \\ \lim_{n \to \infty} u_n(t) = 0, \quad t \in J, \end{aligned} \tag{2.16}$$

and the convergence is uniform in each bounded set.

Proof. From (2.11) and (2.15) we have

$$\begin{aligned} u_1(t) &= H\bigg(t,u_0(t),\int_0^t w_1(t,s,u_0(s))\,ds,\dots,\int_0^t w_m(t,s,u_0(s))\,ds,\\ &\qquad \int_0^T \hat{w}_1(t,s,u_0(s))\,ds,\dots,\int_0^T \hat{w}_m(t,s,u_0(s))\,ds\bigg)\\ &\le H\bigg(t,\overline{u}(t),\int_0^t w_1(t,s,\overline{u}(s))\,ds,\dots,\int_0^t w_m(t,s,\overline{u}(s))\,ds,\\ &\qquad \int_0^T \hat{w}_1(t,s,\overline{u}(s))\,ds,\dots,\int_0^T \hat{w}_m(t,s,\overline{u}(s))\,ds\bigg)+q(t)\\ &\le \overline{u}(t)=u_0(t) \end{aligned} \tag{2.17}$$

for $t\in J$. Further, we obtain (2.16) by induction. But (2.16) implies the convergence of the sequence $\{u_n(t)\}$ to some nonnegative function $\phi(t)$ for $t\in J$. By Lebesgue's theorem and the continuity of H, it follows that the function $\phi(t)$ satisfies (2.13). Now from assumptions (A_5) and (A_6), we have $\phi(t)\equiv 0$, $t\in J$. Hence by the Dini theorem [12], the sequence $\{u_n\}$ converges uniformly in J. □

3. Main results

Theorem 3.1. *If the assumptions (A_1)–(A_6) are satisfied, then there exists a continuous solution $\overline{x}$ of (1.3). The sequence $\{x_n\}$ defined by (2.14) converges uniformly on J to $\overline{x}$, and the following estimates:*

$$D(\overline{x}(t),x_n(t))\le u_n(t),\quad t\in J,\ n=0,1,2,\dots, \tag{3.1}$$

$$D(\overline{x}(t),\hat{0})\le \overline{u}(t),\quad t\in J \tag{3.2}$$

hold. The solution $\overline{x}$ of (1.3) is unique in the class of functions satisfying the condition (3.2).

Proof. We first prove that the sequence $\{x_n(t)\}$, $t\in J$, fulfils the condition

$$D(x_n(t),\hat{0})\le \overline{u}(t),\quad t\in J,\ n=0,1,2,\dots. \tag{3.3}$$

Obviously, we see that $D(x_0(t),\hat{0})=0\le\overline{u}(t)$, $t\in J$. Further, if we suppose that inequality (3.3) is true for $n\ge 0$, then

$$\begin{aligned} &D(x_{n+1}(t),\hat{0})\\ &\quad\le D(x_{n+1}(t),x_1(t))+D(x_1(t),\hat{0})\\ &\quad\le D\Bigg(F\bigg(t,x_n(t),\int_0^t f_1(t,s,x_n(s))\,ds,\dots,\int_0^t f_m(t,s,x_n(s))\,ds,\\ &\qquad\qquad \int_0^T g_1(t,s,x_n(s))\,ds,\dots,\int_0^T g_m(t,s,x_n(s))\,ds\bigg), \end{aligned}$$

$$
\begin{aligned}
&F\bigg(t,\hat{0},\int_0^t f_1(t,s,\hat{0})ds,\ldots,\int_0^t f_m(t,s,\hat{0})ds,\int_0^T g_1(t,s,\hat{0})ds,\ldots,\int_0^T g_m(t,s,\hat{0})ds\bigg)\bigg)\\
&\quad+D\bigg(F\bigg(t,\hat{0},\int_0^t f_1(t,s,\hat{0})ds,\ldots,\int_0^t f_m(t,s,\hat{0})ds,\int_0^T g_1(t,s,\hat{0})ds,\ldots,\int_0^T g_m(t,s,\hat{0})ds\bigg),\hat{0}\bigg)\\
&\le H\bigg(t,D(x_n(t),\hat{0}),D\bigg(\int_0^t f_1(t,s,x_n(s))ds,\int_0^t f_1(t,s,\hat{0})ds\bigg),\ldots,\\
&\qquad D\bigg(\int_0^t f_m(t,s,x_n(s))ds,\int_0^t f_m(t,s,\hat{0})ds\bigg),\\
&\qquad D\bigg(\int_0^T g_1(t,s,x_n(s))ds,\int_0^T g_1(t,s,\hat{0})ds\bigg),\ldots,\\
&\qquad D\bigg(\int_0^T g_m(t,s,x_n(s))ds,\int_0^T g_m(t,s,\hat{0})ds\bigg)\bigg)+q(t)\\
&\le H\bigg(t,D(x_n(t),\hat{0}),\int_0^t D(f_1(t,s,x_n(s)),f_1(t,s,\hat{0}))ds,\ldots,\\
&\qquad \int_0^t D(f_m(t,s,x_n(s)),f_m(t,s,\hat{0}))ds,\int_0^T D(g_1(t,s,x_n(s)),g_1(t,s,\hat{0}))ds,\ldots,\\
&\qquad \int_0^T D(g_m(t,s,x_n(s)),g_m(t,s,\hat{0}))ds\bigg)+q(t)\\
&\le H\bigg(t,D(x_n(t),\hat{0}),\int_0^t w_1(t,s,D(x_n(s),\hat{0}))ds,\ldots,\int_0^t w_m(t,s,D(x_n(s),\hat{0}))ds,\\
&\qquad \int_0^T \hat{w}_1(t,s,D(x_n(s),\hat{0}))ds,\ldots,\int_0^T \hat{w}_m(t,s,D(x_n(s),\hat{0}))ds\bigg)+q(t)\\
&\le H\bigg(t,\overline{u}(t),\int_0^t w_1(t,s,\overline{u}(s))ds,\ldots,\int_0^t w_m(t,s,\overline{u}(s))ds,\\
&\qquad \int_0^T \hat{w}_1(t,s,\overline{u}(s))ds,\ldots,\int_0^T \hat{w}_m(t,s,\overline{u}(s))ds\bigg)+q(t)\\
&\le \overline{u}(t)\quad \text{for } t\in J.
\end{aligned}
\tag{3.4}
$$

Now we obtain (3.3) by induction. Next, we prove that

$$D(x_{n+r}(t),x_n(t))\le u_n(t),\quad t\in J,\ n=0,1,2,\ldots,\ r=0,1,2,\ldots. \tag{3.5}$$

By (3.3), we have

$$D(x_r(t),x_0(t))=D(x_r(t),\hat{0})\le \overline{u}(t)=u_0(t),\quad t\in J,\ r=0,1,2,\ldots. \tag{3.6}$$

Suppose that (3.5) is true for $n, r \geq 0$, then

$$
\begin{aligned}
&D(x_{n+r+1}(t), x_{n+1}(t)) \\
&\quad = D\Bigg(F\Bigg(t, x_{n+r}(t), \int_0^t f_1(t,s,x_{n+r}(s))\,ds, \dots, \int_0^t f_m(t,s,x_{n+r}(s))\,ds, \\
&\qquad\qquad \int_0^T g_1(t,s,x_{n+r}(s))\,ds, \dots, \int_0^T g_m(t,s,x_{n+r}(s))\,ds\Bigg), \\
&\qquad\quad F\Bigg(t, x_n(t), \int_0^t f_1(t,s,x_n(s))\,ds, \dots, \int_0^t f_m(t,s,x_n(s))\,ds, \\
&\qquad\qquad \int_0^T g_1(t,s,x_n(s))\,ds, \dots, \int_0^T g_m(t,s,x_n(s))\,ds\Bigg)\Bigg) \\
&\quad \leq H\Bigg(t, D(x_{n+r}(t), x_n(t)), D\Bigg(\int_0^t f_1(t,s,x_{n+r}(s))\,ds, \int_0^t f_1(t,s,x_n(s))\,ds\Bigg), \dots, \\
&\qquad\quad D\Bigg(\int_0^t f_m(t,s,x_{n+r}(s))\,ds, \int_0^t f_m(t,s,x_n(s))\,ds\Bigg), \\
&\qquad\quad D\Bigg(\int_0^T g_1(t,s,x_{n+r}(s))\,ds, \int_0^T g_1(t,s,x_n(s))\,ds\Bigg), \dots, \\
&\qquad\quad D\Bigg(\int_0^T g_m(t,s,x_{n+r}(s))\,ds, \int_0^T g_m(t,s,x_n(s))\,ds\Bigg)\Bigg) \\
&\quad \leq H\Bigg(t, D(x_{n+r}(t), x_n(t)), \int_0^t D(f_1(t,s,x_{n+r}(s)), f_1(t,s,x_n(s)))\,ds, \dots, \\
&\qquad\quad \int_0^t D(f_m(t,s,x_{n+r}(s)), f_m(t,s,x_n(s)))\,ds, \\
&\qquad\quad \int_0^T D(g_1(t,s,x_{n+r}(s)), g_1(t,s,x_n(s)))\,ds, \dots, \\
&\qquad\quad \int_0^T D(g_m(t,s,x_{n+r}(s)), g_m(t,s,x_n(s)))\,ds\Bigg) \\
&\leq H\Bigg(t, D(x_{n+r}(t), x_n(t)), \int_0^t w_1(t,s,D(x_{n+r}(s), x_n(s)))\,ds, \dots, \\
&\qquad\quad \int_0^t w_m(t,s,D(x_{n+r}(s), x_n(s)))\,ds, \int_0^T \hat{w}_1(t,s,D(x_{n+r}(s), x_n(s)))\,ds, \dots, \\
&\qquad\quad \int_0^T \hat{w}_m(t,s,D(x_{n+r}(s), x_n(s)))\,ds\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&\leq H\bigg(t, u_n(t), \int_0^t w_1(t,s,u_n(s))\,ds, \dots, \int_0^t w_m(t,s,u_n(s))\,ds, \\
&\qquad \int_0^T \hat{w}_1(t,s,u_n(s))\,ds, \dots, \int_0^T \hat{w}_m(t,s,u_n(s))\,ds\bigg) \\
&\leq u_{n+1}(t) \quad \text{for } t \in J.
\end{aligned}
\tag{3.7}
$$

Now we obtain (3.5) by induction.

Because of Lemma 2.6, $\lim_{n\to\infty} u_n(t) = 0$ in J and we have from (3.5) that $x_n \to \overline{x}$ in J. The continuity of $\overline{x}$ follows from the uniform convergence of the sequence $\{x_n\}$ and the continuity of all functions x_n. If $r \to \infty$, then (3.5) gives estimation (3.1). Estimation (3.2) implies (3.3). It is obvious that $\overline{x}$ is a solution of (1.3).

To prove that the solution $\overline{x}$ is a unique solution of (1.3) in the class of functions satisfying the condition (3.2), we suppose that there exists another solution $\hat{x}$ defined in J such that $\overline{x}(t) \neq \hat{x}(t)$ and $\|\hat{x}(t)\| \leq \overline{u}(t)$ for $t \in J$. From (3.1) we get $D(\hat{x}(t), x_n(t)) \leq u_n(t)$, $t \in J$, $n = 0, 1, 2, \dots$ and it follows that $\overline{x}(t) = \hat{x}(t)$ for $t \in J$. This contradiction proves the uniqueness of $\overline{x}$ in the class of functions satisfying the relation (3.2). This completes the proof of the theorem. □

Theorem 3.2. *If the assumptions* (A_1)–(A_4) *are satisfied and the function* $y(t) \equiv 0$, $t \in J$, *is the only nonnegative continuous solution of the inequality*

$$
\begin{aligned}
y(t) \leq H\bigg(&t, y(t), \int_0^t w_1(t,s,y(s))\,ds, \dots, \int_0^t w_m(t,s,y(s))\,ds, \\
&\int_0^T \hat{w}_1(t,s,y(s))\,ds, \dots, \int_0^T \hat{w}_m(t,s,y(s))\,ds\bigg), \quad t \in J,
\end{aligned}
\tag{3.8}
$$

then (1.3) has at most one solution in J.

Proof. We suppose that there exist two solutions $\overline{x}$ and $\hat{x}$ of (1.3) such that $\overline{x}(t) \neq \hat{x}(t)$, $t \in J$. Put $y(t) = D(\overline{x}(t), \hat{x}(t))$, $t \in J$, then

$$
\begin{aligned}
y(t) &= D(\overline{x}(t), \hat{x}(t)) \\
&= D\bigg(F\bigg(t, \overline{x}(t), \int_0^t f_1(t,s,\overline{x}(s))\,ds, \dots, \int_0^t f_m(t,s,\overline{x}(s))\,ds, \\
&\qquad \int_0^T g_1(t,s,\overline{x}(s))\,ds, \dots, \int_0^T g_m(t,s,\overline{x}(s))\,ds\bigg), \\
&\qquad F\bigg(t, \hat{x}(t), \int_0^t f_1(t,s,\hat{x}(s))\,ds, \dots, \int_0^t f_m(t,s,\hat{x}(s))\,ds, \\
&\qquad \int_0^T g_1(t,s,\hat{x}(s))\,ds, \dots, \int_0^T g_m(t,s,\hat{x}(s))\,ds\bigg)\bigg)
\end{aligned}
$$

$$
\begin{aligned}
&\le H\bigg(t, D(\overline{x}(t),\hat{x}(t)), D\bigg(\int_0^t f_1(t,s,\overline{x}(s))\,ds, \int_0^t f_1(t,s,\hat{x}(s))\,ds\bigg),\dots,\\
&\qquad D\bigg(\int_0^t f_m(t,s,\overline{x}(s))\,ds, \int_0^t f_m(t,s,\hat{x}(s))\,ds\bigg),\\
&\qquad D\bigg(\int_0^T g_1(t,s,\overline{x}(s))\,ds, \int_0^T g_1(t,s,\hat{x}(s))\,ds\bigg),\dots,\\
&\qquad D\bigg(\int_0^T g_m(t,s,\overline{x}(s))\,ds, \int_0^T g_m(t,s,\hat{x}(s))\,ds\bigg)\bigg)\\
&\le H\bigg(t, D(\overline{x}(t),\hat{x}(t)), \int_0^t D\big(f_1(t,s,\overline{x}(s)), f_1(t,s,\hat{x}(s))\big)\,ds,\dots,\\
&\qquad \int_0^t D\big(f_m(t,s,\overline{x}(s)), f_m(t,s,\hat{x}(s))\big)\,ds,\\
&\qquad \int_0^T D\big(g_1(t,s,\overline{x}(s)), g_1(t,s,\hat{x}(s))\big)\,ds,\dots,\\
&\qquad \int_0^T D\big(g_m(t,s,\overline{x}(s)), g_m(t,s,\hat{x}(s))\big)\,ds\bigg)\\
&\le H\bigg(t, D(\overline{x}(t),\hat{x}(t)), \int_0^t w_1\big(t,s,D(\overline{x}(s),\hat{x}(s))\big)\,ds,\dots,\\
&\qquad \int_0^t w_m\big(t,s,D(\overline{x}(s),\hat{x}(s))\big)\,ds,\\
&\qquad \int_0^T \hat{w}_1\big(t,s,D(\overline{x}(s),\hat{x}(s))\big)\,ds,\dots,\\
&\qquad \int_0^T \hat{w}_m\big(t,s,D(\overline{x}(s),\hat{x}(s))\big)\,ds\bigg)\\
&\le H\bigg(t, y(t), \int_0^t w_1(t,s,y(s))\,ds,\dots, \int_0^t w_m(t,s,y(s))\,ds,\\
&\qquad \int_0^T \hat{w}_1(t,s,y(s))\,ds,\dots, \int_0^T \hat{w}_m(t,s,y(s))\,ds\bigg)
\end{aligned}
\tag{3.9}
$$

and by (3.8) we conclude that $y(t) \equiv 0$ for $t \in J$, that is, $\overline{x}(t) = \hat{x}(t)$, $t \in J$. This contradiction proves our Theorem 3.2. □

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