EXISTENCE OF SOLUTIONS OF A SPECIAL CLASS OF FUZZY INTEGRAL EQUATIONS

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We prove an existence theorem for a special class of fuzzy integral equations involving fuzzy set-valued mappings. The results are obtained by using the contraction mapping principle.

1. Introduction

Chandrasekhar [5] and Crum [6] considered the following integral equation:

$$H(t) = 1 + H(t)\int_0^1 \frac{t}{t+s}\psi(s)H(s)ds. \tag{1.1}$$

This equation arises in the study of radiation transfer in a semi-infinite atmosphere. The first rigorous proof of existence of solutions of (1.1) was given in [6]. By using operators on a Banach algebra and a fixed point theorem of Darbo for a set contraction map, Legget [8] proved an existence theorem for an equation of the form

$$x = x_0 + xKx, \tag{1.2}$$

where K is a compact operator on the Banach algebra. His abstract theorems are applied to the integral equation of the form

$$x(t) = x_0(t) + x(t)\int_\Omega K(t,s)f\big(s,x(s)\big)ds, \quad t \in \Omega,\ \Omega \subset R^n. \tag{1.3}$$

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Cahlon and Eskin [4] considered the equation

$$H(t) = 1 + H(t)\int_0^1 \frac{t}{t+s}\psi(s)H(s)ds + \int_0^1 P(t,s,H(t),H(s))\,ds. \tag{1.4}$$

This equation is a generalization of (1.3), where P is the perturbation of Chandrasekhar H-equation.

The problem of existence of solutions of fuzzy integral equations has been studied by many authors [1, 2, 9–11, 14–16]. Kaleva [7] and Seikkala [13] have discussed the existence of solutions of fuzzy differential equations. Subrahmanyam and Sudarsanam [14] studied existence results for fuzzy Volterra integral equation of the form

$$x(t) = \phi(t) + \int_0^t g(t,s,x(s))\,ds, \tag{1.5}$$

where as Park et al. [11] proved the existence of solutions of fuzzy integral equation of the form

$$\phi(u) = w_0 + \int_{u_0}^u F(u,s,\phi(s))\,ds \quad \phi(u_0) = w_0. \tag{1.6}$$

Balachandran and Dauer [2] established the local existence of solutions and approximate solutions of the perturbed fuzzy integral equation. Balachandran and Prakash [3] studied the existence of solutions of nonlinear fuzzy Volterra integral equations of the form

$$x(t) = \phi(t) + f\left(t, x(t), \int_0^t g(t,s,x(s))\,ds\right). \tag{1.7}$$

In this paper we prove the existence of solutions of fuzzy integral equations of the form

$$x(t) = \phi(t) + x(t)\int_0^t k(t,s)f(s,x(s))\,ds + \int_0^t g(t,s,x(s))\,ds, \tag{1.8}$$

where $\phi : [0,T] \to E^n$, $k : [0,T]\times[0,T] \to R$, $f : [0,T]\times E^n \to E^n$, and $g : [0,T]\times[0,T]\times E^n \to E^n$ are continuous functions. This equation is a generalization of Chandrasekhar-type equation in fuzzy setting.

The outlay of the paper is as follows. In Section 2 we give some basic definitions for our study and in Section 3 we prove the main theorem on the existence of solutions of fuzzy integral equation (1.8). In Section 4 we state a theorem on the existence of solutions of a generalization of (1.8).

2. Preliminaries

Let $P_k(R^n)$ denote the family of all nonempty, compact, convex subsets of R^n. Addition and scalar multiplication in $P_k(R^n)$ are defined as usual. $\overline{U}$ denotes the closure of U, where U is contained in R^n. Let $I = [0,1] \subseteq R$ be a compact interval and denote

$$E^n = \{u : R^n \longrightarrow [0,1] : u \text{ satisfies (i)–(iv) below}\}, \tag{2.1}$$

where

(i) u is normal, that is, there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,

(ii) u is fuzzy convex,

(iii) u is upper semicontinuous,

(iv) $[u]^0 = cl\{x \in R^n : u(x) > 0\}$ is compact.

For $0 < \alpha \leq 1$ denote $[u]^\alpha = \{x \in R^n : u(x) \geq \alpha\}$. Then from (i)–(iv) it follows that the α-level set $[u]^\alpha \in P_k(R^n)$ for all $0 \leq \alpha \leq 1$.

If $g : R^n \times R^n \to R^n$ is a function, then using Zadeh's extension principle we can extend g to $E^n \times E^n \to E^n$ by the equation

$$\tilde{g}(u,v)(z) = \sup_{z=g(x,y)} \min\{u(x), v(y)\}. \tag{2.2}$$

It is well known that $[\tilde{g}(u,v)]^\alpha = g([u]^\alpha,[v]^\alpha)$ for all $u,v \in E^n$, $0 \leq \alpha \leq 1$, and continuous function g. In addition the above equation gives $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$. The real numbers can be embedded in E^n by the rule $c \to \hat{c}(t)$, where

$$\hat{c}(t) = \begin{cases} 1 & \text{for } t = c, \\ 0 & \text{elsewhere.} \end{cases} \tag{2.3}$$

We can also generalize the multiplication by a real number and for any real number c we get $[cu]^\alpha = c[u]^\alpha$, where $0 \leq \alpha \leq 1$ and $u \in E^n$.

Let $D : E^n \times E^n \to R^+ \cup \{0\}$ be defined by $D(u,v) = \sup_{0\leq\alpha\leq 1} H([u]^\alpha,[v]^\alpha)$, where H is the Hausdorff metric defined in $P_K(R^n)$. Then D is a metric on E^n. Further, (E^n,D) is a complete metric space [7, 12]. Also $D(u+w,v+w) = D(u,v)$ for every $u,v,w \in E^n$. Furthermore, $D(\lambda u,\lambda v) = |\lambda| D(u,v)$ for every $u,v \in E^n$ and $\lambda \in R$.

It can be proved straight away that $D(u+v,w+z) \leq D(u,w)+D(v,z)$ for u, v, w, and $z \in E^n$. (The proof is based on the fact $H(A_1+A_2,B_1+B_2) \leq H(A_1,B_1)+H(A_2,B_2)$, where H is the Hausdorff metric on $P_k(R^n)$ induced by the norm in R^n.)

Definition 2.1 [1]. Let I be $[0,1]$ and for each t in I, let $F(t)$ be a nonempty subset of R^n. Let $\mathcal{F}$ be the set of all point-valued functions f from I to R^n such that f is integrable over I and $f(t) \in F(t)$ for all t in I. Then

$$\int_I F(t)dt = \left\{ \int_I f(t)dt : f \in \mathcal{F} \right\}. \tag{2.4}$$

Definition 2.2 [7]. A mapping $F : I \to E^n$ is strongly measurable if for all $\alpha \in [0,1]$ the set-valued map $F_\alpha : I \to P_k(R^n)$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is Lebesgue measurable when $P_k(R^n)$ has the topology induced by the Hausdorff metric H.

Definition 2.3 [7]. A mapping $F : I \to E^n$ is said to be integrably bounded if there is an integrable function h such that $\|x\| \leq h(t)$ for every $x \in F_0(t)$.

Definition 2.4 [12]. The integral of a fuzzy mapping $F:[0,1]\to E^n$ is defined levelwise by

$$\left[\int_{[0,1]} F(t)dt\right]^{\alpha} = \int_{[0,1]} F_{\alpha}(t)dt$$
$$= \left\{\int_{[0,1]} f(t)dt : f:[0,1]\longrightarrow R^n \text{ is a measurable selection for } F_{\alpha}\right\} \tag{2.5}$$

for all $\alpha\in[0,1]$.

It has been proved by Puri and Ralescu [12] that a strongly measurable and integrably bounded mapping $F:I\to E^n$ is integrable (i.e., $\int_I F(t)dt\in E^n$). The concept of a fuzzy integral generalizes the Aumann integral of a set-valued mapping. The following results are proved in [7].

Theorem 2.5. *If $F:I\to E^n$ is continuous, then it is integrable.*

Theorem 2.6. *Let $F,G:I\to E^n$ be integrable and $\lambda\in R$. Then*
- (i) $\int_I (F(t)+G(t))dt = \int_I F(t)dt + \int_I G(t)dt$,
- (ii) $\int_I \lambda F(t)dt = \lambda\int_I F(t)dt$,
- (iii) $D(F,G)$ *is integrable,*
- (iv) $D(\int_I F(t)dt, \int_I G(t)dt) \le \int_I D(F(t),G(t))dt$.

3. Existence theorem

Theorem 3.1. *Let a and b be positive numbers such that*

$$b = \max_{0\le t\le a} D\left(\psi(t)\int_0^t k(t,s)f(s,\psi(s))ds + \int_0^t g(t,s,\psi(s))ds, \hat{0}\right). \tag{3.1}$$

Suppose that
- (i) $\phi:[0,a]\to E^n$ *is continuous,*
- (ii) $f:[0,T]\times E^n\to E^n$ *and* $k:[0,T]\times[0,T]\to R$ *are continuous and there exists a constant* $L>0$ *such that*

$$D\left(x(t)\int_0^t k(t,s)f(s,x(s))ds, y(t)\int_0^t k(t,s)f(s,y(s))ds\right) \le LD(x,y) \tag{3.2}$$

  for $x,y\in E^n$,
- (iii) $g:U\to E^n$ *is continuous, where* $U=\{(t,s,x): 0\le s\le t\le a,\ x\in E^n \text{ and } D(x,\phi(t))\le b\}$ *and satisfies Lipschitz condition with respect to x on U, that is, there exists a constant* $M>0$ *such that*

$$D(g(t,s,x),g(t,s,y)) \le MD(x,y) \quad \text{if } (t,s,x),(t,s,y)\in U. \tag{3.3}$$

If $c=(\alpha-L)/M$ *for some fixed* $\alpha\in(0,1)$, *then there is a unique solution of (1.7) on* $[0,T]$, *where* $T=\min\{a,b,c\}$.

Proof. Let $\mathscr{C}$ be the space of continuous functions from $[0,T]$ into (E^n,D) with $H_1(\psi,\phi) \le b$, that is, $\mathscr{C} = \{\psi : \psi : [0,T] \to E^n$ is continuous and $H_1(\psi,\phi) \le b\}$, where $H_1(\psi,\phi) = \sup_{0\le t\le T} D(\psi(t),\phi(t))$. Define an operator $A : \mathscr{C} \to \mathscr{C}$ by

$$A\psi(t) = \phi(t) + \psi(t)\int_0^t k(t,s)f(s,\psi(s))\,ds + \int_0^t g(t,s,\psi(s))\,ds. \tag{3.4}$$

To prove $A : \mathscr{C} \to \mathscr{C}$, we have to prove that $A\psi$ is continuous and $H_1(A\psi,\phi) \le b$ whenever $\psi \in \mathscr{C}$. Consider

$$\begin{aligned}
&D(A\psi(t+h),A\psi(t)) \\
&\quad = D\bigg(\phi(t+h) + \psi(t+h)\int_0^{t+h} k(t+h,s)f(s,\psi(s))\,ds + \int_0^{t+h} g(t+h,s,\psi(s))\,ds, \phi(t) \\
&\qquad\quad + \psi(t)\int_0^t k(t,s)f(s,\psi(s))\,ds + \int_0^t g(t,s,\psi(s))\,ds\bigg) \le D(\phi(t+h),\phi(t)) \\
&\qquad + D\bigg(\psi(t+h)\int_0^{t+h} k(t+h,s)f(s,\psi(s))\,ds, \psi(t)\int_0^t k(t,s)f(s,\psi(s))\,ds\bigg) \\
&\qquad + D\bigg(\int_0^{t+h} g(t+h,s,\psi(s))\,ds, \int_0^t g(t,s,\psi(s))\,ds\bigg) \\
&\quad \le \frac{\epsilon}{3} + D\bigg(\psi(t+h)\int_0^{t+h} k(t+h,s)f(s,\psi(s))\,ds, \psi(t)\int_0^t k(t,s)f(s,\psi(s))\,ds\bigg) \\
&\qquad + \int_0^t D(g(t+h,s,\psi(s))\,ds, g(t,s,\psi(s)))\,ds + \int_t^{t+h} D(g(t+h,s,\psi(s))\,ds,\hat{0})\,ds.
\end{aligned} \tag{3.5}$$

Clearly the right-hand side of (3.5) is less than ϵ as $h \to 0$. So $A\psi$ is continuous. Consider

$$\begin{aligned}
H_1(A\psi,\phi) &= \sup_{0\le t\le T} D(A\psi(t),\phi(t)) \\
&= \sup_{0\le t\le T} D\bigg(\phi(t) + \psi(t)\int_0^t k(t,s)f(s,\psi(s))\,ds + \int_0^t g(t,s,\psi(s))\,ds, \phi(t)\bigg) \\
&= \sup_{0\le t\le T} D\bigg(\psi(t)\int_0^t k(t,s)f(s,\psi(s))\,ds + \int_0^t g(t,s,\psi(s))\,ds, \hat{0}\bigg) \\
&\le b.
\end{aligned} \tag{3.6}$$

So $A\psi \in \mathscr{C}$ and A maps $\mathscr{C}$ into itself. We show that $\mathscr{C}$ is a closed subset of $C([0,T],E^n)$ a complete metric space with the metric H_1 (see [7]).

Let (ψ_n) be a sequence in $\mathscr{C}$ converging to ψ in $C([0,T],E^n)$. Consider

$$\begin{aligned} H_1(\psi,\phi) &= \sup_{0\le t\le T} D(\psi(t),\phi(t)) \\ &= \sup_{0\le t\le T} \{D(\psi_n(t),\psi(t)) + D(\psi_n(t),\phi(t))\} \\ &\le H_1(\psi_n,\psi) + H_1(\psi_n,\phi) \\ &\le \epsilon + b \end{aligned} \tag{3.7}$$

for sufficiently large n and all positive ϵ. So $\psi \in \mathscr{C}$. This implies that $\mathscr{C}$ is a closed subset of $C([0,T],E^n)$. Therefore $\mathscr{C}$ is a complete metric space. We prove that A is a contraction mapping. For $\psi_1,\psi_2 \in \mathscr{C}$,

$$\begin{aligned} &H_1(A\psi_1,A\psi_2) \\ &\quad= \sup_{0\le t\le T} D(A\psi_1(t),A\psi_2(t)) \\ &\quad= \sup_{0\le t\le T} D\Big(\phi(t)+\psi_1(t)\int_0^t k(t,s)f(s,\psi_1(s))\,ds + \int_0^t g(t,s,\psi_1(s))\,ds, \phi(t) \\ &\qquad\qquad + \psi_2(t)\int_0^t k(t,s)f(s,\psi_2(s))\,ds + \int_0^t g(t,s,\psi_2(s))\,ds\Big) \\ &\quad\le \sup_{0\le t\le T} \Big\{D\Big(\psi_1(t)\int_0^t k(t,s)f(s,\psi_1(s))\,ds, \psi_2(t)\int_0^t k(t,s)f(s,\psi_2(s))\,ds\Big) \\ &\qquad\qquad + \int_0^t D(g(t,s,\psi_1(s)),g(t,s,\psi_2(s)))\,ds\Big\} \\ &\quad\le (L+MT)H_1(\psi_1,\psi_2) \\ &\quad\le (L+Mc)H_1(\psi_1,\psi_2) \\ &\quad\le \alpha H_1(\psi_1,\psi_2) \quad \text{where } \alpha \in (0,1). \end{aligned} \tag{3.8}$$

So $A : \mathscr{C} \to \mathscr{C}$ is a contraction map. Since $\mathscr{C}$ is a complete metric space and A is a contracting self-map on $\mathscr{C}$, it has a unique fixed point $x \in \mathscr{C}$. This fixed point is the required unique solution to (1.8). □

4. General equations

As a generalization of (1.8) we consider the following fuzzy integral equation:

$$x(t) = \phi(t) + h(t,x(t))\int_0^t k(t,s)f(s,x(s))\,ds + \int_0^t g(t,s,x(s))\,ds, \tag{4.1}$$

where $h : [0,T]\times E^n \to E^n$ is continuous and all other conditions are as before. Now we state without proof an existence theorem for (4.1).

Theorem 4.1. *Let a^* and b^* be positive numbers such that*

$$b^* = \max_{0\le t\le a^*} D\Big(h(t,\psi(t))\int_0^t k(t,s)f(s,\psi(s))\,ds + \int_0^t g(t,s,\psi(s))\,ds, \hat{0}\Big). \tag{4.2}$$

Suppose that

(i) $\phi : [0,a^*] \to E^n$ *is continuous,*

(ii) $f,h : [0,T]\times E^n \to E^n$ *and* $k : [0,T]\times[0,T] \to R$ *are continuous and there exists a constant* $L^* > 0$ *such that*

$$\begin{aligned} &D\Big(h(t,x(t))\int_0^t k(t,s)f(s,x(s))\,ds, h(t,y(t))\int_0^t k(t,s)f(s,y(s))\,ds\Big) \\ &\quad \le L^* D(x,y) \quad \text{for } x,y\in E^n, \end{aligned} \tag{4.3}$$

(iii) $g : U \to E^n$ *is continuous where* $U = \{(t,s,x) : 0\le s\le t\le a^*,\ x\in E^n \text{ and } D(x,\phi(t))\le b^*\}$ *and satisfies Lipschitz condition with respect to x on U, that is, there exists a constant* $M^* > 0$ *such that*

$$D(g(t,s,x),g(t,s,y)) \le M^* D(x,y) \quad \text{if } (t,s,x),(t,s,y)\in U. \tag{4.4}$$

If $c^* = (\alpha - L^*)/M$ *for some fixed* $\alpha\in(0,1)$*, then there is a unique solution of (4.1) on* $[0,T]$*, where* $T = \min\{a^*,b^*,c^*\}$.

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