MODERATE DEVIATIONS FOR BOUNDED SUBSEQUENCES

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We study Davis' series of moderate deviations probabilities for L^p-bounded sequences of random variables ($p > 2$). A certain subseries therein is convergent for the same range of parameters as in the case of martingale difference or i.i.d. sequences.

1. Introduction and main results

Let $(X_n)_{n\geq 1}$ be a sequence of random variables on a probability space $(\Omega,\mathcal{F},P)$ and $p \geq 1$ a fixed real number. We say that $(X_n)_{n\geq 1}$ is L^p-bounded if it has finite pth moments, that is, $\|X_n\|_p \leq C$ for some $C > 0$ and any $n \geq 1$. Let $\varepsilon > 0$; finding the rate of convergence of the moderate deviations probabilities $P[|\sum_{k=1}^n X_k| > \varepsilon a_n]$ with $a_n = (n\log n)^{1/2}$ or $(n\log\log n)^{1/2}$ is known in the literature as Davis' problems. More precisely, let $\delta = \delta(p) \geq 0$ be a function of $p \geq 1$ and consider the series

$$\sum_{n=2}^{\infty}\frac{(\log n)^{\delta}}{n}P\left[\left|\sum_{k=1}^{n}X_k\right| > \varepsilon(n\log n)^{1/2}\right],$$
$$\sum_{n=3}^{\infty}\frac{1}{n(\log n)^{\delta}}P\left[\left|\sum_{k=1}^{n}X_k\right| > \varepsilon(n\log\log n)^{1/2}\right]; \tag{1.1}$$

the convergence of series (2.1) has been studied by Davis (see [7, 8]) and Gut (see [10]) when $(X_n)_{n\geq 1}$ are L^p-bounded i.i.d. sequences, and by Stoica (see [14, 15]) when $(X_n)_{n\geq 1}$ are L^p-bounded martingale difference sequences.

In the sequel, we are interested in Davis' theorems under the only assumption that $(X_n)_{n\geq 1}$ is L^p-bounded. Our results rely on the "subsequence principle," that is, given any sequence of L^p-bounded random variables, one can find a subsequence that satisfies, together with all its further subsequences, the same type of limit laws as do i.i.d. variables (or martingale difference sequences) with similar moment bounds. This principle

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was introduced by Chatterji (see [4–6]) and unifies results by Banach and Saks, Komlòs, Révész, Steinhaus in the context of law of large numbers, iterated logarithm, and central limit theorem; extensions to exchangeable sequences were given by Aldous [1] and Berkes and Péter [3]. Also note that Gut [11] and Asmussen and Kurtz [2] gave necessary and sufficient requirements for subsequences to satisfy the famous Hsu- Robbins-Erdős complete convergence result related to the law of large numbers. Our results go a step further, that is, we replace the i.i.d. assumption by L^p-boundedness, and consider Davis normalizing factors $\sum_{n=2}^{\infty}(\log n)^{\delta}/n$ and $\sum_{n=3}^{\infty}(1/n(\log n)^{\delta})$ instead of complete convergence. We thus have the following.

Theorem 1.1. *For any $p>2$ and L^p-bounded sequence $(X_n)_{n\geq 1}$, there exist $1 \leq n_1 < n_2 \cdots$ such that the series*

$$\sum_{N=2}^{\infty} \frac{(\log N)^{\delta}}{N} P\left[\left|\sum_{k=1}^{N} X_{n_k}\right| > \varepsilon (N\log N)^{1/2}\right] \tag{1.2}$$

is convergent for any $0 \leq \delta < p/2-1$ and any $\varepsilon > 0$.

Theorem 1.2. *For any $p \geq 2$ and L^p-bounded sequence $(X_n)_{n\geq 1}$, there exist $1 \leq n_1 < n_2 \cdots$ such that the series*

$$\sum_{N=3}^{\infty} \frac{1}{N(\log N)^{\delta}} P\left[\left|\sum_{k=1}^{N} X_{n_k}\right| > \varepsilon (N\log\log N)^{1/2}\right] \tag{1.3}$$

is convergent for any $\varepsilon > 0$ if either $\delta > 1$, or $\delta = 1$ and $p > 2$.

If $\delta = 1$, Theorem 1.1 holds under the same hypotheses (i.e., $p > 4$), as in the case of martingale difference sequences (see [15]). In the i.i.d case, Theorem 1.1 holds for L^2-bounded centered sequences (see [7, 10]).

Theorem 1.2 holds under the same hypotheses as in the case of martingale difference sequences (see [14]). In the i.i.d. case, slightly less than a second moment is needed: $E[X_n^2 \log^+ \log^+ |X_n|^{-\eta}] < \infty$ for some $0 < \eta < 1$ (see [8, 10]), and for necessary moment conditions, one may consult [13].

In the case of martingale difference sequences, Theorem 1.1 fails if $\delta \geq p/2-1$ and Theorem 1.2 fails if $0 \leq \delta < 1$ (see [14]), therefore Theorems 1.1 and 1.2 are the best results one can expect in the L^p-bounded case.

2. Proofs

Proof of Theorem 1.1. In the sequel we will make use of the so-called c_r-inequality (see [12, page 57]), which says that

$$E|X+Y|^p \leq 2^{p-1}\left(E|X|^p + E|Y|^p\right) \tag{2.1}$$

for any random variables X, Y and $p > 1$. Throughout the paper, C denotes a constant that depends on p and ε (but not on k, n, N), and may vary from line to line, even within the same line.

As $(X_n)_{n\geq 1}$ is bounded in L^p, according to [9, Corollary IV.8.4], it is weakly sequentially compact. Denote by $(Y_n)_{n\geq 1}$ a subsequence of $(X_n)_{n\geq 1}$ that converges weakly in L^p to some $Y \in L^p$. Subtracting Y from each element of $(Y_n)_{n\geq 1}$, we reduce the problem to a sequence $(Y_n)_{n\geq 1}$ that converges weakly in L^p to 0. Further, for any $n \geq 1$, we choose a simple random variable Z_n (i.e., Z_n takes only a finite number of distinct values), such that

$$\left|\left|Y_n - Z_n\right|\right|_p < \frac{1}{2^n}. \tag{2.2}$$

Using Markov's inequality and (2.1), one has

$$\begin{aligned}
&\sum_{N=2}^{\infty} \frac{(\log N)^{\delta}}{N} P\left[\left|\sum_{k=1}^{N} Y_{n_k}\right| > \varepsilon (N\log N)^{1/2}\right] \\
&\quad \leq \varepsilon^{-p} \sum_{N=2}^{\infty} \frac{(\log N)^{\delta-p/2}}{N^{1+p/2}} E\left|\sum_{k=1}^{N} Y_{n_k}\right|^{p} \\
&\quad \leq C\left(\sum_{N=2}^{\infty} \frac{(\log N)^{\delta-p/2}}{N^{1+p/2}} E\left|\sum_{k=1}^{N} Z_{n_k}\right|^{p} + \sum_{N=2}^{\infty} \frac{(\log N)^{\delta-p/2}}{N^{1+p/2}} E\left|\sum_{k=1}^{N} (Y_{n_k} - Z_{n_k})\right|^{p}\right).
\end{aligned} \tag{2.3}$$

According to (2.1), we have

$$E\left|\sum_{k=1}^{N} (Y_{n_k} - Z_{n_k})\right|^{p} \leq 2^{p-1}\left(E\left|Y_{n_1} - Z_{n_1}\right|^{p} + E\left|\sum_{k=2}^{N} (Y_{n_k} - Z_{n_k})\right|^{p}\right), \tag{2.4}$$

whence by iteration

$$E\left|\sum_{k=1}^{N} (Y_{n_k} - Z_{n_k})\right|^{p} \leq \sum_{k=1}^{N} 2^{k(p-1)} E\left|Y_{n_k} - Z_{n_k}\right|^{p}, \tag{2.5}$$

and assumption (2.2) yields

$$E\left|\sum_{k=1}^{N} (Y_{n_k} - Z_{n_k})\right|^{p} \leq \sum_{k=1}^{N} 2^{k(p-1)-n_k p} \leq \sum_{k=1}^{N} \frac{1}{2^k} \leq 1 \quad \text{for any } N \geq 1, \tag{2.6}$$

(we used that the subsequence $(n_k)_{k\geq 1}$ is strictly increasing, so $n_k \geq k$), therefore the last series in (2.3) converges. To prove Theorem 1.1, it suffices to exibit a subsequence $(n_k)_{k\geq 1}$ such that

$$\sum_{N=2}^{\infty} \frac{(\log N)^{\delta-p/2}}{N^{1+p/2}} E\left|\sum_{k=1}^{N} Z_{n_k}\right|^{p} < \infty. \tag{2.7}$$

One can see that $(Z_n)_{n\geq 1}$ also converges weakly in L^p to 0. Indeed, for any $Q \in L^q$, where $1/p + 1/q = 1$, we have $E(Z_n Q) = E((Z_n - Y_n)Q) + E(Y_n Q)$, and the first term on the right-hand side tends to 0 by Hölder's inequality and (2.1), while the second term tends to 0 because Y_n converges weakly in L^p to 0.

By induction, one may choose a subsequence of natural numbers $1 \le n_1 < n_2 \cdots$ such that

$$E[Z_{n_k} \mid Z_i,\ i \in I] \le \frac{1}{2^k} \quad \text{for each } I \subset \{n_1,\dots,n_{k-1}\}, \tag{2.8}$$

where $E[Z_{n_k} \mid Z_i,\ i \in I]$ denotes the conditional expectation of Z_{n_k} given the σ-algebra $\sigma(Z_i,\ i \in I)$ generated by $(Z_i)_{i\in I}$. This can be done because $\sigma(Z_i,\ i \in I)$ consists of a finite partition of Ω, and as $Z_n \to 0$ weakly in L^p, we have $\int_A Z_n dP \to 0$ for any A in $\sigma(Z_i,\ i \in I)$.

We now prove that $(n_k)_{k\ge1}$ is the required subsequence in (2.7). Indeed, one can write

$$Z_{n_k} = V_k + W_k, \tag{2.9}$$

where $E[V_k \mid V_1,\dots,V_{k-1}] = 0$ and $|W_k| \le 1/2^k$. In particular, $(V_k)_{k\ge1}$ is a martingale difference sequence. Using Minkowski's inequality, we deduce that

$$\left(E\left|\sum_{k=1}^{N} Z_{n_k}\right|^p\right)^{1/p} \le \left(E\left|\sum_{k=1}^{N} V_k\right|^p\right)^{1/p} + \left(E\left|\sum_{k=1}^{N} W_k\right|^p\right)^{1/p}. \tag{2.10}$$

According to Burkholder and Hölder's inequalities, we have

$$E\left|\sum_{k=1}^{N} V_k\right|^p \le CE\left(\sum_{k=1}^{N} V_k^2\right)^{p/2} \le CN^{p/2-1}\sum_{k=1}^{N} E|V_k|^p \le CN^{p/2}. \tag{2.11}$$

Also,

$$E\left|\sum_{k=1}^{N} W_k\right|^p \le \left(\sum_{k=1}^{N} \frac{1}{2^k}\right)^p \le 1 \quad \text{for any } N \ge 1. \tag{2.12}$$

Using (2.9)–(2.12), we obtain

$$\sum_{N=2}^{\infty} \frac{(\log N)^{\delta-p/2}}{N^{1+p/2}} E\left|\sum_{k=1}^{N} Z_{n_k}\right|^p \le C\sum_{N=2}^{\infty} \frac{(\log N)^{\delta-p/2}}{N^{1+p/2}}(N^{1/2}+1)^p \le C\sum_{N=2}^{\infty} \frac{(\log N)^{\delta-p/2}}{N}. \tag{2.13}$$

The latter series in (2.13) is convergent if and only if $\delta < p/2 - 1$, thus (2.7) holds and Theorem 1.1 is proved. □

Proof of Theorem 1.2. With the same notations and method as in the proof of Theorem 1.1, it suffices to prove the following analog of (2.7):

$$\sum_{N=3}^{\infty} \frac{1}{N^{1+p/2}(\log N)^{\delta}(\log\log N)^{p/2}} E\left|\sum_{k=1}^{N} Z_{n_k}\right|^p < \infty. \tag{2.14}$$

Using (2.9)–(2.12), the series in (2.14) is dominated by

$$C\sum_{N=3}^{\infty} \frac{(N^{1/2}+1)^p}{N^{1+p/2}(\log N)^{\delta}(\log\log N)^{p/2}} \le C\sum_{N=3}^{\infty} \frac{1}{N(\log N)^{\delta}(\log\log N)^{p/2}}. \tag{2.15}$$

The latter series in (2.15) is convergent if and only if either $\delta > 1$, or $\delta = 1$ and $p > 2$; thus (2.14) holds and Theorem 1.2 is proved. □

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