ORBITAL STABILITY OF STANDING WAVES FOR A CLASS OF SCHRÖDINGER EQUATIONS WITH UNBOUNDED POTENTIAL

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Received 26 August 2004; Revised 20 October 2004; Accepted 9 November 2004

This paper is concerned with the nonlinear Schrödinger equation with an unbounded potential $i\varphi_t = -\triangle\varphi + V(x)\varphi - \mu|\varphi|^{p-1}\varphi - \lambda|\varphi|^{q-1}\varphi$, $x \in \mathbb{R}^N$, $t \geq 0$, where $\mu > 0$, $\lambda > 0$, and $1 < p < q < 1 + 4/N$. The potential $V(x)$ is bounded from below and satisfies $V(x) \to \infty$ as $|x| \to \infty$. From variational calculus and a compactness lemma, the existence of standing waves and their orbital stability are obtained.

1. Introduction

In this paper, we consider the nonlinear Schrödinger equation with an unbounded potential

$$i\varphi_t = -\triangle\varphi + V(x)\varphi - \mu|\varphi|^{p-1}\varphi - \lambda|\varphi|^{q-1}\varphi, \quad x \in \mathbb{R}^N,\ t \geq 0, \tag{1.1}$$

where $\mu > 0$, $\lambda > 0$, and $1 < p < q < 1 + 4/N$. The potential $V(x)$ is bounded from below and satisfies $V(x) \to \infty$ as $|x| \to \infty$. Equation (1.1) has its physical background. For example, when $V(x) = |x|^2$, it models the Bose-Einstein condensate with attractive interparticle interactions under magnetic trap [2, 7, 11, 17, 20].

When $|D^\alpha V|$ is bounded for all $|\alpha| \geq 2$, in terms of the smoothness of the time 0 of Schrödinger kernel for potentials of quadratic growth provided by Fujiwara [9], Oh [13] established the local well-posedness of (1.1) in the corresponding energy space. Since Yajima [19] showed that for superquadratic potentials, the Schrödinger kernel is nowhere C^1, we see that quadratic potentials are the highest-order potential for local well-posedness of (1.1). Thus the result of Oh [13], the local well-posedness of nonlinear Schrödinger equation with the potential function $V(x)$, is indeed sharp.

We are interested in the following standing waves of (1.1):

$$\varphi(t,x) = e^{iwt}u(x), \tag{1.2}$$

Hindawi Publishing Corporation
Journal of Applied Mathematics and Stochastic Analysis
Volume 2006, Article ID 57676, Pages 1–7
DOI 10.1155/JAMSA/2006/57676

where $w \in R$ is a parameter and $u(x)$ is the solution of the nonlinear elliptic equation

$$-\triangle u + V(x)u + wu - \mu|u|^{p-1}u - \lambda|u|^{q-1}u = 0. \tag{1.3}$$

The interesting topics to investigate standing waves are pursued strongly by many physicians and mathematicians [4, 3, 12, 14, 16].

For (1.3), Ding and Ni [8] by using "mountain pass" and comparison arguments got the existence of positive solutions. Rabinowitz [15] and Zhang [20, 21] also studied the existence of the solutions for (1.3) by the method of variation. Hirose and Ohta [10] studied the uniqueness of the solution for (1.3).

In this paper, for $1 < p < q < 1 + 4/N$, we establish the existence of the standing waves with the ground state of (1.1) by variational calculus which originates in Berestycki [1], Cazenave and Lions [6], Weinstein [18], and Zhang [20–23]. Furthermore, we prove the standing waves are orbitally stable.

This paper is organized as follows. In the second section, we give some necessary preliminaries which include the compactness lemma. In the third section, we prove the existence of the standing waves. And in the last section, we obtain their orbital stability.

2. Preliminaries

For (1.1), we impose the initial value as follows:

$$\varphi(x,0) = \varphi_0(x), \quad x \in \mathbb{R}^N. \tag{2.1}$$

In the course of nature, we set

$$H := \left\{ u \in H^1(\mathbb{R}^N) : \int V(x)|u|^2 dx < \infty \right\}. \tag{2.2}$$

Here and hereafter, for simplicity, we denote $\int_{\mathbb{R}^N} dx$ by $\int dx$. H becomes a Hilbert space, continuously embedded in $H^1(\mathbb{R}^N)$, when endowed with the inner product

$$\langle \varphi, \psi \rangle_H = \int \nabla\varphi \nabla\overline{\psi} + \varphi\overline{\psi} + \left(V(x) - \inf V(x)\right)\varphi\overline{\psi}\, dx, \tag{2.3}$$

whose associated norm is denoted by $\|\cdot\|_H$.

Lemma 2.1 [5, 13]. *Let $V(x)$ satisfy that $\inf V(x) > -\infty$ and for each $|\alpha| \geq 2$, $|D^\alpha V|$ is bounded, $1 < p < q < 1 + 4/N$, and $\varphi_0 \in H$. Then there exists a unique solution $\varphi(t,x)$ of the Cauchy problem (1.1), (2.1) in $([0,\infty); H)$, and $\varphi(t,\cdot)$ satisfies the following two conservation laws of the mass*

$$M(\varphi) = \int |\varphi|^2 dx = \int |\varphi_0|^2 dx = M(\varphi_0) \tag{2.4}$$

and energy

$$E(\varphi) = \int |\nabla\varphi|^2 + V(x)|\varphi|^2 - \frac{2\mu}{p+1}|\varphi|^{p+1} - \frac{2\lambda}{q+1}|\varphi|^{q+1} dx = E(\varphi_0) \tag{2.5}$$

for all $t \in [0,\infty)$.

Lemma 2.2. *If $V(x) \to \infty$ as $|x| \to \infty$, let $1 \le p < (N+2)/(N-2)$ when $N \ge 3$ and $1 \le p < \infty$ when $N = 1,2$. Then the embedding $H \hookrightarrow L^{p+1}(\mathbb{R}^N)$ is compact.*

Proof. We firstly show it for $p = 1$.

Since $H \hookrightarrow H^1(\mathbb{R}^N)$ continuously, it follows from the Sobolev embedding theorem that $H \hookrightarrow L^{p+1}(\mathbb{R}^N)$ continuously. Now let $\{u_n\}_n \subset H$ be a sequence such that

$$u_n \rightharpoonup 0 \quad \text{weakly in } H. \tag{2.6}$$

Then we have

$$u_n \rightharpoonup 0 \quad \text{weakly in } H^1(\mathbb{R}^N). \tag{2.7}$$

Moreover, we have $M := \sup_n \|u_n\|_H < \infty$. Let $\varepsilon > 0$. Then there exists $B > 0$ such that $1/V(x) \le \varepsilon$ for $|x| \ge B$. For B, from (2.7), we have

$$u_n \longrightarrow 0 \quad \text{in } L^2(\{|x| \le B\}). \tag{2.8}$$

It follows that there exists $m > 0$ such that

$$\int_{|x|\le B} |u_n|^2 dx \le \varepsilon \quad \text{for } n \ge m. \tag{2.9}$$

Then when $n \ge m$, we get

$$\begin{aligned} \int |u_n|^2 dx &= \int_{|x|\le B} |u_n|^2 dx + \int_{|x|\ge B} |u_n|^2 dx \\ &\le \varepsilon + \varepsilon \int_{|x|\ge B} V(x) |u_n|^2 dx \le \varepsilon + \varepsilon C M^2. \end{aligned} \tag{2.10}$$

Here and hereafter C denotes various positive constant. Thus we get that

$$u_n \longrightarrow 0 \quad \text{in } L^2(\mathbb{R}^N). \tag{2.11}$$

It follows that the embedding $H \hookrightarrow L^2(\mathbb{R}^N)$ is compact.

For $p > 1$, using the conclusion of $p = 1$ and the Gagliardo-Nirenberg inequality,

$$\|u\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} \le C \|\nabla u\|^{N(p-1)/2}_{L^2(\mathbb{R}^N)} \|u\|^{p+1-N(p-1)/2}_{L^2(\mathbb{R}^N)}, \tag{2.12}$$

we can get the conclusion immediately. □

3. The existence of standing waves

Firstly, we define a variational problem as follows:

$$d_\rho := \inf_{\{u \in H\setminus\{0\}: \int |u|^2 dx = \rho\}} E(u) \quad \text{for any } \rho > 0. \tag{3.1}$$

Theorem 3.1. *If $1 < p < q < 1 + 4/N$, then*

$$d_\rho = \min_{\{u \in H\setminus\{0\}: \int |u|^2 dx = \rho\}} E(u) \quad \textit{for any } \rho > 0. \tag{3.2}$$

Proof. Choose the minimizing sequence $\{u_n\}_{n\in\mathbb{N}}$ of the variational problem (3.1). Therefore, we have

$$u_n \in H\backslash\{0\}, \quad E(u_n) \longrightarrow d \quad \text{as } n \longrightarrow \infty, \tag{3.3}$$

$$\int |u_n|^2 dx = \rho. \tag{3.4}$$

By the Gagliardo-Nirenberg inequality and (3.4), for $1 < p < q < 1+4/N$, one has

$$\int |u_n|^{p+1} dx \le C\left(\int |\nabla u_n|^2 dx\right)^{\theta_1}, \qquad \int |u_n|^{q+1} dx \le C\left(\int |\nabla u_n|^2 dx\right)^{\theta_2}, \tag{3.5}$$

where $0 < \theta_1 < \theta_2 < 1$. Hence, from (3.3) and (3.5), we have

$$\begin{aligned} C &\ge \int |\nabla u_n|^2 + V(x)|u_n|^2 - \frac{2\mu}{p+1}|u_n|^{p+1} - \frac{2\lambda}{q+1}|u_n|^{q+1} dx \\ &\ge \frac{1}{2}\int |\nabla u_n|^2 dx - C\left(\int |\nabla u_n|^2 dx\right)^{\theta_1} + \frac{1}{2}\int |\nabla u_n|^2 dx - C\left(\int |\nabla u_n|^2 dx\right)^{\theta_2} \\ &\quad + \int (V(x) - \inf V(x))|u_n|^2 dx + \int \inf V(x)|u_n|^2 dx. \end{aligned} \tag{3.6}$$

Let $f(x) = x - Cx^\theta$ and $x > 0$, where $\theta \in (0,1)$ and $C > 0$. One has
(1^0) when $x = 0$ or $x = C^{1/(1-\theta)}$, $f(x) = 0$;
(2^0) $f'(x) = 1 - C\theta x^{\theta-1}$ and $f'(C^{1/(1-\theta)}) = 1 - \theta > 0$;
(3^0) $f''(x) = C\theta(1-\theta)x^{\theta-2} > 0$ as $x > 0$.
From the Taylor expansion of $f(x)$,

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(\xi)}{2}(x - x_0)^2, \tag{3.7}$$

where ξ is between x_0 and x, and choosing $x_0 = C^{1/(1-\theta)}$, one has

$$f(x) \ge (1-\theta)x - (1-\theta)C^{1/(1-\theta)}. \tag{3.8}$$

Therefore, by (3.4), (3.6), and (3.8), it yields that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in H. Therefore, there exists $u \in H$ such that the subsequence of $\{u_n\}_{n\in\mathbb{N}}$ which we still denote by $\{u_n\}_{n\in\mathbb{N}}$ satisfies

$$u_n \rightharpoonup u \quad \text{in } H. \tag{3.9}$$

By Lemma 2.2, one has

$$\begin{gathered} u_n \longrightarrow u \quad \text{in } L^2(\mathbb{R}^N), \\ u_n \longrightarrow u \quad \text{in } L^{p+1}(\mathbb{R}^N), \quad \text{in } L^{q+1}(\mathbb{R}^N). \end{gathered} \tag{3.10}$$

Therefore, it follows from (3.4) and (3.10) that

$$\int |u|^2 dx = \rho, \tag{3.11}$$

which implies that $E(u) \geq d_\rho$. From (3.10) and with

$$F(u) := \int |\nabla u|^2 + \big(V(x) - \inf V(x)\big)|u|^2 dx \tag{3.12}$$

being coercive and convex, one has

$$F(u) \leq \lim_{n\to\infty} \inf F(u_n). \tag{3.13}$$

From (3.3), (3.9), (3.10), (3.11), and (3.13), it follows that $E(u) = d_\rho$. The proof is complete. □

For any $\rho > 0$, let Ω_ρ denote the set of the minimizers of the variational problem (3.2). Then for any $u \in \Omega_\rho$, by Theorem 3.1, there must exist a Lagrange multiplier w such that

$$-\triangle u + V(x)u + wu - \mu|u|^{p-1}u - \lambda|u|^{q-1}u = 0. \tag{3.14}$$

It follows that $\varphi(t,x) = e^{iwt}u(x)$ is the standing wave solution of (1.1), which also called ground state since u is a minimizer of (3.2). Thus $e^{iwt}u(x)$ is the orbit of u. It is obvious that for any $t \geq 0$, if u is a solution of (3.2), then $e^{iwt}u$ is also a solution of (3.2), which yields $e^{iwt}u \in \Omega_\rho$.

4. Orbital stability of standing waves

Now in terms of Cazenave and Lion's argument [6], we have the following orbital stability.

Theorem 4.1. *Assume that $V(x)$ satisfies that* $\inf V > -\infty$, $V(x) \to \infty$ *as* $|x| \to \infty$ *and for each* $|\alpha| \geq 2$, $|D^\alpha V|$ *is bounded. Let* $1 < p < q < 1 + 4/N$. *Then the standing waves of the Cauchy problem (1.1), (2.1) are orbitally stable. In other words, for arbitrary* $\varepsilon > 0$, *there exists a* $\sigma > 0$ *such that for any* $\varphi_0 \in H$, *if*

$$\inf_{u\in\Omega_\rho} ||\varphi_0 - u||_H < \sigma, \tag{4.1}$$

then

$$\inf_{u\in\Omega_\rho} ||\varphi(x,t) - u(x)||_H < \varepsilon \quad \forall t \geq 0. \tag{4.2}$$

Proof. Firstly, for any $\varphi_0 \in H$, from Lemma 2.1, the corresponding solution $\varphi(x,t)$ of the Cauchy problem (1.1), (2.1) is global and bounded in H. Now arguing by contradiction, if the conclusion of the theorem does not hold, then there exist a $\varepsilon_0 > 0$, a sequence $\{\varphi_0^n\}_{n\in\mathbb{N}} \subset H$ such that

$$\inf_{u\in\Omega_\rho} ||\varphi_0^n - u||_H < \frac{1}{n}, \tag{4.3}$$

and a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that

$$\inf_{u\in\Omega_\rho} ||\varphi_n(t_n,\cdot) - u(\cdot)||_H \geq \varepsilon_0, \tag{4.4}$$

where φ_n denotes the solution of the Cauchy problem (1.1), (2.1) with the initial value φ_0^n.

From (4.3) and Lemma 2.2, we have

$$\begin{gathered} M(\varphi_0^n) = \int |\varphi_0^n|^2 dx \longrightarrow \int |u|^2 dx, \\ E(\varphi_0^n) \longrightarrow E(u). \end{gathered} \tag{4.5}$$

It follows from (4.5) and the conservation laws in Lemma 2.1 that $\{\varphi_n(t,\cdot)\}_{n\in\mathbb{N}}$ is a minimizing sequence for the problem (3.2). Therefore, there exists a $u \in \Omega_\rho$ such that

$$||\varphi_n(t_n,\cdot) - u||_H \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{4.6}$$

This is contradictory with (4.4). The proof is complete. □

Acknowledgment

This work is supported by National Natural Science Foundation (10271084), SZD0406, and the Emphasis Scientific Research Foundation of Sichuan Province.

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