# DUALITY FOR MULTIOBJECTIVE VARIATIONAL CONTROL AND MULTIOBJECTIVE FRACTIONAL VARIATIONAL CONTROL PROBLEMS WITH PSEUDOINVEXITY

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*Received 4 March 2005; Revised 28 January 2006; Accepted 28 January 2006*

A class of multiobjective variational control and multiobjective fractional variational control problems is considered, and the duality results are formulated. Under pseudoinvexity assumptions on the functions involved, weak, strong, and converse duality theorems are proved.



## 1. Introduction

The relationship between mathematical programming and classical calculus of variation was explored and extended by Hanson [6]. Thereafter variational programming problems have attracted some attention in literature. Duality for multiobjective variational problems has been of much interest in the recent years, and several contributions have been made to its development (see, e.g., Bector and Husain [2], Nahak and Nanda [9], Mishra and Mukherjee [7]). Using parametric equivalence, Bector et al. [1] formulated a dual program for a multiobjective fractional program involving continuously differentiable convex functions. Recently Nahak and Nanda [10] proved the duality theorems of multiobjective variational control problems under $(F,\rho)$-convexity assumptions.

In this paper, under pseudoinvexity assumptions on the functions involved, duality theorems are proved for multiobjective variational control problems. The duality of multiobjective fractional variational control problems is also considered by relating the primal problem to a parametric multiobjective variational control problem.

## 2. Notations and preliminaries

Let $I=[a,b]$ be a real interval and let $f: I\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}^p$ and let $g: I\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}^m$ be continuously differentiable functions. Consider the function $f(t,x(t),\dot{x}(t),u(t),\dot{u}(t))$, where $x: I\to\mathbb{R}^n$ with derivative $\dot{x}$ and $u: I\to\mathbb{R}^m$ with derivative $\dot{u}$. Here $t$ is the independent variable, $u(t)$ is the control variable and $x(t)$

Hindawi Publishing Corporation
Journal of Applied Mathematics and Stochastic Analysis
Volume 2006, Article ID 62631, Pages 1–15
DOI 10.1155/JAMSA/2006/62631

is the state variable. $u$ is related to $x$ via the state equation $h(t,x(t),\dot{x}(t),u(t),\dot{u}(t)) = 0$, where $h : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$. For $x, y \in \mathbb{R}^n$ by $x \leq y$, we mean $x_i \leq y_i$, for all $i$. All vectors will be taken as column vectors. The symbol $(\cdot)^T$ denotes for the transpose. For a real valued $k(t,x(t),\dot{x}(t),u(t),\dot{u}(t))$, denote the partial derivative of $k$ with respect to $t$, $x$, and $\dot{x}$, respectively, by $k_t$, $k_x$, and $k_{\dot{x}}$ such that $k_x = (\partial k/\partial x_1,\dots,\partial k/\partial x_n)$, $k_{\dot{x}} = (\partial k/\partial \dot{x}_1,\dots,\partial k/\partial \dot{x}_n)$. Similarly, we write the partial derivative of the vector functions $f$ and $g$ using matrices with $p$ and $m$ rows instead of one row. Partial derivatives with respect to $u$ and $\dot{u}$ are defined analogously. The control problem is to transfer the state variable from an initial state $\alpha$ at $x = a$ to a final state $\beta$ at $x = b$ so as to extremize a given functional subject to constraints on the control and state variables. Let $S(I,\mathbb{R}^n)$ denote the space of piecewise smooth functions $x$ with norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differentiation operator $D$ is given by

$$v = Dx \Longleftrightarrow x(t) = \alpha + \int_a^t v(s)ds, \tag{2.1}$$

and $\alpha$ is a given boundary value. Therefore $D = d/dt$ except at discontinuities. Consider the following multiobjective variational control primal problem as follows.

$$\begin{aligned} \text{Minimize } & \int_a^b f(t,x(t),\dot{x}(t),u(t),\dot{u}(t))dt \\ &= \left(\int_a^b f^1(t,x(t),\dot{x}(t),u(t),\dot{u}(t))dt,\dots,\int_a^b f^p(t,x(t),\dot{x}(t),u(t),\dot{u}(t))dt\right) \end{aligned} \tag{P}$$

subject to

$$x(a) = \alpha, \qquad x(b) = \beta, \tag{2.2}$$

$$g(t,x(t),\dot{x}(t),u(t),\dot{u}(t)) \geq 0, \quad t \in I, \tag{2.3}$$

$$h(t,x(t),\dot{x}(t),u(t),\dot{u}(t)) = 0, \quad t \in I. \tag{2.4}$$

Let $X$ denote the set of feasible points of (P).

*Definition 2.1.* A point $(x^*,u^*)$ in $X$ is said to be an efficient solution of (P) if for all $(x,u)$ in $X$,

$$\begin{aligned} &\int_a^b f^i(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t))dt \\ &\quad \geq \int_a^b f^i(t,x(t),\dot{x}(t),u(t),\dot{u}(t))dt, \quad \forall i \in \{1,\dots,p\}, \\ &\quad \Longrightarrow \int_a^b f^i(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t))dt \\ &\quad = \int_a^b f^i(t,x(t),\dot{x}(t),u(t),\dot{u}(t))dt, \quad \forall i \in \{1,\dots,p\}. \end{aligned} \tag{2.5}$$

We introduce the following problem (D) as the dual of (P).

$$\begin{aligned}\text{Maximize } &\int_a^b \Big[f\big(t,x(t),\dot{x}(t),u(t),\dot{u}(t)\big)-\mu^T g\big(t,x(t),\dot{x}(t),u(t),\dot{u}(t)\big)\\ &\qquad -\lambda^T h\big(t,x(t),\dot{x}(t),u(t),\dot{u}(t)\big)\Big]dt\\ =&\Bigg(\int_a^b \Big[f^1\big(t,x(t),\dot{x}(t),u(t),\dot{u}(t)\big)-\mu^T g\big(t,x(t),\dot{x}(t),u(t),\dot{u}(t)\big)\\ &\qquad -\lambda^T h(t,x(t),\dot{x}(t),u(t),\dot{u}(t))\Big]dt,\dots,\\ &\int_a^b \Big[f^p\big(t,x(t),\dot{x}(t),u(t),\dot{u}(t)\big)-\mu^T g\big(t,x(t),\dot{x}(t),u(t),\dot{u}(t)\big)\\ &\qquad -\lambda^T h\big(t,x(t),\dot{x}(t),u(t),\dot{u}(t)\big)\Big]dt\Bigg)\end{aligned} \tag{D}$$

subject to

$$x(a)=\alpha,\qquad x(b)=\beta, \tag{2.6}$$

$$\begin{aligned}&\Big[\tau^T f_x\big(t,x(t),\dot{x}(t),u(t),\dot{u}(t)\big)-\mu^T g_x\big(t,x(t),\dot{x}(t),u(t),\dot{u}(t)\big)-\lambda^T h_x\big(t,x(t),\dot{x}(t),u(t),\dot{u}(t)\big)\Big]\\ &\quad = D\Big[\tau^T f_{\dot{x}}\big(t,x(t),\dot{x}(t),u(t),\dot{u}(t)\big)-\mu^T g_{\dot{x}}\big(t,x(t),\dot{x}(t),u(t),\dot{u}(t)\big)\\ &\qquad -\lambda^T h_{\dot{x}}\big(t,x(t),\dot{x}(t),u(t),\dot{u}(t)\big)\Big],\end{aligned} \tag{2.7}$$

$$\begin{aligned}&\Big[\tau^T f_u\big(t,x(t),\dot{x}(t),u(t),\dot{u}(t)\big)-\mu^T g_u\big(t,x(t),\dot{x}(t),u(t),\dot{u}(t)\big)\\ &\quad -\lambda^T h_u\big(t,x(t),\dot{x}(t),u(t),\dot{u}(t)\big)\Big]=0,\end{aligned} \tag{2.8}$$

$$\sum_{i=1}^p \tau_i = 1,\qquad \mu(t)\geq 0, \tag{2.9}$$

where $\mu\in\mathbb{R}^m$, $\lambda\in\mathbb{R}^n$, and $\mathbb{R}^p{}_+$ denotes the nonnegative orthant of $\mathbb{R}^p$. Let $H$ denote the set of feasible points of (D).

*Definition 2.2.* For a scalar-valued function $k(t,x(t),\dot{x}(t),u(t),\dot{u}(t))$, the functional $\int_a^b k(t,x(t),\dot{x}(t),u(t),\dot{u}(t))dt$ is said to be pseudoinvex in $x$, $\dot{x}$, and $u$ on $[a,b]$ with respect to $\eta$ and $\xi$ if there exist vector functions $\eta(t,x(t),x^*(t),\dot{x}(t),\dot{x}^*(t),u(t),\dot{u}(t))\in\mathbb{R}^n$ with $\eta=0$ at $t$ if $x(t)=x^*(t)$ and $\xi(t,x(t),x^*(t),\dot{x}(t),\dot{x}^*(t),u(t),u^*(t))\in\mathbb{R}^m$ such that

$$\begin{aligned}\int_a^b &\Big[\eta^T\big(t,x(t),x^*(t),\dot{x}(t),\dot{x}^*(t),u(t),u^*(t)\big)k_x\big(t,x(t),\dot{x}(t),u(t),\dot{u}(t)\big)\\ &+\frac{d}{dt}\eta^T\big(t,x(t),x^*(t),\dot{x}(t),\dot{x}^*(t),u(t),u^*(t)\big)k_{\dot{x}}\big(t,x(t),\dot{x}(t),u(t),\dot{u}(t)\big)\end{aligned}$$

$$+\xi(t,x(t),x^*(t),\dot{x}(t),\dot{x^*}(t),u(t),u^*(t))k_u(t,x(t),\dot{x}(t),u(t),\dot{u}(t))\Big]dt \geq 0$$

$$\Longrightarrow \int_a^b [k(t,x^*,\dot{x}^*,u^*,\dot{u^*}) - k(t,x,\dot{x},u,\dot{u})]dt \geq 0. \tag{2.10}$$

A vector function is said to be pseudoinvex with respect to $\eta$ and $\xi$ if all its components are pseudoinvex with respect to the same $\eta$ and $\xi$.

## 3. Duality theorems

We wil now prove that problems (P) and (D) are a dual pair subject to pseudoinvexity conditions on the objective and constraint functions.

Theorem 3.1 (weak duality). *Let $(x,u) \in X$ and $(x^*,\dot{u}^*,\lambda,\mu,\tau) \in H$. If $\int_a^b[\tau^T f - \lambda^T h - \mu^T g]$ is pseudoinvex with respect to $\eta$ and $\xi$, then the following cannot hold:*

$$\begin{aligned}
&\int_a^b f^i(t,x(t),\dot{x}(t),u(t),\dot{u}(t))dt \\
&\quad\leq \int_a^b \Big[f^i(t,x^*(t),\dot{x}^*,u^*(t),\dot{u}^*(t)) - \lambda^T h(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)) \\
&\qquad - \mu^T g^i(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t))\Big]dt, \quad \forall i \in \{1,2,\ldots,p\}, \\
&\int_a^b f^i(t,x(t),\dot{x}(t),u(t),\dot{u}(t))dt \\
&\quad\leq \int_a^b \Big[f^i(t,x^*(t),\dot{x}^*,u^*(t),\dot{u}^*(t)) - \lambda^T h(t,x^*(t),\dot{x}*(t),u^*(t),\dot{u}^*(t)) \\
&\qquad - \mu^T g(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t))\Big]dt, \quad \textit{for at least one } j \in \{1,2,\ldots,p\}.
\end{aligned} \tag{3.1}$$

*Proof.* Let $(x,u)$ satisfy (2.2)–(2.4), $(x^*,\dot{u}^*,\lambda,\mu)$ satisfy (2.6)–(2.9). Now

$$\begin{aligned}
&\int_a^b \Big\{\eta(t,x,x^*,\dot{x},\dot{x^*},u,u^*)\Big[\tau^T f_x(t,x^*,\dot{x}^*,u^*,\dot{u}^*)\Big] + \frac{d}{dt}\eta(t,x,x^*,\dot{x},\dot{x^*},u,u^*) \\
&\quad\times\Big[\tau^T f_{\dot{x}}(t,x^*,\dot{x}^*,u^*,\dot{u}^*)\Big] + \xi(t,x,x^*,\dot{x},\dot{x^*},u,u^*)\Big[\tau^T f_u(t,x^*,\dot{x}^*,u^*,\dot{u}^*)\Big] \\
&\quad+\eta(t,x,x^*,\dot{x},\dot{x^*},u,u^*)\Big[-\lambda^T h_x(t,x^*,\dot{x}^*,u^*,\dot{u}^*)\Big] + \frac{d}{dt}\eta(t,x,x^*,\dot{x},\dot{x^*},u,u^*) \\
&\quad\times\Big[-\lambda^T h_{\dot{x}}(t,x^*,\dot{x}^*,u^*,\dot{u}^*)\Big] + \xi(t,x,x^*,\dot{x},\dot{x^*},u,u^*)\Big[-\lambda^T h_u(t,x^*,\dot{x}^*,u^*,\dot{u}^*)\Big] \\
&\quad+\eta(t,x,x^*,\dot{x},\dot{x^*},u,u^*)\Big[-\mu^T g_x(t,x^*,\dot{x}^*,u^*,\dot{u}^*)\Big] + \frac{d}{dt}\eta(t,x,x^*,\dot{x},\dot{x^*},u,u^*)
\end{aligned}$$

$$\times\Big[-\mu^T g_{\dot{x}}(t,x^*,\dot{x}^*,u^*,\dot{u}^*)\Big]+\xi(t,x,x^*,\dot{x},\dot{x^*},u,u^*)[-\mu^T g_u(t,x^*,\dot{x}^*,u^*,\dot{u}^*)]\Big\}dt$$

$$\begin{aligned}
&=\int_a^b\Big\{\eta(t,x,x^*,\dot{x},\dot{x}^*,u,u^*)\Big[\tau^T f_x(t,x^*,\dot{x}^*,u^*,\dot{u}^*)-\lambda^T h_x(t,x^*,\dot{x}^*,u^*,\dot{u}^*)\\
&\qquad\qquad-\mu^T g_x(t,x^*,\dot{x}^*,u^*,\dot{u}^*)\Big]\\
&\qquad+\frac{d}{dt}\eta(t,x,x^*,\dot{x},\dot{x^*},u,u^*)\Big[\tau^T f_{\dot{x}}(t,x^*,\dot{x}^*,u^*,\dot{u}^*)-\lambda^T h_{\dot{x}}(t,x^*,\dot{x}^*,u^*,\dot{u}^*)\\
&\qquad\qquad-\mu^T g_{\dot{x}}(t,x^*,\dot{x}^*,u^*,\dot{u}^*)\Big]\\
&\qquad+\xi(t,x,x^*,\dot{x},\dot{x^*},u,u^*)[\tau^T f_u(t,x^*,\dot{x}^*,u^*,\dot{u}^*)-\lambda^T h_u(t,x^*,\dot{x}^*,u^*,\dot{u}^*)\\
&\qquad\qquad-\mu^T g_u(t,x^*,\dot{x}^*,u^*,\dot{u}^*)]\Big\}dt\\
&=\int_a^b\eta(t,x,x^*,\dot{x},\dot{x^*},u,u^*)\frac{d}{dt}\Big[\tau^T f_{\dot{x}}(t,x^*,\dot{x}^*,u^*,\dot{u}^*)-\lambda^T h_{\dot{x}}(t,x^*,\dot{x}^*,u^*,\dot{u}^*)\\
&\qquad\qquad-\mu^T g_{\dot{x}}(t,x^*,\dot{x}^*,u^*,\dot{u}^*)\Big]dt\\
&\quad-\int_a^b\frac{d}{dt}\eta(t,x,x^*,\dot{x},\dot{x^*},u,u^*)\Big[\tau^T f_{\dot{x}}(t,x^*,\dot{x}^*,u^*,\dot{u}^*)-\lambda^T h_{\dot{x}}(t,x^*,\dot{x}^*,u^*,\dot{u}^*)\\
&\qquad\qquad-\mu^T g_{\dot{x}}(t,x^*,\dot{x}^*,u^*,\dot{u}^*)\Big]dt\quad\text{(by (2.7) and (2.8))}\\
&=\eta(t,x,x^*,\dot{x},\dot{x^*},u,u^*)\Big[\tau^T f_{\dot{x}}(t,x^*,\dot{x}^*,u^*,\dot{u}^*)-\lambda^T h_{\dot{x}}(t,x^*,\dot{x}^*,u^*,\dot{u}^*)\\
&\qquad\qquad-\mu^T g_{\dot{x}}(t,x^*,\dot{x}^*,u^*,\dot{u}^*)\Big]\\
&\quad\times\Big|_a^b-\int_a^b\frac{d}{dt}\eta(t,x,x^*,\dot{x},\dot{x^*},u,u^*)\\
&\qquad\times\Big[\tau^T f_{\dot{x}}(t,x^*,\dot{x}^*,u^*,\dot{u}^*)-\lambda^T h_{\dot{x}}(t,x^*,\dot{x}^*,u^*,\dot{u}^*)-\mu^T g_{\dot{x}}(t,x^*,\dot{x}^*,u^*,\dot{u}^*)\Big]dt\\
&\quad+\int_a^b\frac{d}{dt}\eta(t,x,x^*,\dot{x},\dot{x^*},u,u^*)\Big[\tau^T f_{\dot{x}}(t,x^*,\dot{x}^*,u^*,\dot{u}^*)-\lambda^T h_{\dot{x}}(t,x^*,\dot{x}^*,u^*,\dot{u}^*)\\
&\qquad\qquad-\mu^T g_{\dot{x}}(t,x^*,\dot{x}^*,u^*,\dot{u}^*)\Big]dt=0
\end{aligned}\tag{3.2}$$

(by integration by parts and since $\eta(t,x^*,x^*,\dot{x^*},\dot{x^*},u,u^*)=0$). So by pseudoinvexity of $\int_a^b[\tau^T f-\lambda^T h-\mu^T g]dt$, we have

$$\begin{aligned}
&\int_a^b\Big\{\tau^T f(t,x(t),\dot{x}(t),u(t),\dot{u}(t))-\lambda^T h(t,x(t),\dot{x}(t),u(t),\dot{u}(t))\\
&\qquad-\mu^T g(t,x(t),\dot{x}(t),u(t),\dot{u}(t))\Big\}dt\\
&\qquad\ge\int_a^b\Big[\tau^T f(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t))-\lambda^T h(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t))\\
&\qquad\qquad-\mu^T g(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t))\Big]dt,
\end{aligned}\tag{3.3}$$

conditions (2.3) and (2.4) give

$$\begin{aligned}
&\int_a^b \tau^T f(t,x(t),\dot{x}(t),u(t),\dot{u}(t))\,dt \\
&\qquad \geq \int_a^b \Big[\tau^T f(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)) - \lambda^T h(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)) \\
&\qquad\qquad - \mu^T g(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t))\Big]dt.
\end{aligned}\tag{3.4}$$

So the following cannot hold:

$$\begin{aligned}
&\int_a^b f^i(t,x(t),\dot{x}(t),u(t),\dot{u}(t))\,dt \\
&\qquad \leq \int_a^b \Big[f^i(t,x^*(t),\dot{x}^*,u^*(t),\dot{u}^*(t)) - \lambda^T h(t,x^*(t)\dot{x}*(t),u^*(t),\dot{u}^*(T))\Big]dt \\
&\qquad\qquad - \mu^T g(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t))\,dt, \quad \forall i \in \{1,2,\dots,p\}, \\
&\int_a^b f^i(t,x(t),\dot{x}(t),u(t),\dot{u}(t))\,dt \\
&\qquad \leq \int_a^b \Big[f^i(t,x^*(t),\dot{x}^*,u^*(t),\dot{u}^*(t)) - \lambda^T h(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)) \\
&\qquad\qquad - \mu^T g(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t))\Big]dt, \quad \text{for at least one } j \in \{1,2,\dots,p\}.
\end{aligned}\tag{3.5}$$

□

Assume that the necessary constraints for the existence of multipliers at an extreme value of (P) are satisfied, thus for every efficient $(x^*,u^*)$ of (P) there exists a piecewise smooth $\mu_0 : I \to \mathbb{R}^m$, $\mu(t)$ such that

$$\begin{aligned}
F = {} & \mu_0^T f(t,x(t),\dot{x}(t),u(t),\dot{u}(t)) - \mu^T g(t,x(t),\dot{x}(t),u(t),\dot{u}(t)) \\
& - \lambda^T h(t,x(t),\dot{x}(t),u(t),\dot{u}(t))
\end{aligned}\tag{3.6}$$

satisfies

$$\begin{gathered}
F_x = \frac{d}{dt}F_{\dot{x}}, \qquad F_u = 0, \\
\mu^T g = 0, \qquad \mu(t) \geq 0.
\end{gathered}\tag{3.7}$$

It is assumed from now on that the minimizing solution $(x^*,u^*)$ of (P) is normal, that is, $\mu_0$ is nonzero, so that without loss of generality, we can take $\mu_0 = 1$.

Theorem 3.2 (strong duality). *Under the pseudoinvexity conditions of Theorem 3.1, if $(x^*,u^*)$ is an efficient solution for (P), then there exists a piecewise smooth $\mu : I \to \mathbb{R}^m$,* *such that $(x^*,u^*,\lambda,\mu,\tau)$ is an efficient solution of (D) and the extreme values of (P) and (D) are equal.*

*Proof.* Since $(x^*,u^*)$ is an efficient solution of (P), there exists, $\mu(t): I \to \mathbb{R}^m$, such that for $t \in I$,

$$\begin{aligned}
&\Big[\tau^T f_x(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)) - \mu^T g_x(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)) \\
&\quad - \lambda^T(t) h_x(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t))\Big] \\
&\qquad = \frac{d}{dt}\Big[\tau^T f_{\dot{x}}(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)) - \mu^T g_{\dot{x}}(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)) \\
&\qquad\quad - \lambda^T(t) h_{\dot{x}}(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t))\Big],
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
&\Big[\tau^T f_u(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)) - \mu^T g_u(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)) \\
&\quad - \lambda^T(t) h_u(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t))\Big] = 0,
\end{aligned} \tag{3.9}$$

$$\mu^T g(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)) = 0, \tag{3.10}$$

$$\sum_{i=1}^{p} = 1, \quad \mu \geq 0. \tag{3.11}$$

From (3.8), (3.9), and (3.11), it follows that $(x^*,u^*,\lambda,\mu)$ is feasible for (D). From (3.10) it follows that extreme values of (P) and (D) are equal. By Theorem 3.1, $(x^*,u^*,\lambda,\mu,\tau)$ is efficient for (D). □

For validating the converse duality theorem (Theorem 3.3), we make the assumption that $X_2$ denotes the space of piecewise twice the differentiable functions $x: I \to \mathbb{R}^n$ for which $x(a) = x(b) = 0$ is equipped with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty + \|D^2x\|_\infty$, and $U_2$ denotes the space of $u: I \to \mathbb{R}^m$ with norm $\|u\| = \|u\|_\infty$, defining $(D = d/dt)$ as before. The problem (D) may be rewritten in the following form.

$$\text{Minimize } -\phi(x,u,\lambda,\mu) = (-\phi^1(x,u,\lambda,\mu), -\phi^2(x,u,\lambda,\mu), \ldots, -\phi^p(x,u,\lambda,\mu)) \tag{3.12}$$

subject to

$$\begin{gathered}
x(a) = \alpha, \qquad x(b) = \beta, \\
\Theta(t,x,\dot{x},\ddot{x},\lambda,u,\dot{u},\mu,\dot{\mu}) = 0, \quad t \in I, \\
\mu(t) \geq 0, \quad t \in I,
\end{gathered} \tag{3.13}$$

where

$$\begin{aligned}
\phi^i(x,u,\mu,\lambda) = \int_a^b &\big[f^i(t,x(t),\dot{x}(t),u(t),\dot{u}(t)) - \mu^T g(t,x(t),\dot{x}(t),u(t),\dot{u}(t)) \\
&- \lambda^T h(t,x(t),\dot{x}(t),u(t),\dot{u}(t))\big]dt, \quad i = 1,2,\ldots,p,
\end{aligned}$$

$$
\begin{aligned}
\Theta &\equiv \Theta(t,x,\dot{x},\ddot{x},\lambda,u,\dot{u},\mu,\dot{\mu}) \\
&= \Big[\tau^T f_x(t,x(t),\dot{x}(t),u(t),\dot{u}(t)) - \mu^T g_x(t,x(t),\dot{x}(t),u(t),\dot{u}(t)) \\
&\qquad - \lambda^T(t) h_x(t,x(t),\dot{x}(t),u(t),\dot{u}(t))\Big] \\
&\qquad - D\Big[\tau^T f_{\dot{x}}(t,x(t),\dot{x}(t),u(t),\dot{u}(t)) - \mu^T g_{\dot{x}}(t,x(t),\dot{x}(t),u(t),\dot{u}(t)) \\
&\qquad\qquad - \lambda^T(t) h_{\dot{x}}(t,x(t),\dot{x}(t),u(t),\dot{u}(t))\Big] = 0, \quad t \in I
\end{aligned}
\tag{3.14}
$$

with $\ddot{x} = D^2 x(t)$.

Consider $\Theta(t,x,\dot{x},\ddot{x},\lambda,u,\dot{u},\mu,\dot{\mu})$ as defined above and a map $\psi : X_2 \times Y \times \mathbb{R}^p{}_+ \to A$, where $Y$ is the space of piecewise differentiable function $u : I \to \mathbb{R}^m$, and $A$ is a Banach space. A Fritz John theorem [4, 5] for infinite dimensional multiobjective programming problem may be applied to problem (D) along with the analysis outlined in [8] or [3] for the derivation of optimality conditions. It suffices to assume that the Frechet derivative $\psi' = (\psi_x, \psi_y, \psi_\lambda)$ has a (*weak**) closed range.

Theorem 3.3 (converse duality). *If $(x^*, u^*, \lambda, \mu)$ is an efficient solution of (D), and if*

(i) *$\psi'$ has a (weak*) closed range;*

(ii) *$f$, $g$ and $h$ are twice continuously differentiable;*

(iii) *$\int_a^b [f_x^i - D f_{\dot{x}}^i]\,dt$, $i = 1,2,\ldots,p$, is linearly independent; and*

(iv) *$(\beta(t)^T \Theta_x - D\beta(t)^T \Theta_{\dot{x}} + D^2 \beta(t)^T \Theta_{\dot{x}})\beta(t) = 0 \Rightarrow \beta(t) = 0, t \in I$,*

*then $(x^*, u^*)$ is an efficient solution of (P), and the objective values of (P) and (D) are equal.*

*Proof.* See Bector and Husain [2]. □

## 4. Fractional variational control problems

Now we are in a position to study duality in multiobjective fractional variational control problems. Following Mishra and Mukherjee [7], we give the multiobjective fractional control problem as follows.

(Primal) Minimize

$$
\begin{aligned}
&\frac{\int_a^b f(t,x(t),\dot{x}(t),u(t),\dot{u}(t))\,dt}{\int_a^b g(t,x(t),\dot{x}(t),u(t),\dot{u}(t))\,dt} \\
&\quad = \left( \frac{\int_a^b f^1(t,x(t),\dot{x}(t),u(t),\dot{u}(t))\,dt}{\int_a^b g^1(t,x(t),\dot{x}(t),u(t),\dot{u}(t))\,dt}, \ldots, \frac{\int_a^b f^p(t,x(t),\dot{x}(t),u(t),\dot{u}(t))\,dt}{\int_a^b g^p(t,x(t),\dot{x}(t),u(t),\dot{u}(t))\,dt} \right)
\end{aligned}
\tag{P$_1$}
$$

subject to

$$
\begin{gathered}
x(a)=\alpha, \qquad x(b)=\beta, \\
h^j(t,x(t),\dot{x}(t),u(t),\dot{u}(t))=0, \quad j=1,2,\ldots,n, \\
k^j(t,x(t),\dot{x}(t),u(t),\dot{u}(t))\geq 0, \quad j=1,2,\ldots,m.
\end{gathered} \tag{4.1}
$$

We assume $g(t,x(t),\dot{x}(t),u(t),\dot{u}(t))>0$ and $f(t,x(t),\dot{x}(t),u(t),\dot{u}(t))\geq 0$.

The equivalent parametric form of the problem is the following.

Maximize

$$
\begin{aligned}
&\int_a^b [f(t,x(t),\dot{x}(t),u(t),\dot{u}(t))-v^T g(t,x(t),\dot{x}(t),u(t),\dot{u}(t))]dt \\
&\quad=\bigg(\int_a^b [f^1(t,x(t),\dot{x}(t),u(t),\dot{u}(t))-v^T g(t,x(t),\dot{x}(t),u(t),\dot{u}(t))]dt,\ldots, \\
&\qquad\int_a^b [f^p(t,x(t),\dot{x}(t),u(t),\dot{u}(t))-v^T g(t,x(t),\dot{x}(t),u(t),\dot{u}(t))]dt\bigg)
\end{aligned} \tag{$\mathrm{P}_v$}
$$

subject to

$$
\begin{gathered}
x(a)=\alpha, \qquad x(b)=\beta, \\
h^j(t,x(t),\dot{x}(t),u(t),\dot{u}(t))=0, \quad j=1,2,\ldots,n, \\
k^j(t,x(t),\dot{x}(t),u(t),\dot{u}(t))\geq 0, \quad j=1,2,\ldots,m.
\end{gathered} \tag{4.2}
$$

Lemma 4.1. *Let $(x^*,u^*)$ be efficient for ($P_1$). Then there exists $v\in\mathbb{R}^p{}_+$ such that $(x^*,u^*)$ is also efficient for ($P_v$).*

*Proof.* See [3]. □

*Remark 4.2.* The converse of Lemma 4.1 also holds provided we assume

$$
v_i{}^*=\frac{\int_a^b f^i(t,x^*,\dot{x}^*,u^*,\dot{u}^*)dt}{\int_a^b g^i(t,x^*,\dot{x}^*,u^*,\dot{u}^*)dt}. \tag{4.3}
$$

So the dual of the above primal problem is given by ($\mathrm{D}_1$).

(Dual) Maximize

$$
\int_a^b \Big\{\tau^T f(t,x,\dot{x},u,\dot{u})-v^T g(t,x,\dot{x},u,\dot{u})-\lambda^T h(t,x,\dot{x},u,\dot{u})-\mu^T k(t,x,\dot{x},u,\dot{u})\Big\}dt \tag{$\mathrm{D}_1$}
$$

subject to

$$x(a) = \alpha, \qquad x(b) = \beta, \tag{4.4}$$

$$\begin{aligned}&\big[\tau^T f_x(t,x,\dot{x},u,\dot{u}) - \nu^T g_x(t,x,\dot{x},u,\dot{u}) - \lambda(t)^T h_x(t,x,\dot{x},u,\dot{u}) - \mu(t)^T k_x(t,x,\dot{x},u,\dot{u})\\ &\quad = \frac{d}{dt}\big[\tau^T f_{\dot{x}}(t,x,\dot{x},u,\dot{u}) - \nu^T g_{\dot{x}}(t,x,\dot{x},u,\dot{u}) - \lambda(t)^T h_{\dot{x}}(t,x,\dot{x},u,\dot{u})\\ &\qquad - \mu(t)^T k_{\dot{x}}(t,x,\dot{x},u,\dot{u})\big],\end{aligned} \tag{4.5}$$

$$\tau^T f_u(t,x,\dot{x},u,\dot{u}) - \nu^T g_u(t,x,\dot{x},u,\dot{u}) - \lambda(t)^T h_u(t,x,\dot{x},u,\dot{u}) - \mu(t)^T k_u(t,x,\dot{x},u,\dot{u}) = 0, \tag{4.6}$$

$$\sum_{i=1}^{p} \tau_i = 1, \quad \mu(t) \geq 0. \tag{4.7}$$

Theorem 4.3 (weak duality). *Let $(x,u)$ be feasible for $(P_1)$ and $(x^*,u^*,\mu,\nu,\tau)$ be feasible for $(D_1)$. If $\int_a^b[\tau^T f - \nu^T g - \lambda^T h - \mu^T k]dt$ is pseudoinvex with respect to $\eta$ and $\xi$, then the following cannot hold:*

$$\begin{aligned}&\int_a^b [f^i(t,x,\dot{x},u,\dot{u}) - \nu^T g(t,x,\dot{x},u,\dot{u})]dt\\ &\quad \leq \int_a^b \big[f^i(t,x^*,\dot{x}^*,u^*,\dot{u}^*) - \nu^T g(t,x^*,\dot{x}^*,u^*,\dot{u}^*) - \lambda^T h(t,x^*\dot{x}*,u^*,\dot{u}^*)\\ &\qquad - \mu^T k(t,x^*,\dot{x}^*,u^*,\dot{u}^*)\big]dt, \quad \forall i \in \{1,2,\dots,p\},\\ &\int_a^b [f^i(t,x,\dot{x},u,\dot{u}) - \nu^T g(t,x,\dot{x},u,\dot{u})]dt\\ &\quad < \int_a^b \big[f^i(t,x^*,\dot{x}^*,u^*,\dot{u}^*) - \nu^T g(t,x^*,\dot{x}^*,u^*,\dot{u}^*) - \lambda^T h(t,x^*,\dot{x}^*,u^*,\dot{u}^*)\\ &\qquad - \mu^T k^i(t,x^*,\dot{x}^*,u^*,\dot{u}^*)\big]dt, \quad \text{for at least one } j \in \{1,2,\dots,p\}.\end{aligned} \tag{4.8}$$

*Proof.* Let $(x,u)$ be feasible for $(P_1)$ and $(x^*,u^*,\mu,\nu,\tau)$ feasible for $(D_1)$. For simplification, denote $Z_1 = (t,x(t),x^*(t),\dot{x}(t),\dot{x}^*,u(t),u^*(t))$,

$$\begin{aligned}&\int_a^b \Big\{\eta(Z_1)\tau^T f_x(t,x^*(t),\dot{x^*}(t),u(t),\dot{u^*}(t)) + \frac{d}{dt}\eta(Z_1)[\tau^T f_{\dot{x}}(t,x^*(t),\dot{x^*}(t),u^*(t),\dot{u^*}(t))]\\ &\quad + \xi(Z_1)\tau^T f_u(t,x^*(t),\dot{x^*}(t),u^*(t),\dot{u^*}(t))\\ &\quad + \eta(Z_1)[-\nu^T g_x(t,x^*(t),\dot{x^*}(t),u^*(t),\dot{u^*}(t))]\\ &\quad + \frac{d}{dt}\eta(Z_1)[-\nu^T g_{\dot{x}}(t,x^*(t),\dot{x^*}(t),u^*(t),\dot{u^*}(t))]\end{aligned}$$

$$
\begin{aligned}
&+\xi(Z_1)\Big(-\nu^T g_u\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)\Big)\\
&+\eta(Z_1)\Big(-\lambda^T(t)h_x\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)\Big)\\
&+\frac{d}{dt}\eta(Z_1)\Big[-\lambda^T(t)h_{\dot{x}}\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)\Big]\\
&+\xi(Z_1)\Big[-\lambda^T(t)h_u\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)\Big]\\
&+\eta(Z_1)\Big[-\mu^T(t)k_x\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)\Big]\\
&+\frac{d}{dt}\eta(Z_1)\Big[-\mu^T(t)k_{\dot{x}}\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)\Big]\\
&+\xi(Z_1)\Big[-\mu^T(t)k_u\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)\Big]\Big\}dt\\
&=\int_a^b\eta(Z_1)\Big\{\tau^T f_x\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)-\nu^T g_x\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)\\
&\qquad\qquad-\lambda^T(t)h_x\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)\\
&\qquad\qquad-\mu^T(t)k_x\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)\Big\}dt\\
&\quad+\int_a^b\frac{d}{dt}\eta(Z_1)\Big[\tau^T f_{\dot{x}}\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)-\nu^T g_{\dot{x}}\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)\\
&\qquad\qquad-\lambda^T(t)h_{\dot{x}}\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)\\
&\qquad\qquad-\mu^T(t)k_{\dot{x}}\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)\Big]dt\\
&\quad+\int_a^b\xi(Z_1)\Big[\tau^T f_u\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)-\nu^T g_u\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)\\
&\qquad\qquad-\lambda^T(t)h_u\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)\\
&\qquad\qquad-\mu^T(t)k_u\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)\Big]dt\\
&=\int_a^b\eta(Z_1)\frac{d}{dt}\Big[\tau^T f_{\dot{x}}\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)-\nu^T g_{\dot{x}}\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)\\
&\qquad\qquad-\lambda^T(t)h_{\dot{x}}\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)\\
&\qquad\qquad-\mu^T(t)k_{\dot{x}}\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)\Big]dt\\
&\quad+\int_a^b\frac{d\eta}{dt}\Big[\tau^T f_{\dot{x}}\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)-\nu g_{\dot{x}}\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)\\
&\qquad\qquad-\lambda^T(t)h_{\dot{x}}\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)\\
&\qquad\qquad-\mu^T(t)k_{\dot{x}}\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)\Big]dt,\quad(\text{by (4.5) and (4.6)})\\
&=\eta\Big[\tau^T f_{\dot{x}}\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)-\nu g_{\dot{x}}\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)\\
&\qquad-\lambda^T(t)h_{\dot{x}}\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)-\mu^T(t)k_{\dot{x}}\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)\Big]\Big|_a^b\\
&\quad-\int_a^b\frac{d\eta}{dt}\Big[\tau^T f_{\dot{x}}\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)-\nu g_{\dot{x}}\big(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)\big)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad - \lambda^T(t) h_{\dot{x}}(t, x^*(t), \dot{x^*}(t), u^*(t), \dot{u^*}(t)) \\
&\qquad - \mu^T(t) k_{\dot{x}}(t, x^*(t), \dot{x^*}(t), u^*(t), \dot{u^*}(t)) \Big] dt \\
&+ \int_a^b \frac{d\eta}{dt} \Big[ \tau^T f_{\dot{x}}(t, x^*(t), \dot{x^*}(t), u^*(t), \dot{u^*}(t)) - v g_{\dot{x}}(t, x^*(t), \dot{x^*}(t), u^*(t), \dot{u^*}(t)) \\
&\qquad - \lambda^T(t) h_{\dot{x}}(t, x^*(t), \dot{x^*}(t), u^*(t), \dot{u^*}(t)) \\
&\qquad - \mu^T(t) k_{\dot{x}}(t, x^*(t), \dot{x^*}(t), u^*(t), \dot{u^*}(t)) \Big] dt = 0.
\end{aligned} \tag{4.9}
$$

(By integration by parts and since $(\eta(t, x^*, x^*, \dot{x^*}, x^*, u, u^*) = 0.)$

So by pseudoinvexity of $\int_a^b [\tau^T f - v^T g - \lambda^T h - \mu^T k] dt$, we have

$$
\begin{aligned}
&\int_a^b \Big[ \tau^T f(t, x, \dot{x}, u, \dot{u}) - v^T g(t, x, \dot{x}, u, \dot{u}) - \lambda^T(t) h(t, x, \dot{x}, u, \dot{u}) - \mu^T(t) k(t, x, \dot{x}, u, \dot{u}) \Big] dt \\
&\qquad \geq \int_a^b \Big[ \tau^T f(t, x^*, \dot{x^*}, u^*, \dot{u^*}) - v^T g(t, x^*, \dot{x^*}, u^*, \dot{u^*}) - \lambda^T(t) h(t, x^*, \dot{x^*}, u^*, \dot{u^*}) \\
&\qquad\qquad - \mu^T(t) k(t, x^*, \dot{x^*}, u^*, \dot{u^*}) \Big] dt,
\end{aligned} \tag{4.10}
$$

but $\int_a^b -\mu^T(t) k(t, \dot{x}, u, \dot{u}) \leq 0$, hence

$$
\begin{aligned}
&\int_a^b [\tau^T f(t, x, \dot{x}, u, \dot{u}) - v^T g(t, x, \dot{x}, u, \dot{u})] dt \\
&\qquad \geq \int_a^b \Big[ \tau^T f(t, x^*, \dot{x^*}, u^*, \dot{u^*}) - v^T g(t, x^*, \dot{x^*}, u^*, \dot{u^*}) - \lambda^T(t) h(t, x^*, \dot{x^*}, u^*, \dot{u^*}) \\
&\qquad\qquad - \mu^T(t) k(t, x^*, \dot{x^*}, u^*, \dot{u^*}) \Big] dt.
\end{aligned} \tag{4.11}
$$

Hence the following cannot hold:

$$
\begin{aligned}
&\int_a^b \Big[ f^i(t, x, \dot{x}, u, \dot{u}) - v^T g(t, x, \dot{x}, u, \dot{u}) \Big] dt \\
&\qquad \leq \int_a^b \Big[ f^i(t, x^*, \dot{x}^*, u^*, \dot{u}^*) - v^T g(t, x^*, \dot{x}^*, u^*, \dot{u}^*) - \lambda^T h(t, x^* \dot{x}*, u^*, \dot{u}^*) \\
&\qquad\qquad - \mu^T k^i(t, x^*, \dot{x}^*, u^*, \dot{u}^*) \Big] dt, \quad \forall i \in \{1, 2, \ldots, p\}, \\
&\int_a^b \Big[ f^i(t, x, \dot{x}, u, \dot{u}) - v^T g(t, x, \dot{x}, u, \dot{u}) \Big] dt \\
&\qquad < \int_a^b \Big[ f^i(t, x^*, \dot{x}^*, u^*, \dot{u}^*) - v^T g(t, x^*, \dot{x}^*, u^*, \dot{u}^*) - \lambda^T h(t, x^* \dot{x}*, u^*, \dot{u}^*) \\
&\qquad\qquad - \mu^T k^i(t, x^*, \dot{x}^*, u^*, \dot{u}^*) \Big] dt, \quad \text{for at least one } j \in \{1, 2, \ldots, p\}.
\end{aligned} \tag{4.12}
$$

□

Once weak duality has been established, strong and converse dualities follow. For completeness, we state the results for strong and converse dualities. We assume that the necessary constraints for the existence of multipliers at an extreme value of ($P_1$) are satisfied. Thus for every efficient $(x^*, u^*)$ to ($P_1$), there exists a piecewise smooth $\lambda_0 : I \to \mathbb{R}^m$, such that

$$\begin{aligned} F = \lambda_0^T f(t,x(t),\dot{x}(t),u(t),\dot{u}(t)) - \nu^T g(t,x(t),\dot{x}(t),u(t),\dot{u}(t)) \\ - \lambda^T h(t,x(t),\dot{x}(t),u(t),\dot{u}(t)) - \mu^T k(t,x(t),\dot{x}(t),u(t),\dot{u}(t)) \end{aligned} \tag{4.13}$$

satisfies

$$\begin{aligned} F_x = \frac{d}{dt}F_{\dot{x}}, \qquad F_u = 0, \\ \mu^T k = 0, \qquad \mu(t) \geq 0. \end{aligned} \tag{4.14}$$

It is assumed from now on that the minimizing solution $(x^*, u^*)$ of (P) is normal, that is, $\lambda_0$ is nonzero, so that without loss of generality, we can take $\lambda_0 = 1$.

Theorem 4.4 (strong duality). *Under the pseudoinvexity and the condition of Theorem 4.3, if $(x^*,u^*)$ is an efficient solution for ($P_1$), then there exists a piecewise smooth $\lambda(t)$: $I \to \mathbb{R}^n$, such that $(x^*(t),u^*(t),\lambda(t),\mu(t),\nu)$ is an efficient solution of ($D_1$) and the extreme values of ($P_1$) and ($D_1$) are equal.*

*Proof.* Very similar to that of Theorem 3.2.

For the the converse duality theorem (Theorem 4.5), we make the assumption as before (see Theorem 3.3), that is, $X_2$ denotes the space of piecewise twice differentiable functions $x : I \to \mathbb{R}^n$ for which $x(a) = x(b) = 0$ is equipped with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty + \|D^2x\|_\infty$, and $U_2$ denotes the space of $u : I \to \mathbb{R}^m$ with norm $\|u\| = \|\cdot\|_\infty$, defining ($D = d/dt$) as before. The problem ($D_1$) may be rewritten in the following form.

$$\text{Minimize } -\phi(x,u,\lambda,\mu) = (-\phi^1(x,u,\lambda,\mu), -\phi^2(x,u,\lambda,\mu), \ldots, -\phi^p(x,u,\lambda,\mu)) \tag{4.15}$$

subject to

$$\begin{aligned} x(a) = \alpha, \qquad x(b) = \beta, \\ \Theta(t,x,\dot{x},\ddot{x},\lambda,\mu,u,\dot{u}) = 0, \quad t \in I, \\ \mu(t) \geq 0, \quad t \in I, \end{aligned} \tag{4.16}$$

where

$$\begin{aligned} \phi^i(x,u,\lambda,\mu) = \int_a^b \Big[ f^i(t,x(t),\dot{x}(t),\lambda,\mu,u(t),\dot{u}(t)) - \nu^T g(t,x(t),\dot{x}(t),\lambda,\mu,u(t),\dot{u}(t)) \\ - \lambda^T h(t,x(t),\dot{x}(t),\lambda,\mu,u(t),\dot{u}(t)) \\ - \mu^T(t) k(t,x(t),\dot{x}(t),\lambda,\mu,u(t),\dot{u}(t)) \Big] dt, \quad i = 1,2,\ldots,p, \end{aligned}$$

$$\Theta \equiv \Theta(t,x(t),\dot{x}(t),\ddot{x}(t),\lambda,\mu,u(t),\dot{u}(t))$$

$$
\begin{aligned}
&= \Big[\tau^T f_x(t,x(t),\dot{x}(t),u(t),\dot{u}(t)) - \nu^T g_x(t,x(t),\dot{x}(t),u(t),\dot{u}(t)) \\
&\qquad - \lambda^T(t) h_x(t,x(t),\dot{x}(t),u(t),\dot{u}(t)) - \mu^T k_x(t,x(t),\dot{x}(t),u(t),\dot{u}(t))\Big] \\
&= D\Big[\tau^T f_{\dot{x}}(t,x(t),\dot{x}(t),u(t),\dot{u}(t)) - \nu^T g_{\dot{x}}(t,x(t),\dot{x}(t),u(t),\dot{u}(t)) \\
&\qquad - \lambda^T(t) h_{\dot{x}}(t,x(t),\dot{x}(t),u(t),\dot{u}(t)) - \mu^T k_{\dot{x}}(t,x(t),\dot{x}(t),u(t),\dot{u}(t))\Big]
\end{aligned}
\tag{4.17}
$$

with $t \in I$ and $\ddot{x} = D^2 x(t)$. □

Now, following Bector and Husain [2] and analogously for the control case, we are in a position to deal with the converse duality.

Theorem 4.5 (converse duality). *If $(x^*, u^*, \lambda, \mu, \nu)$ is an efficient solution of $(D_1)$, and if*

(i) *$\psi'$ has a (weak$^*$) closed range;*

(ii) *$f$, $g$ and $h$ are twice continuously differentiable;*

(iii) *$\{\int_a^b (f_x^i - \nu_i{}^T g_x^i) - D(f_{\dot{x}}^i - \nu_i{}^T g_x{}^i)dt$, $i = 1,2,\ldots,p$, is linearly independent; and*

(iv) *$(\beta(t)^T \Theta_x - D\beta(t)^T \Theta_{\dot{x}} + D^2 \beta(t)^T \Theta_{\dot{x}})\beta(t) = 0 \Rightarrow \beta(t) = 0$, $t \in I$,*

*then $(x^*, u^*)$ is an efficient solution of $(P_1)$, and the objective values of $(P_1)$ and $(D_1)$ are equal.*

*Proof.* Very similar to that of Theorem 3.3. □

## Acknowledgments

The author wishes to thank the referee and the associate editor for their valuable suggestions which improved the presentation of the paper. The work of the author was supported by DST, Government of India Grant no. SR/FTP/MS-23/2001. The second part of the work of the author was supported by ISIRD project, IIT Kharagpur, Grant no. IIT/SRIC/ISIRD/2004-5.

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