A POSITIVE SOLUTION FOR SINGULAR DISCRETE BOUNDARY VALUE PROBLEMS WITH SIGN-CHANGING NONLINEARITIES

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This paper presents new existence results for the singular discrete boundary value problem $-\Delta^2 u(k-1) = g(k,u(k)) + \lambda h(k,u(k))$, $k \in [1,T]$, $u(0) = 0 = u(T+1)$. In particular, our nonlinearity may be singular in its dependent variable and is allowed to change sign.

1. Introduction

Let a,b $(b > a)$ be nonnegative integers. We define the discrete interval $[a,b] = \{a,a+1,\dots,b\}$. All other intervals will carry its standard meaning, for example, $[0,\infty)$ denotes the set of nonnegative real numbers. The symbol Δ denotes the forward difference operator with step size 1, that is, $\Delta u(k) = u(k+1) - u(k)$. Furthermore for a positive m, Δ^m is defined as $\Delta^m u(k) = \Delta^{m-1}(\Delta u(k))$. In this paper, we will study positive solutions of the second-order discrete boundary value problem

$$\begin{gathered} -\Delta^2 u(k-1) = g\big(k,u(k)\big) + \lambda h\big(k,u(k)\big), \quad k \in [1,T], \\ u(0) = 0 = u(T+1), \end{gathered} \tag{1.1}$$

where $\lambda > 0$ is a constant and $T > 2$ is a positive integer. Here, $g : [1,T] \times (0,\infty) \to \mathbb{R}$ and $h : [1,T] \times [0,\infty) \to [0,\infty)$ are continuous. As a result, our nonlinearity may be singular at $u = 0$ and may change sign.

By a solution u of the boundary value problem (1.1), we mean $u : [0,T+1] \to \mathbb{R}$, u satisfies the difference equation (1.1) on $[1,T]$ and the stated boundary data.

We will let $C[0,T+1]$ denote the class of map u continuous on $[0,T+1]$ (discrete topology), with norm $|u|_\infty = \max_{k\in[0,T+1]} |u(k)|$.

2. Main results

The main result of the paper is the following.

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Theorem 2.1. *Suppose the following conditions hold:*

(G) *there exist* $g_i : [1,T]\times(0,\infty)\to(0,\infty)$ $(i=1,2)$ *continuous functions such that*

$$\begin{aligned}
&g_i(k,\cdot) \text{ is strictly decreasing for } k\in[1,T],\\
&-g_1(k,u)\le g(k,u)\le g_2(k,u) \quad \text{for } (k,u)\in[1,T]\times(0,\infty),\\
&\int_0^1 g_1(k,s)ds<\infty \quad \text{for } k\in[1,T],\\
&\forall s_0>0,\quad \sup_{s_0\le s}\left|\frac{\partial}{\partial s}g_2(\cdot,s)\right|\in C[1,T];
\end{aligned} \tag{2.1}$$

(H) *there exist* $h_i : [1,T]\times[0,\infty)\to(0,\infty)$ $(i=1,2)$ *continuous functions such that*

$$\begin{aligned}
&h_i(k,\cdot) \text{ increasing for } k\in[1,T],\\
&h_1(k,u)\le h(k,u)\le h_2(k,u) \quad \text{for } (k,u)\in[1,T]\times(0,\infty),\\
&\lim_{u\to\infty}\frac{h_2(k,u)}{u}=0 \quad \text{for } k\in[1,T],\\
&\text{there exists } \bar{s}>0 \text{ such that } h_1(k,\bar{s})>0 \text{ for all } k\in[1,T].
\end{aligned} \tag{2.2}$$

Then there exists $\lambda_0\ge 0$ *such that for every* $\lambda\ge\lambda_0$, *problem (1.1) has at least one solution* $u\in C[0,T+1]$ *and* $u(k)>0$ *for* $k\in[1,T]$. *Moreover, there exists* $c_i=c_i(\lambda,g,h,\phi_1)>0$ $(i=1,2)$ *such that*

$$c_1\phi_1(k)\le u(k)\le c_2(\phi_1(k)+1) \quad \text{for } k\in[0,T+1], \tag{2.3}$$

where ϕ_1 *is defined in Lemma 2.2.*

It is worth remarking here that an estimate for λ_0 will be given in the proof of Lemma 2.11.

We first give some lemmas which will help us to prove Theorem 2.1.

Lemma 2.2 [1]. *Consider the following eigenvalue problem:*

$$\begin{gathered}
-\Delta^2u(k-1)=\lambda u(k), \quad k\in[1,T],\\
u(0)=u(T+1)=0.
\end{gathered} \tag{2.4}$$

Then the eigenvalues are

$$\lambda_m=4\sin^2\frac{m\pi}{2(T+1)}, \quad 1\le m\le T, \tag{2.5}$$

and the corresponding eigenfunctions are

$$\phi_m(k)=\sin\frac{mk\pi}{T+1} \quad \text{for } k\in[0,T+1],\ 1\le m\le T. \tag{2.6}$$

Lemma 2.3 [3]. *Let* $G_a(k,l)$ *be Green's function of the BVP*

$$\begin{gathered}
-\Delta^2u(k-1)+a(t)u(t)=0 \quad \text{for } t\in[1,T],\\
u(0)=0, \qquad u(T+1)=0.
\end{gathered} \tag{2.7}$$

Then

$$0 < G_a(k,l) \le G_a(l,l) \quad \textit{for every } (k,l) \in [1,T] \times [1,T], \tag{2.8}$$

where $a \in C[1,T]$ *and* $a(k) \ge 0$ *for* $k \in [1,T]$.

Remark 2.4. If $a(k) \equiv 0$ for $k \in [1,T]$, then

$$G_0(k,l) = \frac{1}{T+1}\begin{cases} l(T+1-k), & l \in [0,k-1], \\ k(T+1-l), & l \in [k,T+1], \end{cases} \quad \text{for } k \in [1,T]. \tag{2.9}$$

Next we consider the boundary value problem

$$\begin{gathered} -\Delta^2 u(k-1) + a(k)u(k) = f(k), \quad k \in [1,T], \\ u(0) = 0 = u(T+1), \end{gathered} \tag{2.10}$$

where $a, f \in C[1,T]$ and $a(k) \ge 0$ for $k \in [1,T]$.

Let $A : C[1,T] \to C[1,T]$ be the operator defined by

$$Au(k) := \sum_{l=1}^{T} G_a(k,l)u(l). \tag{2.11}$$

It is easy to see that A is a completely continuous operator (see [3]).

Note that if $u \in C[0,T+1]$, $u(0) = u(T+1) = 0$, and

$$u(k) = A(f)(k) \quad \text{for } k \in [1,T], \tag{2.12}$$

then u is a solution of (2.10).

From Lemma 2.3, we have the following lemma.

Lemma 2.5. *The following statements hold:*
(i) *for any* $f \in C[1,T]$, *(2.10) is uniquely solvable and* $u = A(f)$;
(ii) *if* $f(k) \ge 0$ *for* $k \in [1,T]$, *then the solution of (2.10) is nonnegative.*

Corollary 2.6. *If* $f_1(k) \le f_2(k)$ *for* $k \in [1,T]$, *then* $A(f_1)(k) \le A(f_2)(k)$ *for* $k \in [1,T]$.

Lemma 2.7. *Suppose (G) and (H) hold. Let* $n_0 \in \mathbb{N}$. *Assume that for every* $n > n_0$, *there exist* $a_n \in C[1,T]$, $0 \le a_n$, *and there exist* $\overline{u}, \overline{u}_n, \hat{u}_n, \hat{u} \in C[0,T+1]$ *such that*

$$0 < \overline{u}(k) \le \overline{u}_n(k) \le \hat{u}_n(k) \le \hat{u}(k) \quad \textit{for } k \in [1,T], \tag{2.13}$$

and $\hat{u}(0) = \hat{u}(T+1) = 0$. *If*

$$\begin{aligned} &-\Delta^2 \overline{u}_n(k-1) + a_n(k)\overline{u}_n(k) \\ &\quad \le g\left(k, \frac{1}{n} + v(k)\right) + \lambda h(k,v(k)) + a_n(k)v(k) \quad \textit{for } k \in [1,T], \end{aligned} \tag{2.14}$$

$$\begin{aligned} &-\Delta^2 \hat{u}_n(k-1) + a_n(k)\hat{u}_n(k) \\ &\quad \ge g\left(k, \frac{1}{n} + v(k)\right) + \lambda h(k,v(k)) + a_n(k)v(k) \quad \textit{for } k \in [1,T], \end{aligned} \tag{2.15}$$

where $\lambda \geq 0$ *and* $v \in [\overline{u}_n, \hat{u}_n] = \{u \in C[0,T+1],\ \overline{u}_n(k) \leq u(k) \leq \hat{u}_n(k)$ *for* $k \in [0,T+1]\}$, *then problem (1.1) has a solution* $u \in C[0,T+1]$ *such that* $\overline{u}(k) \leq u(k) \leq \hat{u}(k)$ *for* $k \in [0,T+1]$.

Proof. Fix $v \in [\overline{u}, \hat{u}]$. From Lemma 2.5, there exists $\Psi(v) \in C[0,T+1]$ such that

$$\begin{gathered} -\Delta^2\Psi(v)(k-1) + a_n(k)\Psi(v)(k) \\ = g\left(k, \frac{1}{n} + v(k)\right) + \lambda h(k, v(k)) + a_n(k)v(k) \quad \text{for } k \in [1,T], \\ \Psi(v)(0) = \Psi(v)(T+1) = 0. \end{gathered} \tag{2.16}$$

Then

$$\Psi(v)(k) = A\left(g\left(\cdot, \frac{1}{n} + v\right) + \lambda h(\cdot, v) + a_n v\right)(k) \quad \text{for } k \in [1,T]. \tag{2.17}$$

Note also that $\Psi : C[0,T+1] \to C[0,T+1]$ is a completely continuous operator. By Corollary 2.6, we have

$$\overline{u}_n(k) \leq \Psi(v)(k) \leq \hat{u}_n(k) \quad \text{for } k \in [0,T+1]. \tag{2.18}$$

From Schauder's fixed point theorem (note that $\Psi z : [\overline{u}, \hat{u}] \to [\overline{u}, \hat{u}]$), there exists $u_n \in C[0,T+1]$ such that $\overline{u}_n(k) \leq u_n(k) \leq \hat{u}_n(k)$ and $\Psi(u_n)(k) = u_n(k)$ for $k \in [1,T]$. Note that

$$\begin{gathered} -\Delta^2 u_n(k-1) = g\left(k, \frac{1}{n} + u_n(k)\right) + \lambda h(k, u_n(k)) \quad \text{for } k \in [1,T], \\ u_n(0) = u_n(T+1) = 0. \end{gathered} \tag{2.19}$$

Let $m := \min\{\overline{u}(k) : k \in [1,T]\} > 0$ and $M := \max\{\hat{u}(k) : k \in [1,T]\}$. Then

$$m \leq u_n(k) \leq M \quad \text{for } k \in [1,T],\ n = 1,2,\dots, \tag{2.20}$$

and for $k \in [1,T]$, we have

$$\left| g\left(k, \frac{1}{n} + u_n(k)\right) + \lambda h(k, u_n(k)) \right| \leq g_2(k,m) + \lambda h_2(k,M). \tag{2.21}$$

From the Arzela-Ascoli theorem, there exist a $u \in C[0,T+1]$ and a subsequence $\{u_{n_m}\}_{m\in\mathbb{N}}$ converging to u in $C[0,T+1]$, and of course

$$u(k) = \lim_{m\to\infty} u_{n_m}(k) \quad \text{for } k \in [0,T+1]. \tag{2.22}$$

Observe that $u_{n_m} \in [\overline{u}, \hat{u}]$, so $u(0) = u(T+1) = 0$ and $u \in C[0, T+1]$ with $u > 0$ in $[1, T]$. Also,

$$\begin{aligned} u(k) &= \lim_{m\to\infty} \sum_{l=1}^{T} G_0(k,l)\left[g\left(l, \frac{1}{n} + u_{n_m}(l)\right) + \lambda h(l, u_{n_m}(l))\right] \\ &= \sum_{l=1}^{T} G_0(k,l)[g(l,u(l)) + \lambda h(l,u(l))]. \end{aligned} \tag{2.23}$$

As a result

$$\begin{gathered} -\Delta^2 u(k-1) = g(k,u(k)) + \lambda h(k,u(k)) \quad \text{for } k \in [1,T], \\ u(0) = u(T+1) = 0. \end{gathered} \tag{2.24}$$

□

LEMMA 2.8. *Let $\psi : [1,T] \times (0,\infty) \to (0,\infty)$ be a continuous function with $\psi(k,\cdot)$ strictly decreasing. Then the problem*

$$\begin{gathered} -\Delta^2 \omega(k-1) = \psi\left(k, \omega + \frac{1}{n}\right) \quad \textit{for } k \in [0,T], \\ \omega(0) = \omega(T+1) = 0 \end{gathered} \tag{2.25}$$

has a solution $\omega_n \in C[0,T+1]$ such that

$$\omega_n(k) \le \omega_{n+1}(k) \le 1 + \omega_1(k) \quad \textit{for } k \in [0,T+1],\ n \in \mathbb{N}. \tag{2.26}$$

If $\omega(k) = \lim_{n\to\infty} \omega_n(k)$ for $k \in [0,T+1]$, then

$$\begin{gathered} \omega \in C[0,T+1], \quad \omega(k) > 0, \quad \textit{for } k \in [1,T], \\ -\Delta^2 \omega(k-1) = \psi(k,\omega) \quad \textit{for } k \in [1,T], \\ \omega(0) = \omega(T+1) = 0. \end{gathered} \tag{2.27}$$

Proof. There exists $\chi_1 \in C[0,T+1]$ such that

$$\begin{gathered} -\Delta^2 \chi_1(k-1) = \psi(k,1), \\ \chi_1(0) = \chi_1(T+1) = 0, \\ \chi_1(k) > 0 \quad \text{for } k \in [1,T]. \end{gathered} \tag{2.28}$$

Notice that

$$\begin{gathered} -\Delta^2 \chi_1(k-1) = \psi(k,1) \ge \psi(k, 1 + \chi_1(k)), \\ 0 \le \psi(k, 1 + 0). \end{gathered} \tag{2.29}$$

By a standard upper-lower solution method [2, page 264], there exists $\omega_1 \in C[0,T+1]$ such that

$$\begin{gathered} -\Delta^2 \omega_1(k-1) = \psi(k, 1 + \omega_1(k)) \quad \text{for } k \in [1,T], \\ \omega_1(0) = \omega_1(T+1) = 0. \end{gathered} \tag{2.30}$$

Suppose that there exists $\omega_n \in C[0,T+1]$ such that

$$\begin{gathered} -\Delta^2\omega_n(k-1) = \psi\left(k, \frac{1}{n} + \omega_n(k)\right), \\ \omega_n(0) = \omega_n(T+1) = 0, \\ \omega_n(k) > 0 \quad \text{for } k \in [1,T]. \end{gathered} \tag{2.31}$$

We know that there exist $\chi_{n+1} \in C[0,T+1]$ such that

$$\begin{gathered} -\Delta^2\chi_{n+1}(k-1) = \psi\left(k, \frac{1}{n+1}\right), \\ \chi_{n+1}(0) = \chi_{n+1}(T+1) = 0, \\ \chi_{n+1}(k) > 0 \quad \text{for } k \in [1,T]. \end{gathered} \tag{2.32}$$

Then

$$\begin{gathered} -\Delta^2\chi_{n+1}(k-1) = \psi\left(k, \frac{1}{n+1}\right) \geq \psi\left(k, \frac{1}{n+1} + \chi_{n+1}(k)\right), \\ -\Delta^2\omega_n(k-1) = \psi\left(k, \frac{1}{n} + \omega_n(k)\right) \leq \psi\left(k, \frac{1}{n+1} + \omega_n(k)\right) \quad \text{for } k \in [1,T], \\ \omega_n(0) = \omega_n(T+1) = 0, \\ \omega_n(k) = \sum_{l=1}^{T} G_0(k,l)\psi\left(l, \frac{1}{n} + \omega_n(l)\right) \leq \sum_{l=1}^{T} G_0(k,l)\psi\left(l, \frac{1}{n+1}\right) = \chi_{n+1}(k) \quad \text{for } k \in [1,T]. \end{gathered} \tag{2.33}$$

By a standard upper-lower solution method, there exist $\omega_{n+1} \in C[0,T+1]$ such that

$$\begin{gathered} -\Delta^2\omega_{n+1}(k-1) = \psi\left(k, \frac{1}{n+1} + \omega_{n+1}\right) \quad \text{for } k \in [1,T], \\ \omega_{n+1}(0) = \omega_{n+1}(T+1) = 0, \\ \omega_n(k) \leq \omega_{n+1}(k) \quad \text{for } k \in [0,T+1]. \end{gathered} \tag{2.34}$$

Next we prove

$$\omega_{n+1}(k) + \frac{1}{n+1} \leq \omega_n(k) + \frac{1}{n} \quad \text{for } k \in [0,T+1]. \tag{2.35}$$

To see this, we consider the problem

$$\begin{gathered} -\Delta^2 v(k-1) = \psi(k,v) \quad \text{for } k \in [1,T], \\ v(0) = v(T+1) = \frac{1}{n}. \end{gathered} \tag*{$(2.36)_n$}$$

Then $v_n(k) = 1/n + \omega_n(k)$ for $k \in [0,T+1]$ is a solution of $(2.36)_n$. We next prove

$$v_{n+1}(k) \leq v_n(k) \quad \text{for } k \in [0,T+1]. \tag{2.37}$$

Since $v_{n+1}(0) = 1/(n+1) < 1/n = v_n(0)$, $v_{n+1}(1) = 1/(n+1) < 1/n = v_n(1)$, we need only to prove that

$$v_{n+1}(k) \le v_n(k) \quad \text{for } k \in [1,T]. \tag{2.38}$$

If this is not true, then there exist $m \in [1,T]$ with $v_{n+1}(m) > v_n(m) > 0$. Let σ be the point where $v_{n+1}(k) - v_n(k)$ assumes its maximum over $[1,T]$. Certainly, $v_{n+1}(\sigma) - v_n(\sigma) > 0$. Let $y(k) = v_{n+1}(k) - v_n(k)$. Now $y(\sigma) \ge y(\sigma+1)$ and $y(\sigma) \ge y(\sigma-1)$ imply that

$$2y(\sigma) \ge y(\sigma+1) + y(\sigma-1), \tag{2.39}$$

that is,

$$y(\sigma+1) + y(\sigma-1) - 2y(\sigma) \le 0. \tag{2.40}$$

Thus

$$\Delta^2 y(\sigma-1) \le 0. \tag{2.41}$$

On the other hand, since $v_{n+1}(\sigma) > v_n(\sigma)$, we have

$$\begin{aligned} \Delta^2 y(\sigma-1) &= \Delta^2 v_{n+1}(\sigma-1) - \Delta^2 v_n(\sigma-1) \\ &= -\psi(\sigma, v_{n+1}(\sigma)) + \psi(\sigma, v_n(\sigma)) \\ &= \psi(\sigma, v_n(\sigma)) - \psi(\sigma, v_{n+1}(\sigma)) > 0. \end{aligned} \tag{2.42}$$

This is a contradiction. Thus $v_{n+1}(k) \le v_n(k)$ for $k \in [1,T]$, and so

$$0 < \frac{1}{n+1} + \omega_{n+1} \le \omega_n + \frac{1}{n}. \tag{2.43}$$

Also notice that

$$\omega_1(k) \le \omega_n(k) \le \omega_{n+1}(k) \le 1 + \omega_1(k) \quad \text{for } k \in [0,T+1],\ n \in \mathbb{N}. \tag{2.44}$$

Now with

$$\omega(k) = \lim_{n\to\infty} \omega_n(k) = \sup_{n\in\mathbb{N}} \omega_n(k) \quad \text{for } k \in [0,T+1], \tag{2.45}$$

we have

$$\begin{gathered} 0 < \omega_1(k) \le \omega(k) \le 1 + \omega_1(k) \quad \text{for } k \in [1,T], \\ \omega(0) = \omega(T+1) = 0. \end{gathered} \tag{2.46}$$

Also for $k \in [1,T]$, we have

$$\begin{aligned}\omega(k) &= \lim_{n\to\infty}\omega_n(k)\\ &= \lim_{n\to\infty}\sum_{l=1}^{T}G(k,l)\psi\left(l,\frac{1}{n}+\omega_n(l)\right)\\ &= \sum_{l=1}^{T}G(k,l)\psi(l,\omega(l)),\end{aligned} \tag{2.47}$$

so

$$\begin{aligned}-\Delta^2\omega(k-1) &= \psi(k,\omega) \quad \text{for } k\in[0,T],\\ \omega(0) &= \omega(T+1)=0.\end{aligned} \tag{2.48}$$

□

LEMMA 2.9. *Suppose that* $m:[1,T]\times[0,\infty)\to[0,\infty)$ *is a continuous function such that*

$$\begin{gathered} m(k,\cdot) \text{ is increasing},\\ \lim_{u\to+\infty}\frac{m(k,u)}{u}=0 \quad \text{for } k\in[1,T].\end{gathered} \tag{2.49}$$

There exist $R_0>0$ *and* $\tilde{v}\in C[0,T+1]$ *with* $0\le\tilde{v}\le R_0\phi_1$ *and*

$$\begin{aligned}-\Delta^2\tilde{v}(k-1) &= m(k,\tilde{v}) \quad \text{for } k\in[1,T],\\ \tilde{v}(0) &= \tilde{v}(T+1)=0.\end{aligned} \tag{2.50}$$

Proof. We first prove that

$$\lim_{\mathbb{R}\to\infty}\frac{\sum_{l=1}^{T}G_0(k,l)m(l,v(l))}{R\phi_1(k)}=0 \quad \text{for } k\in[1,T], \tag{2.51}$$

for all $v\in C[0,T+1]$ with $0\le v(i)\le R\phi_1(i)$ for $i\in[0,T+1]$.

From (2.49), for all $\sigma>0$, there exist $s_\sigma>0$ such that

$$m(k,s)\le\sigma s \quad \text{for } k\in[1,T] \text{ and } s_\sigma\le s. \tag{2.52}$$

As a result,

$$m(k,v(k))\big|_{0\le v(k)\le R\phi_1(k)}\le m(k,s_\sigma)+\sigma v(k)\le m(k,s_\sigma)+\sigma R\phi_1(k) \quad \text{for } k\in[1,T], \tag{2.53}$$

so

$$\begin{aligned}\frac{1}{\phi_1(k)}\sum_{l=1}^{T}G_0(k,l)m(l,v(l)) &\le \frac{1}{\phi_1(k)}\left[\sum_{l=1}^{T}G_0(k,l)m(l,s_\sigma)+R\sigma\sum_{l=1}^{T}G_0(k,l)\phi_1(l)\right]\\ &= \frac{1}{\phi_1(k)}\sum_{l=1}^{T}G_0(k,l)m(l,s_\sigma)+\frac{R\sigma}{\lambda_1},\end{aligned} \tag{2.54}$$

and consequently

$$\frac{1}{R\phi_1(k)}\sum_{l=1}^{T}G_0(k,l)m(l,v(l)) \le \frac{1}{R\phi_1(k)}\sum_{l=1}^{T}G_0(k,l)m(l,s_\sigma)+\frac{\sigma}{\lambda_1}, \tag{2.55}$$

so (2.51) follows. Thus there exist $R_0>0$ such that if $v\in C[0,T+1]$ and $0\le v(i)\le R_0\phi_1(i)$ for $i\in[0,T+1]$, then

$$\frac{1}{R_0\phi_1(k)}\sum_{l=1}^{T}G_0(k,l)m(l,v(l)) \le 1 \quad \text{for } k\in[1,T], \tag{2.56}$$

and so

$$0\le\sum_{l=1}^{T}G_0(k,l)m(l,v(l)) \le R_0\phi_1(k) \quad \text{for } k\in[1,T]. \tag{2.57}$$

Let $\Phi: C[1,T]\to C[1,T]$ be the operator defined by

$$(\Phi v)(k) := \sum_{l=1}^{T}G_0(k,l)m(l,v(l)) \quad \text{for } v\in C[1,T],\ k\in[1,T]. \tag{2.58}$$

It is easy to see that Φ is a completely continuous operator. Also if $v\in C[0,T+1]$ and $0\le v(k)\le R_0\phi_1(k)$ for $k\in[1,T]$, then $0\le\Phi(v)(k)\le R_0\phi_1(k)$ for $k\in[1,T]$, so Schauder's fixed point theorem guarantees that there exists $\tilde{v}\in[0,R_0\phi_1]$ with $\Phi(\tilde{v})=\tilde{v}$, that is,

$$-\Delta^2\tilde{v}(k-1)=m(k,\tilde{v}), \qquad \tilde{v}(0)=\tilde{v}(T+1)=0. \tag{2.59}$$

□

Corollary 2.10. *Let $\psi(k,s)$, $m(k,s)$, $(\omega_n)_{n\in\mathbb{N}}$, and $R_0>0$ be as in Lemmas 2.8 and 2.9. Then there exist $\{\tilde{v}_n\}_{n\in\mathbb{N}}\subset C[0,T+1]$ and $0\le\tilde{v}_n\le R_0\phi_1$ such that*

$$\begin{gathered} -\Delta^2\tilde{v}_n(k-1)=m(k,\omega_n+\tilde{v}_n) \quad \text{for } k\in[1,T], \\ \tilde{v}_n(0)=\tilde{v}_n(T+1)=0, \\ -\Delta^2(w_n+\tilde{v}_n)(k-1)\ge\psi\left(k,\frac{1}{n}+\omega_n+\tilde{v}_n\right)+m(k,\omega_n+\tilde{v}_n) \quad \text{for } k\in[1,T]. \end{gathered} \tag{2.60}$$

Proof. Let $n\in\mathbb{N}$ be fixed. Then $m(k,\omega_n+s)$ satisfies the conditions of Lemma 2.9, so there exists $\tilde{v}_n\in C[0,T+1]$ with $0\le\tilde{v}_n\le R_0\phi_1$ such that (2.60) holds and

$$\begin{aligned} -\Delta^2(w_n+\tilde{v}_n)(k-1) &= -\Delta^2 w_n(k-1)-\Delta^2\tilde{v}_n(k-1)=\psi\left(k,\frac{1}{n}+\omega_n\right)+m(k,\omega_n+\tilde{v}_n) \\ &\ge\psi\left(k,\frac{1}{n}+\omega_n+\tilde{v}_n\right)+m(k,\omega_n+\tilde{v}_n) \quad \text{for } k\in[1,T]. \end{aligned} \tag{2.61}$$

□

Lemma 2.11. *Suppose (G) and (H) hold. Then there exist $\lambda_0 \geq 0$, $c > 0$ such that for all $\lambda \geq \lambda_0$, there exist $R_c > c$, $\overline{u} \in C([0,T+1])$ with $c\phi_1(k) \leq \overline{u}(k) \leq R_c\phi_1(k)$ and*

$$\begin{gathered} -\Delta^2\overline{u}(k-1) = -g_1(k,\overline{u}) + \lambda h_1(k,\overline{u}) \quad \text{for } k \in [1,T], \\ \overline{u}(0) = \overline{u}(T+1) = 0. \end{gathered} \tag{2.62}$$

Proof. Let us consider the operator $T_\lambda : C[1,T] \to C[1,T]$ given by

$$T_\lambda(v)(k) := \frac{1}{\phi_1(k)} \sum_{l=1}^{T} G_0(k,l)\big[-g_1\big(l,v(l)\phi_1(l)\big) + \lambda h_1\big(l,v(l)\phi_1(l)\big)\big] \quad \text{for } k \in [1,T]. \tag{2.63}$$

By (H), there exists $\bar{s} \geq 0$ such that $0 < h_1(k,\bar{s})$ for $k \in [1,T]$. We let

$$c = 2\frac{\bar{s}+1}{|\phi_1|_\infty}, \qquad \Theta = \left\{k \in [1,T] : \frac{|\phi_1|_\infty}{2} < \phi_1(k)\right\}. \tag{2.64}$$

Note that Θ is nonempty. If $k \in \Theta$, $v \in C[0,T+1]$, and $c \leq v$, we have

$$\bar{s} = \frac{c|\phi_1|_\infty}{2} - 1 \leq \frac{c|\phi_1|_\infty}{2} \leq c\phi_1(k) \leq v(k)\phi_1(k), \tag{2.65}$$

so

$$h_1(k,\bar{s}) \leq h_1\big(k, v(k)\phi_1(k)\big), \tag{2.66}$$

for all $v \in C[0,T+1]$ with $c \leq v$. Let

$$\rho = \min_{k\in[1,T]} \frac{1}{\phi_1(k)} \sum_{l\in\Theta} G_0(k,l)h_1(l,\bar{s}) > 0, \tag{2.67}$$

and note for $v \in C[0,T+1]$ with $c \leq v$ that

$$\begin{aligned} \frac{1}{\phi_1(k)} \sum_{l=1}^{T} G_0(k,l)h_1\big(l,v(l)\phi_1(l)\big) &\geq \frac{1}{\phi_1(k)} \sum_{l\in\Theta} G_0(k,l)h_1\big(l,v(l)\phi_1(l)\big) \\ &\geq \frac{1}{\phi_1(k)} \sum_{l\in\Theta} G_0(k,l)h_1(l,\bar{s}) \quad \big(\text{see (2.66)}\big) \\ &\geq \min_{k\in[1,T]} \frac{1}{\phi_1(k)} \sum_{l\in\Theta} G_0(k,l)h_1(l,\bar{s}) \\ &= \rho \quad \forall k \in [1,T], \end{aligned} \tag{2.68}$$

that is,

$$\frac{\phi_1(k)}{\sum_{l=1}^{T} G_0(k,l)h_1\big(l,v(l)\phi_1(l)\big)} \leq \frac{1}{\rho}. \tag{2.69}$$

On the other hand, for all $v \in C[0,T+1]$ with $v \geq c$, we have

$$\begin{aligned} c+\frac{1}{\phi_1(k)}\sum_{l=1}^{T}G_0(k,l)g_1\big(l,v(l)\phi_1(l)\big) \\ \leq c+\frac{1}{\phi_1(k)}\sum_{l=1}^{T}G_0(k,l)g_1\big(l,c\phi_1(l)\big) \leq c+\frac{1}{\phi_1(k)}\sum_{l=1}^{T}G_0(k,l)g_1(l,c\mu), \end{aligned} \tag{2.70}$$

where $\mu = \min_{1\leq l\leq T}\phi_1(l)$. Thus, for all $v \in C[0,T+1]$ with $v(k) \geq c$, we have

$$\begin{aligned} \frac{c+(1/\phi_1(k))\sum_{l=1}^{T}G_0(k,l)g_1\big(l,v(l)\phi_1(l)\big)}{\Big(\sum_{l=1}^{T}G_0(k,l)h_1\big(l,v(l)\phi_1(l)\big)\Big)/\phi_1(k)} \\ \leq \frac{1}{\rho}\left(c+\frac{1}{\phi_1(k)}\sum_{l=1}^{T}G_0(k,l)g_1(l,c\mu)\right) \quad \text{for } k \in [1,T]. \end{aligned} \tag{2.71}$$

Let

$$\lambda_0 := \sup\left\{\left|\frac{c+(1/\phi_1(k))\sum_{l=1}^{T}G_0(k,l)g_1\big(l,v(l)\phi_1(l)\big)}{\Big(\sum_{l=1}^{T}G_0(k,l)h_1\big(l,v(l)\phi_1(l)\big)\Big)/\phi_1(k)}\right|_* : v \in C[0,T+1],\ c \leq v\right\} < \infty, \tag{2.72}$$

where $|u|_* = \max[1,T]|u(k)|$. Then, for all $\lambda \geq \lambda_0$, $v \in C[0,T+1]$, and $c \leq v$, we have for $k \in [1,T]$ that

$$\frac{c+(1/\phi_1(k))\sum_{l=1}^{T}G_0(k,l)g_1\big(l,v(l)\phi_1(l)\big)}{\Big(\sum_{l=1}^{T}G_0(k,l)h_1\big(l,v(l)\phi_1(l)\big)\Big)/\phi_1(k)} \leq \lambda, \tag{2.73}$$

that is,

$$c+\frac{1}{\phi_1(k)}\sum_{l=1}^{T}G_0(k,l)g_1\big(l,v(l)\phi_1(l)\big) \leq \frac{\lambda}{\phi_1(k)}\sum_{l=1}^{T}G_0(k,l)h_1\big(l,v(l)\phi_1(l)\big), \tag{2.74}$$

so

$$\begin{aligned} c &\leq \frac{1}{\phi_1(k)}\sum_{l=1}^{T}G_0(k,l)\Big(-g_1\big(l,v(l)\phi_1(l)\big)+\lambda h_1\big(l,v(l)\phi_1(l)\big)\Big) \\ &= T_\lambda(v)(k) \quad \text{for } k \in [1,T]. \end{aligned} \tag{2.75}$$

On the other hand, for all $v \in C[0,T+1]$ with $v \geq c$, we have

$$\begin{aligned} \frac{1}{\phi_1(k)}\sum_{l=1}^{T}G_0(k,l)g_1\big(l,v(l)\phi_1(l)\big) \leq \frac{1}{\phi_1(k)}\sum_{l=1}^{T}G_0(k,l)g_1\big(l,c\phi_1(l)\big) \\ \leq \max_{k\in[1,T]}\frac{1}{\phi_1(k)}\sum_{l=1}^{T}G_0(k,l)g_1\big(l,c\phi_1(l)\big), \end{aligned} \tag{2.76}$$

so

$$\lim_{\mathbb{R}\to\infty}\frac{1}{R}\left[\frac{1}{\phi_1(k)}\sum_{l=1}^{T}G_0(k,l)g_1\big(l,v(l)\phi_1(l)\big)\right]=0, \tag{2.77}$$

for all $v\in C[0,T+1]$ with $v\geq c$ and $k\in[1,T]$. Essentially the same reasoning as in the proof of (2.51) yields (note that $\lim_{u\to\infty}(h_1(k,u)/u)=0$ for $k\in[1,T]$)

$$\lim_{\mathbb{R}\to\infty}\frac{1}{R}\left[\frac{1}{\phi_1(k)}\sum_{l=1}^{T}G_0(k,l)h_1\big(l,v(l)\phi_1(l)\big)\right]=0 \tag{2.78}$$

for all $v\in C[0,T+1]$ with $0\leq v(i)\leq R$ and $i\in[1,T]$. Thus if $\lambda\geq\lambda_0$, there exists $R_c>c$ with $T_\lambda([c,R_c])\subset[c,R_c]$.

It is easy to see that T_λ is a completely continuous operator, so Schauder's fixed point theorem guarantees that there exists $\overline{v}\in[c,R_c]$ with $T_\lambda(\overline{v})=\overline{v}$, that is,

$$\overline{v}(k)\phi_1(k)=\sum_{l=1}^{T}G_0(k,l)\big(-g_1\big(l,\overline{v}(l)\phi_1(l)\big)+\lambda h_1\big(l,\overline{v}(l)\phi_1(l)\big)\big). \tag{2.79}$$

The function $\overline{u}=\phi_1\overline{v}$ satisfies (2.62). □

Proof of Theorem 2.1. Let $\lambda_0>0$, $c>0$, and $\overline{u}\in(C[0,T+1])$ be defined as in Lemma 2.11. Also let

$$\begin{gathered}\psi(k,s)=g_2(k,s)+\lambda h_1\big(k,\overline{u}(k)\big)\quad\text{for } k\in[1,T],\\ m(k,s)=\lambda h_2(k,s),\end{gathered} \tag{2.80}$$

where $\lambda\geq\lambda_0$.

From (G), we notice that ψ satisfies the assumptions of Lemma 2.8. As a result, there exist $\omega,\omega_n\in C[0,T+1]$ such that

$$\begin{gathered}-\Delta^2\omega_n(k-1)=g_2\left(k,\frac{1}{n}+\omega_n\right)+\lambda h_1\big(k,\overline{u}(k)\big)\quad\text{for } k\in[1,T],\\ \omega_n(0)=\omega_n(T+1)=0,\\ \omega(k)=\lim_{n\to\infty}\omega_n(k)\quad\text{for } k\in[0,T+1].\end{gathered} \tag{2.81}$$

From (H), we notice that m satisfies the assumptions of Lemma 2.9. As a result from Corollary 2.10, there exist $R_0>0$ and $\widetilde{v}_n\in C([0,T+1])$, $0\leq\widetilde{v}_n(k)\leq R_0\phi_1(k)$ for $k\in[0,T+1]$ such that

$$\begin{gathered}-\Delta^2\widetilde{v}_n(k-1)=\lambda h_2\big(k,\omega_n+\widetilde{v}_n\big)\quad\text{for } k\in[1,T],\\ \widetilde{v}_n(0)=\widetilde{v}_n(T+1)=0,\\ -\Delta^2\big(\omega_n+\widetilde{v}_n\big)(k-1)\geq g_2\left(k,\frac{1}{n}+\omega_n+\widetilde{v}_n\right)+\lambda h_1\big(k,\overline{u}(k)\big)+\lambda h_2\big(k,\omega_n+\widetilde{v}_n\big)\quad\text{for } k\in[1,T].\end{gathered} \tag{2.82}$$

Let

$$\widehat{u}_n(k) = \omega_n(k) + \widetilde{v}_n(k) \quad \text{for } k \in [0, T+1]. \tag{2.83}$$

Then, $\widehat{u}_n \in C[0, T+1]$, $\widehat{u}_n(1) = \widehat{u}_n(T+1) = 0$.

We let

$$\widehat{u}(k) = \omega(k) + R_0\phi_1(k) \quad \text{for } k \in [0, T+1], \tag{2.84}$$

so

$$0 \le \widehat{u}_n(k) \le \widehat{u}(k) \quad \text{for } k \in [0, T+1]. \tag{2.85}$$

From Lemma 2.11, we have

$$\begin{aligned} -\Delta^2\overline{u}(k-1) &= -g_1(k, \overline{u}(k)) + \lambda h_1(k, \overline{u}(k)) \\ &\le \lambda h_1(k, \overline{u}(k)) \\ &\le \lambda h_1(k, \overline{u}(k)) + g_2\left(k, \frac{1}{n} + \widehat{u}_n(k)\right) + \lambda h_2(k, \widehat{u}_n(k)) \\ &\le -\Delta^2\widehat{u}_n(k-1) \quad \text{for } k \in [1, T], \end{aligned} \tag{2.86}$$

that is,

$$-\Delta^2(\overline{u} - \widehat{u}_n)(k-1) \le 0. \tag{2.87}$$

A standard argument (see the argument to show (2.35)) yields

$$\overline{u}(k) \le \widehat{u}_n(k) \quad \text{for } k \in [1, T]. \tag{2.88}$$

Let

$$a_n(k) = \sup\left\{ \left| \frac{\partial}{\partial s} g_2\left(k, \frac{1}{n} + s\right) \right| : 0 < s \right\}, \tag{2.89}$$

and notice that $s \to g_2(k, 1/n + s) + a(k)s$ is increasing. Let $\overline{u}_n = \overline{u}$. From (2.85) and (2.88), we have

$$\overline{u}(k) = \overline{u}_n(k) \le \widehat{u}_n(k) \le \widehat{u}(k) \quad \text{for } k \in [0, T+1]. \tag{2.90}$$

Also for $v \in C[1, T]$ with $\overline{u}_n(k) \le v(k) \le \widehat{u}_n(k)$, $k \in [1, T]$, we have

$$\begin{aligned} -\Delta^2\overline{u}_n(k-1) + a_n(k)\overline{u}_n(k) &= -g_1(k, \overline{u}_n(k)) + \lambda h_1(k, \overline{u}_n(k)) + a_n(k)\overline{u}_n(k) \\ &\le -g_1(k, v(k)) + \lambda h_1(k, v(k)) + a_n(k)v(k) \\ &\le -g_1\left(k, \frac{1}{n} + v(k)\right) + \lambda h_1(k, v(k)) + a_n(k)v(k) \\ &\le g\left(k, \frac{1}{n} + v(k)\right) + \lambda h(k, v) + a_n(k)v(k) \quad \text{for } k \in [1, T], \end{aligned} \tag{2.91}$$

so (2.14) holds.

Also for $v \in C[1,T]$ with $\overline{u}_n(k) \le v(k) \le \hat{u}_n(k)$, $k \in [1,T]$, we have

$$\begin{aligned}
&-\Delta^2\hat{u}_n(k-1)+a_n(k)\hat{u}_n(k)\\
&\qquad \ge g_2\left(k,\frac{1}{n}+\hat{u}_n(k)\right)+\lambda h_1\left(k,\overline{u}(k)\right)+\lambda h_2\left(k,\hat{u}_n(k)\right)+a_n(k)\hat{u}_n(k)\\
&\qquad \ge g_2\left(k,\frac{1}{n}+\hat{u}_n(k)\right)+a_n(k)\hat{u}_n(k)+\lambda h_2\left(k,\hat{u}_n(k)\right)\\
&\qquad \ge g_2\left(k,\frac{1}{n}+v(k)\right)+a_n(k)v(k)+\lambda h_2\left(k,v(k)\right)\\
&\qquad \ge g\left(k,\frac{1}{n}+v(k)\right)+\lambda h\left(k,v(k)\right)+a_n(k)v(k) \quad \text{for } k\in[1,T],
\end{aligned} \tag{2.92}$$

so (2.15) holds. Lemma 2.7 guarantees that there exists a solution $u \in C[0,T+1]$ to (1.1) with

$$\overline{u}(k) \le u(k) \le \hat{u}(k) \quad \text{for } k \in [0,T+1]. \tag{2.93}$$

Moreover, because $\hat{u}(k) \le |\omega|_\infty + R_0\phi_1(k) \le (|\omega|_\infty + R_0)(1+\phi_1(k))$ and $c\phi_1(k) \le \overline{u}(k)$ (see Lemma 2.11), the estimates asserted in the theorem follow. □

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