EXISTENCE RESULTS FOR SECOND-ORDER IMPULSIVE FUNCTIONAL DIFFERENTIAL INCLUSIONS

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The existence of solutions on a compact interval to second-order impulsive functional differential inclusions is investigated. Several new results are obtained by using Sadovskii's fixed point theorem.

1. Introduction

In this paper, we consider the existence of solutions, defined on a compact interval, for second-order functional differential inclusions of the form

$$\big(p(t)y'(t)\big)' \in F(t,y_t), \quad \text{a.e. } t \in [0,T]\setminus\{t_1,\dots,t_m\}, \tag{1.1}$$

$$\Delta y|_{t=t_k} = I_k\big(y(t_k^-)\big), \quad k=1,\dots,m, \tag{1.2}$$

$$\Delta y'|_{t=t_k} = J_k\big(y(t_k^-)\big), \quad k=1,\dots,m, \tag{1.3}$$

subject to initial conditions

$$y(t)=\phi(t), \quad t\in[-r,0], \qquad y'(0)=\eta \tag{1.4}$$

or boundary conditions

$$y(t)=\phi(t), \quad t\in[-r,0], \qquad y(T)=\eta, \tag{1.5}$$

respectively, where $F:[0,T]\times D \to \mathcal{P}(\mathbb{R}^n)$ is a multivalued map, $D=\{\psi:[-r,0]\to\mathbb{R}^n;\ \psi$ is continuous everywhere except for a finite number of points $\tilde{t}$ at which $\psi(\tilde{t}^-)$ and $\psi(\tilde{t}^+)$ exist with $\psi(\tilde{t}^-)=\psi(\tilde{t})\}$, $\phi\in D$, $p:[0,T]\to\mathbb{R}_+$ is continuous, $\eta\in\mathbb{R}^n$, $\mathcal{P}(\mathbb{R}^n)$ is the family of all nonempty subsets of $\mathbb{R}^n$, $0<r<\infty$, $0=t_0<t_1<\cdots<t_m<t_{m+1}=T$, $I_k,J_k:\mathbb{R}^n\to\mathbb{R}^n (k=1,\dots,m)$ are continuous functions. $y(t_k^-)$ and $y(t_k^+)$ represent the left and

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right limits of $y(t)$ at $t = t_k$, respectively, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, and $\Delta y'|_{t=t_k} = y'(t_k^+) - y'(t_k^-)$.

For any continuous function y defined on $[-r,T]\backslash\{t_1,\dots,t_m\}$ and any $t \in [0,T]$, we denote by y_t the element of D defined by

$$y_t(\theta) = y(t+\theta), \quad \theta \in [-r,0], \tag{1.6}$$

where $y_t(\cdot)$ represents the history of the state from time $t-r$, up to the present time t.

Many evolution processes are characterized by the fact that at certain moments of time, they experience a change of state abruptly. These processes are subject to short-term perturbations whose duration is negligible in comparison with the duration of the process. Consequently, it is natural to assume that these perturbations act instantaneously, that is, in the form of impulses. It is known, for example, that many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, and optimal control models in economics, do exhibit impulsive effects. For the theory of impulsive differential equations, we refer to the monograph of Lakshmikantham et al. [14].

Recently, there have appeared a few existence results on solutions of impulsive functional differential inclusions, among which we would like to mention [9] by Chang and Li. In [9], the authors established some interesting results for the problem (1.1)–(1.4) under the condition

$$\left| I_k(x) \right| \le c_k|x|, \quad k = 1,2,\dots,m \tag{1.7}$$

or

$$\lim_{|x|\to\infty} \frac{I_k(x)}{x} = 0, \quad k = 1,2,\dots,m. \tag{1.8}$$

Obviously, their results have dropped the boundedness of the impulsive functions (which is required in many papers, such as [1, 7]). However, the multivalued map F needs to be integrally bounded in [9], that is, there exists a function $M \in L^1([0,T],\mathbb{R}_+)$ such that

$$\|F(t,u)\| = \sup\{|v| : v \in F(t,u)\} \le M(t) \quad \text{for a.e. } t \in [0,T],\ u \in D. \tag{1.9}$$

We remark that it is a stronger condition (see Remark 3.7).

The main purpose of this paper is to investigate the existence of solutions to the problems (1.1)–(1.4) and (1.1)–(1.3) with boundary condition (1.5) by Sadovskii's fixed point theorem under the following conditions.

(H1) There exist nondecreasing functions $l_k : \mathbb{R}_+ \to \mathbb{R}_+$ such that $|I_k(x)| \le l_k(|x|)$, $k = 1,\dots,m$ for each $x \in \mathbb{R}^n$, and

$$\liminf_{r\to+\infty} \frac{l_k(r)}{r} = \beta_k < \infty. \tag{1.10}$$

(H2) There exist nondecreasing functions $L_k : \mathbb{R}_+ \to \mathbb{R}_+$ such that $|J_k(x)| \le L_k(|x|)$, $k = 1,\dots,m$ for each $x \in \mathbb{R}^n$, and

$$\liminf_{r\to+\infty} \frac{L_k(r)}{r} = \gamma_k < \infty. \tag{1.11}$$

We note that (H1) and (H2) cover many special cases (see Corollaries 3.3–3.6 in Section 3), especially those including the conditions used in [9]. In particular, the multivalued map F in our results is not required to be integrally bounded and may be nonconvex valued.

For more results about the differential inclusions, we refer to the papers [1–7, 9, 12] and the references therein.

This paper will be organized as follows. In Section 2, we will recall briefly some basic definitions and preliminary facts from multivalued analysis which will be used in the sequel. In Sections 3 and 4, we will prove our main results for the problems (1.1)–(1.4) and (1.1)–(1.3) with conditions (1.5), respectively.

2. Preliminaries

In this section, we introduce some notations, definitions, and lemmas which are used throughout this paper.

Let $C([-r,0],\mathbb{R}^n)$ be the Banach space of continuous functions from $[-r,0]$ into $\mathbb{R}^n$ with the norm

$$\|\phi\| := \sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}, \tag{2.1}$$

and let $C([0,T],\mathbb{R}^n)$ denote the Banach space of all continuous functions from $[0,T]$ into $\mathbb{R}^n$ normed by

$$\|y\|_\infty := \sup\{|y(t)| : t \in [0,T]\}. \tag{2.2}$$

By $L^1([0,T],\mathbb{R}^n)$, we denote the space of continuous function $y:[0,T]\to\mathbb{R}^n$ which is Lebesgue integrable and equipped with the norm

$$\|y\|_{L^1} = \int_0^T |y(t)|\,dt \quad \forall y \in L^1([0,T],\mathbb{R}^n). \tag{2.3}$$

Let X be a nonempty closed subset of $\mathbb{R}^n$, and $N : X \to \mathcal{P}(\mathbb{R}^n)$ be a multivaled map with nonempty closed values. N is lower semicontinuous (l.s.c.) on X if the set $\{x \in X : N(x) \cap C \neq \emptyset\}$ is open for any open set C in $\mathbb{R}^n$.

Let A denote a subset of $[0,T]\times\mathbb{R}^n$. We say A is $\mathcal{L}\otimes\mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $\mathcal{N}\times\mathfrak{D}$ where $\mathcal{N}$ is Lebsegue measurable in $[0,T]$ and $\mathfrak{D}$ is Borel measurable in $\mathbb{R}^n$. A subset S of $L^1([0,T],\mathbb{R}^n)$ is decomposable if, for all $u,v\in S$ and all measurable subsets $\mathcal{N}$ of $[0,T]$, the function $u\chi_{\mathcal{N}} + v_{[0,T]-\mathcal{N}} \in S$, where χ denotes the characteristic function.

In order to define the solution of (1.1)–(1.4), we introduce the following space:

$$\begin{aligned} PC = \{y : [0,T] \longrightarrow \mathbb{R}^n : y_k \in C([t_k,t_{k+1}],\mathbb{R}^n),\ k = 0,\dots,m, \\ \text{and there exist } y(t_k^-) \text{ and } y(t_k^+) \text{ with } y(t_k^-) = y(t_k),\ k = 1,\dots,m\}, \end{aligned} \tag{2.4}$$

which is a Banach space with the norm

$$\|y\|_{PC} = \max\{||y_k||_{(t_k,t_{k+1}]},\ k = 0,\dots,m\}, \tag{2.5}$$

where y_k is the restriction of y to $(t_k, t_{k+1}]$, $k = 0,\dots,m$.

Set $\Omega = D \cup PC$. Then Ω is a Banach space normed by

$$\|y\|_\Omega = \max\{\|y\|_D, \|y\|_{PC}\}, \quad \text{for each } y \in \Omega. \tag{2.6}$$

Let $F : [0,T] \times D \to \mathcal{P}(\mathbb{R}^n)$ be a multivalued map with nonempty compact values. Assign to F the multivalued operator

$$\mathcal{F} : \Omega \longrightarrow \mathcal{P}(L^1([0,T],\mathbb{R}^n)) \tag{2.7}$$

by letting

$$\mathcal{F}(y) = \{\omega \in L^1([0,T],\mathbb{R}^n) : \omega(t) \in F(t,y_t) \text{ for a.e. } t \in [0,T]\}. \tag{2.8}$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated with F.

The multivalued map N has a fixed point if there exists $x \in X$ such that $x \in N(x)$. For more details on multivalued maps, see the books of Deimling [10] and Hu and Papageorgiou [13].

Definition 2.1. A function $y \in \Omega$ is said to be a solution of (1.1)–(1.4) if y satisfies the differential inclusion (1.1) a.e. on $[0,T]\backslash\{t_1,\dots,t_m\}$ and the conditions (1.2)–(1.4).

Definition 2.2. A function $y \in \Omega$ is said to be a solution of (1.1)–(1.3) if y satisfies the differential inclusion (1.1) a.e. on $[0,T]\backslash\{t_1,\dots,t_m\}$ and the conditions (1.2)-(1.3), (1.5).

Definition 2.3. Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,T],\mathbb{R}^n))$ be a multivlaued operator. N has property (BC) if

(i) N is lower semicontinuous (l.s.c.);

(ii) N has nonempty closed and decomposable values.

Definition 2.4. Let $F : [0,T] \times D \to \mathcal{P}(\mathbb{R}^n)$ be a multivalued map with nonempty compact values. F is of lower semicontinuous type (l.s.c. type) if its associated Niemytzki operator $\mathcal{F}$ is lower semicontinuous and has nonempty closed and decomposable values.

The following lemmas are of great importance in the proof of our main results.

Lemma 2.5 [8]. *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,T],\mathbb{R}^n))$ be a multivlaued operator which has property (BC). Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1([0,T],\mathbb{R}^n)$ such that $g(y) \in N(y)$ for every $y \in Y$.*

Lemma 2.6 [15]. *Let P be a condensing operator on a Banach space X, that is, P is continuous and takes bounded sets into bounded sets, and $\alpha(P(B)) \le \alpha(B)$ for every bounded set B of X with $\alpha(B) > 0$. If $P(D) \subset D$ for a convex, closed, and bounded set H of X, then P has a fixed point in D (where $\alpha(\cdot)$ denotes Kuratowski's measure of noncompactness).*

We remark here that a completely continuous operator is the easiest example of a condensing map.

3. Initial value problems

By Sadovskii's fixed point theorem, combined with the selection theorem of Bressan and Colombo for lower semicontinuous multivalued operators with decomposable values, we will establish some new existence results for the problem (1.1)–(1.4). In addition to (H1)-(H2), we need the following hypotheses.

(H3) Let $F:[0,T]\times D\to\mathcal{P}(\mathbb{R}^n)$ be a nonempty, compact valued multivalued map such that (i) $(t,u)\mapsto F(t,u)$ is $\mathcal{L}\otimes\mathcal{B}$ measurable; and (ii) $u\mapsto F(t,u)$ is lower semicontinuous for a.e. $t\in[0,T]$.

(H4) For each $r>0$, there exists $h_r\in L^1([0,T],\mathbb{R}_+)$ such that

$$||F(t,u)||=\sup\{|f|:f\in F(t,u)\}\leq h_r(t) \tag{3.1}$$

for a.e. $t\in[0,T]$ and $u\in D$ with $\|u\|_D\leq r$, and

$$\liminf_{r\to+\infty}\frac{1}{r}\int_0^T h_r(t)dt=\alpha<\infty. \tag{3.2}$$

In the proof of our main theorem, we introduce the following lemma.

Lemma 3.1 [11]. *Let $F:[0,T]\times D\to\mathcal{P}(\mathbb{R}^n)$ be a multivalued map with nonempty, compact values. Assume* (H3) *and* (H4) *hold. Then F is of l.s.c. type.*

Theorem 3.2. *Assume that (H1)–(H4) hold. Then the problem (1.1)–(1.4) has at least one solution on $[-r,T]$, provided that*

$$\frac{T}{p_0}\alpha+\sum_{k=1}^{m}\left(\beta_k+\frac{T-t_k}{p_0}p(t_k)\gamma_k\right)<1, \tag{3.3}$$

where $p_0=\min\{p(t):t\in[0,T]\}$.

Proof. Note that (H3), (H4), and Lemma 3.1 imply that F is of l.s.c. type. Then, from Lemma 2.5, there exists a continuous function $f:\Omega\to L^1([0,T],\mathbb{R}^n)$ such that $f(y)\in\mathcal{F}(y)$ for all $y\in\Omega$.

We consider the problem

$$\begin{gathered}
\big(p(t)y'(t)\big)'=f(y_t),\quad \text{a.e. } t\in[0,T]\backslash\{t_1,\dots,t_m\},\\
\Delta y|_{t=t_k}=I_k\big(y(t_k^-)\big),\quad k=1,\dots,m,\\
\Delta y'|_{t=t_k}=J_k\big(y(t_k^-)\big),\quad k=1,\dots,m,\\
y(t)=\phi(t),\quad t\in[-r,0],\qquad y'(0)=\eta,
\end{gathered} \tag{3.4}$$

It is clear that if $y\in\Omega$ is a solution of (3.4), then y is a solution to the problem (1.1)–(1.4). Transform the problem (3.4) into a fixed point problem. Consider the operator

$\Gamma:\Omega\to\Omega$, defined by

$$\Gamma(y)(t)=\begin{cases}\phi(t), & t\in[-r,0],\\ \phi(0)+p(0)\eta\int_0^t\frac{ds}{p(s)}+\int_0^t\frac{1}{p(s)}\int_0^s f(y_u)\,du\,ds & \\ \quad+\sum_{0<t_k<t}\left[I_k(y(t_k^-))+J_k(y(t_k^-))\times\int_{t_k}^t\frac{p(t_k)}{p(s)}ds\right], & t\in[0,T].\end{cases}\tag{3.5}$$

Next we will show that Γ satisfies all the conditions of Lemma 2.5. The proof will be given in several steps.

Step 1. For each constant $r>0$, let $B_r=\{y\in\Omega:\|y\|_\Omega\le r\}$, clearly B_r is a bounded closed convex set in Ω. We claim that there exists a positive number r such that $\Gamma(B_r)\subseteq B_r$.

If it is not true, then for each positive number r, there exists a function $y^r(\cdot)\in B_r$ and $t^r\in[0,T]$ such that $\|\Gamma(y^r(t^r))\|>r$. However, on the other hand, we have

$$\begin{aligned}r&<|\Gamma(y^r)(t^r)|\\&=\Bigg|\phi(0)+p(0)\eta\int_0^{t^r}\frac{ds}{p(s)}+\int_0^{t^r}\frac{1}{p(s)}\int_0^s f(y_u^r)\,du\,ds\\&\qquad+\sum_{0<t_k<t}\left[I_k(y^r(t_k^-))+J_k(y^r(t_k^-))\times\int_{t_k}^t\frac{p(t_k)}{p(s)}ds\right]\Bigg|\\&\le\|\phi\|+p(0)|\eta|\frac{T}{p_0}+\frac{T}{p_0}\int_0^T|f(y_s^r)|\,ds\\&\qquad+\sum_{k=1}^m|I_k(y^r(t_k^-))|+\sum_{k=1}^m\left|J_k(y^r(t_k^-))\times\int_{t_k}^{t^r}\frac{p(t_k)}{p(s)}ds\right|\\&\le\|\phi\|+p(0)|\eta|\frac{T}{p_0}+\frac{T}{p_0}\int_0^T h_r(s)ds+\sum_{k=1}^m l_k(r)+\sum_{k=1}^m\frac{T-t_k}{p_0}p(t_k)L_k(r)\\&=M+\frac{T}{p_0}\int_0^T h_r(s)ds+\sum_{k=1}^m l_k(r)+\sum_{k=1}^m\frac{T-t_k}{p_0}p(t_k)L_k(r),\end{aligned}\tag{3.6}$$

where M is independent of r. Dividing both sides by r and taking the lower limit, we get

$$\frac{T}{p_0}\alpha+\sum_{k=1}^m\left(\beta_k+\frac{T-t_k}{p_0}p(t_k)\gamma_k\right)\ge1,\tag{3.7}$$

which contradicts (3.3). Hence for some positive number r, we have $\Gamma(B_r)\subseteq B_r$.

Step 2. Γ is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in B_r. Then

$$\begin{aligned}
&|\Gamma(y_n)(t)-\Gamma(y)(t)| \\
&\quad \le \left| \int_0^t \frac{1}{p(s)}\int_0^s f(y_{nu})\,du\,ds - \int_0^t \frac{1}{p(s)}\int_0^s f(y_u)\,du\,ds \right| \\
&\qquad + \sum_{0<t_k<t} |I_k(y_n(t_k^-)) - I_k(y(t_k^-))| \\
&\qquad + \sum_{0<t_k<t} \int_{t_k}^t \frac{p(t_k)}{p(s)} |J_k(y_n(t_k^-)) - J_k(y(t_k^-))|\,ds \\
&\quad \le \frac{T}{p_0}\int_0^T \left| f(y_{nu}) - \int_0^s f(y_u) \right| du + \sum_{k=1}^m |I_k(y_n(t_k^-)) - I_k(y(t_k^-))| \\
&\qquad + \sum_{k=1}^m \frac{T-t_k}{p_0} p(t_k) |J_k(y_n(t_k^-)) - J_k(y(t_k^-))|.
\end{aligned} \tag{3.8}$$

Since the functions f, I_k, J_k are continuous, by the dominated convergence theorem, we have

$$\|\Gamma(y_n)-\Gamma(y)\| = \sup_{t\in[0,T]} |\Gamma(y_n)(t)-\Gamma(y)(t)| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{3.9}$$

that is, Γ is continuous on B_r.

Step 3. $\Gamma(B_r)$ is uniformly bounded.

Let $B_r := \{y \in \Omega : \|y\|_\Omega \le r\}$ be a bounded set in Ω and $y \in B_r$, there exists $f \in \mathcal{F}(y)$ such that

$$\begin{aligned}
\Gamma(y)(t) &= \phi(0) + p(0)\eta \int_0^t \frac{ds}{p(s)} + \int_0^t \frac{1}{p(s)} \int_0^s f(y_u)\,du\,ds \\
&\quad + \sum_{0<t_k<t} \left[I_k(y(t_k^-)) + J_k(y(t_k^-)) \times \int_{t_k}^t \frac{p(t_k)}{p(s)}\,ds \right], \quad t \in [0,T].
\end{aligned} \tag{3.10}$$

From (H1), (H2), and (H4), we have for each $t \in [0,T]$

$$\begin{aligned}
|\Gamma(y)(t)| &\le \|\phi\| + p(0)|\eta|\frac{T}{p_0} + \frac{T}{p_0}\int_0^T |f(y_s)|\,ds \\
&\quad + \sum_{k=1}^m |I_k(y(t_k^-))| + \sum_{k=1}^m \left| J_k(y(t_k^-)) \times \int_{t_k}^t \frac{p(t_k)}{p(s)}\,ds \right| \\
&\le \|\phi\| + p(0)|\eta|\frac{T}{p_0} + \frac{T}{p_0}\int_0^T h_r(s)\,ds \\
&\quad + \sum_{k=1}^m l_k(|y(t_k^-)|) + \sum_{k=1}^m \frac{T-t_k}{p_0} p(t_k) L_k(|y(t_k^-)|).
\end{aligned} \tag{3.11}$$

Then, for each $y \in B_r$, we have

$$\begin{aligned} ||\Gamma(y)|| \le \|\phi\| + p(0)|\eta|\frac{T}{p_0} + \frac{T}{p_0}||h_r||_{L^1} \\ + \sum_{k=1}^{m}\left(l_k(r) + \frac{T-t_k}{p_0}p(t_k)L_k(r)\right). \end{aligned} \tag{3.12}$$

Hence $\Gamma(B_r)$ is uniformly bounded in Ω.

Step 4. Γ maps bounded sets into equicontinuous sets in Ω.

Let $\tau_1, \tau_2 \in J$, $0 < \tau_1 \le \tau_2$, and $y \in B_r = \{y \in \Omega : \|y\|_\Omega \le r\}$ be a bounded subset of Ω. Then for each $t \in [0,T]$, we have

$$\begin{aligned} \Gamma(y)(t) = \phi(0) + p(0)\eta\int_0^t \frac{ds}{p(s)} + \int_0^t \frac{1}{p(s)}\int_0^s f(y_u)\,du\,ds \\ + \sum_{0<t_k<t}\left[I_k(y(t_k^-)) + J_k(y(t_k^-)) \times \int_{t_k}^t \frac{p(t_k)}{p(s)}ds\right]. \end{aligned} \tag{3.13}$$

Thus,

$$\begin{aligned} &|\Gamma(y)(\tau_2) - \Gamma(y)(\tau_1)| \\ &\quad = \left|\int_0^{\tau_2}\frac{1}{p(s)}\int_0^s f(y_u)\,du\,ds + \sum_{0<t_k<\tau_2}\left[I_k(y(t_k^-)) + J_k(y(t_k^-)) \times \int_{t_k}^{\tau_2}\frac{p(t_k)}{p(s)}ds\right]\right. \\ &\qquad \left. - \int_0^{\tau_1}\frac{1}{p(s)}\int_0^s f(y_u)\,du\,ds - \sum_{0<t_k<\tau_1}\left[I_k(y(t_k^-)) + J_k(y(t_k^-)) \times \int_{t_k}^{\tau_1}\frac{p(t_k)}{p(s)}ds\right]\right| \\ &\quad \le \left|\int_{\tau_1}^{\tau_2}\frac{1}{p(s)}\int_0^s f(y_u)\,du\,ds\right| + \sum_{\tau_1<t_k<\tau_2}\left|I_k(y(t_k^-)) + J_k(y(t_k^-)) \times \int_{t_k}^{\tau_1}\frac{p(t_k)}{p(s)}ds\right| \\ &\qquad + \sum_{0<t_k<\tau_2}\left|J_k(y(t_k^-)) \times \int_{\tau_1}^{\tau_2}\frac{p(t_k)}{p(s)}ds\right| \\ &\quad \le \int_0^T h_r(s)ds\int_{\tau_1}^{\tau_2}\frac{1}{p(s)}ds + \sum_{0<t_k<\tau_2-\tau_1}\left(l_k(r) + \frac{T-t_k}{p_0}p(t_k)L_k(r)\right) \\ &\qquad + \sum_{0<t_k<\tau_2}L_k(r)\int_{\tau_1}^{\tau_2}\frac{p(t_k)}{p(s)}ds. \end{aligned} \tag{3.14}$$

As $\tau_2 \to \tau_1$, the right hand side of the above inequality tends to zero. The equicontinuity for the cases $\tau_1 < \tau_2 \le 0$ and $\tau_1 \le 0 \le \tau_2$ are obvious.

As a consequence of Steps 1 to 4 together with the Ascoli-Arzela theorem, we can conclude that $\Gamma : \Omega \to \Omega$ is completely continuous, therefore, a condensing map. In view of Lemma 2.6, Γ has a fixed point on B_r, which in turn is a solution of the problem (1.1)–(1.4). □

From the proof of Theorem 3.2, we immediately obtain the following corollaries.

Corollary 3.3. *Suppose that (H3)-(H4) and the following conditions hold.*

(H5) *There exist constants c_k such that $|I_k(x)| \leq c_k|x|$, $k = 1,\ldots,m$, for each $x \in \mathbb{R}^n$.*

(H6) *There exist constants d_k such that $|J_k(x)| \leq d_k|x|$, $k = 1,\ldots,m$, for each $x \in \mathbb{R}^n$.*

Then the problem (1.1)–(1.4) has at least one solution on $[-r,T]$, provided that

$$\frac{T}{p_0}\alpha + \sum_{k=1}^{m}\left(c_k + \frac{T-t_k}{p_0}p(t_k)d_k\right) < 1. \tag{3.15}$$

Corollary 3.4. *Suppose that (H3)-(H4) and the following conditions are satisfied.*

(H5′) *There exist constants c_k' such that $|I_k(x)| \leq c_k'$, $k = 1,\ldots,m$, for each $x \in \mathbb{R}^n$.*

(H6′) *There exist constants d_k' such that $|J_k(x)| \leq d_k'$, $k = 1,\ldots,m$, for each $x \in \mathbb{R}^n$.*

Then the problem (1.1)–(1.4) has at least one solution on $[-r,T]$, provided that

$$\frac{T}{p_0}\alpha < 1. \tag{3.16}$$

Corollary 3.5. *In addition to (H3)-(H4), suppose that the following conditions are satisfied.*

(H7) *There exist constants $a_k \in \mathbb{R}$, $b_k \in \mathbb{R}_+$, $\alpha_k \in [0,1)$, $k = 1,\ldots,m$, such that*

$$\left|I_k(y)\right| \leq a_k + b_k|y|^{\alpha_k}. \tag{3.17}$$

(H8) *There exist constants $\overline{a}_k \in \mathbb{R}$, $\overline{b}_k \in \mathbb{R}_+$, $\overline{\alpha}_k \in [0,1)$, $k = 1,\ldots,m$, such that*

$$\left|J_k(y)\right| \leq \overline{a}_k + \overline{b}_k|y|^{\overline{\alpha}_k}. \tag{3.18}$$

Then the problem (1.1)–(1.4) has at least one solution on $[-r,T]$ if (3.16) holds.

Corollary 3.6. *Assume that (H3)-(H4) and the following conditions hold:*

(H9) $\lim_{|x|\to\infty}(I_k(x)/x) = 0$, $k = 1,\ldots,m$,

(H10) $\lim_{|x|\to\infty}(J_k(x)/x) = 0$, $k = 1,\ldots,m$.

Then the problem (1.1)–(1.4) has at least one solution on $[-r,T]$ if (3.16) holds.

Proof. As in the proof of Theorem 3.2, we can define the operator $N : \Omega \to \Omega$,

$$N(y)(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ \phi(0) + p(0)\eta\displaystyle\int_0^t \frac{ds}{p(s)} + \int_0^t \frac{1}{p(s)}\int_0^s f(y_u)\,du\,ds & \\ \quad +\sum_{0<t_k<t}\left[I_k(y(t_k^-)) + J_k(y(t_k^-)) \times \displaystyle\int_{t_k}^t \frac{p(t_k)}{p(s)}ds\right], & t \in [0,T]. \end{cases} \tag{3.19}$$

In view of (H1)-(H2) and (3.16), there exist positive constants ε_k, ε_k', $k = 1,\dots,m$, and C such that

$$\begin{gathered} |I_k(y)| \le \varepsilon_k |y|, \qquad |J_k(y)| \le \varepsilon_k' |y|, \quad \forall |y| > C, \\ \frac{T}{p_0}\alpha + \sum_{k=1}^m \left(\varepsilon_k + \frac{T-t_k}{p_0} p(t_k)\varepsilon_k'\right) < 1. \end{gathered} \tag{3.20}$$

Let

$$\begin{aligned} E_1 &= \{t : t \in [0,T],\ |y| \le C\}, \qquad & E_2 &= \{t : t \in [0,T],\ |y| > C\}, \\ C_1 &= \max\{|I_k(y(t))|,\ t \in E_1\}, \qquad & C_2 &= \max\{|J_k(y(t))|,\ t \in E_1\}. \end{aligned} \tag{3.21}$$

Note that

$$\begin{aligned} \left|\sum_{k=1}^m I_k(y(t_k^-))\right| &\le \sum_{t_k \in E_1} |I_k(y(t_k^-))| + \sum_{t_k \in E_2} |I_k(y(t_k^-))| \le mC_1 + \sum_{k=1}^m \varepsilon_k |y|, \\ \left|\sum_{k=1}^m J_k(y(t_k^-)) \times \int_{t_k}^t \frac{p(t_k)}{p(s)} ds\right| &\le \frac{T-t_k}{p_0} p(t_k) \left(\sum_{t_k \in E_1} |J_k(y(t_k^-))| + \sum_{t_k \in E_2} |J_k(y(t_k^-))|\right) \\ &\le \frac{T-t_k}{p_0} p(t_k) \left(mC_2 + \sum_{k=1}^m \varepsilon_k' |y|\right). \end{aligned} \tag{3.22}$$

By an analogous reasoning in Theorem 3.2, we can show that N has a fixed point. The details are omitted here. □

Remark 3.7. If the multivalued map F in the problem (1.1)–(1.4) is integrally bounded, then Theorem 3.2 and Corollaries 3.3–3.6 hold with $\alpha = 0$. In this case, Theorem 3.2 still extends the corresponding theorem in [9], that is to say, the main existence theorem in [9] is also the immediate corollaries of Theorem 3.2. So, our results improve those of Chang and Li [9] greatly.

4. Boundary value problems

In this section, we give some existence results for the problem (1.1)–(1.3), (1.5).

Theorem 4.1. *Assume that (H1)–(H4) hold. Then the problem (1.1)–(1.3), (1.5) has at least one solution on $[-r,T]$, provided that*

$$w(T)\alpha + 2\sum_{k=1}^m \left(\beta_k + \frac{T-t_k}{p_0} p(t_k)\gamma_k\right) < 1, \tag{4.1}$$

where $w(t) = \int_0^t (1/p(s))ds$.

Proof. As in Theorem 3.2, we transform the problem into a fixed point problem. Consider the multivalued operator $\Theta : \Omega \to \Omega$, defined by

$$\Theta(y)(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ \phi(0) + \dfrac{\eta - \phi(0)}{w(T)} w(t) + \displaystyle\int_0^T G(t,s) f(y_s)\,ds + \sum\nolimits_{0<t_k<t} I_k(y(t_k^-)) & \\ \quad - \dfrac{w(t)}{w(T)} \sum\nolimits_{k=1}^m I_k(y(t_k^-)) + \displaystyle\int_0^t \frac{1}{p(s)} \Bigg[\sum_{0<t_k<s} p(t_k) J_k(y(t_k^-)) \Bigg] ds & \\ \quad - \dfrac{w(t)}{w(T)} \displaystyle\int_0^T \Big[\sum\nolimits_{0<t_k<s} p(t_k) J_k(y(t_k^-)) \Big] ds, & t \in [0,T], \end{cases} \tag{4.2}$$

where $G(t,s)$ is the Green's function for the corresponding homogeneous problem which is given by the formula

$$G(t,s) = \begin{cases} w(s)\left(\dfrac{w(t)}{w(T)} - 1\right) & \text{if } 0 \le s \le t \le T, \\ w(t)\left(\dfrac{w(s)}{w(T)} - 1\right) & \text{if } 0 \le t < s \le T. \end{cases} \tag{4.3}$$

Obviously, we have

$$\sup\{|G(t,s)| : t,s \in [0,T]\} \le w(T). \tag{4.4}$$

We can show that Θ has a fixed point as in the proof of Theorem 3.2. The details are omitted here. □

From Theorem 4.1, we can also obtain Corollaries 4.2–4.5.

Corollary 4.2. *Suppose that (H3)–(H6) hold. Then the problem (1.1)–(1.3), (1.5) has at least one solution on* $[-r,T]$, *provided that*

$$w(T)\alpha + 2\sum_{k=1}^m \left(c_k + \frac{T - t_k}{p_0} p(t_k) d_k \right) < 1. \tag{4.5}$$

Corollary 4.3. *Suppose that (H3)-(H4) and (H5′)–(H6′) are satisfied. Then the problem (1.1)–(1.3), (1.5) has at least one solution on* $[-r,T]$, *provided that*

$$w(T)\alpha < 1. \tag{4.6}$$

Corollary 4.4. *In addition to (H3)-(H4), suppose that (H7)-(H8) are satisfied. Then the problem (1.1)–(1.3), (1.5) has at least one solution on* $[-r,T]$ *if (4.6) holds.*

Corollary 4.5. *Suppose that (H3)-(H4) and (H9)-(H10) hold. Then the problem (1.1)–(1.3), (1.5) has at least one solution on* $[-r,T]$ *if (4.6) holds.*

Remark 4.6. If the multivalued map F in the problem (1.1)–(1.3), (1.5) is integrally bounded, then Theorem 4.1 and Corollaries 4.2–4.5 hold with $\alpha = 0$.

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