ATTRACTIVITY OF NONLINEAR IMPULSIVE DELAY DIFFERENTIAL EQUATIONS

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The attractivity of nonlinear differential equations with time delays and impulsive effects is discussed. We obtain some criteria to determine the attracting set and attracting basin of the impulsive delay system by developing an impulsive delay differential inequality and introducing the concept of nonlinear measure. Examples and their simulations illustrate the effectiveness of the results and different asymptotical behaviors between the impulsive system and the corresponding continuous system.

1. Introduction

The stability and attractivity of impulsive differential equations have been deeply investigated in the monographs of Baĭnov and Simeonov [1], Lakshmikantham et al. [6], Samoĭlenko and Perestyuk [16], Borisenko et al. [3]. The recent work has provided a full discussion of this subject for impulsive delay differential equations (see, e.g., Yan and Shen [19], Liu and Ballinger [10], Liu et al. [12], Yu [20], Zhang and Sun [21], etc.). Most of these results on asymptotic behavior are valid locally in the neighborhood of the equilibrium state, but do not estimate the range of the stable region and domain of attraction (referring to the definition given by Lakshmikantham and Leela [7], Šiljak [17], Kolmanovskii and Nosov [5]). That is, we do not know how far initial conditions can be allowed to vary without disrupting the asymptotic properties of the equilibrium state. Furthermore, it may be difficult to know whether the equilibrium state exists in nonlinear impulsive delay systems. In this case, it should be important and interesting to estimate the region attracting solutions of the impulsive systems. Therefore, a general problem of the attractivity is to discuss the attracting set and attracting basin for the impulsive systems. Some significant progress has been made in the techniques and methods of determining the attracting set and attracting basin (domain of attraction) for the continuous systems described by ordinary differential equations [7, 17] and functional differential

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equations [5, 9, 13, 15, 18]. However, so far the corresponding problems for impulsive delay differential equations have not been considered.

In this paper, by developing an impulsive delay differential inequality and introducing the concept of nonlinear measure, we study the attractivity for a class of nonlinear impulsive delay differential equations. The criteria present a feasible and effective approach to estimate the attracting set, attracting basin, and asymptotically stable region of the impulsive systems by solving an algebraic equation. Examples and their simulations are given to demonstrate the effectiveness of our results.

2. Preliminaries

Let $\mathbb{N}$ be the set of all positive integers, let $\mathbb{R}^n$ be the real n-dimensional vector space with a norm $\|\cdot\|$, and let $\mathbb{R}^{m\times n}$ be the set of $m\times n$ real matrices. $\mathbb{R}^+ = [0,+\infty)$ and I denotes the $n\times n$ unit matrix.

Let $\tau > 0$ be the upper bound of time delays and let $t_1 < \cdots < t_k < t_{k+1} < \cdots$ $(k \in \mathbb{N})$ be the fixed points with $\lim_{k\to\infty} t_k = \infty$ (called impulsive moments).

For a function $x = (x_1,\dots,x_n)^T : \mathbb{R} \to \mathbb{R}^n$, we define

$$D^+x(t) = \limsup_{s\to 0^+} \frac{x(t+s)-x(t)}{s}, \qquad x(t^-) = \lim_{s\to 0^-} x(t+s), \qquad x(t^+) = \lim_{s\to 0^+} x(t+s),$$
$$[x(t)]_\tau = \left([x_1(t)]_\tau,\dots,[x_n(t)]_\tau\right)^T, \qquad [x_i(t)]_\tau = \sup_{-\tau\le s\le 0}\{x_i(t+s)\}. \tag{2.1}$$

$C[X,Y]$ denotes the space of continuous mappings from the topological space X to the topological space Y. Especially, let $C \triangleq C[[-\tau,0],\mathbb{R}^n]$.

$\mathrm{PC}[J,\mathbb{R}^n] \triangleq \{\psi : J \to \mathbb{R}^n \mid \psi(t)$ is continuous at $t \ne t_k$, $\psi(t_k^+)$ and $\psi(t_k^-)$ exist, $\psi(t_k) = \psi(t_k^+)$, for $t_k \in J\}$, where $J \subset \mathbb{R}$ is an interval.

$\mathrm{PC} \triangleq \{\phi : [-\tau,0] \to \mathbb{R}^n \mid \phi(t^+) = \phi(t)$ for $t \in [-\tau,0)$, $\phi(t^-)$ exists for $t \in (-\tau,0]$, $\phi(t^-) = \phi(t)$ for all but at most a finite number of points $t \in (-\tau,0]\}$. PC is a space of piecewise right-hand continuous functions with the norm $\|\phi\|_\tau = \sup_{-\tau\le s\le 0}\{\|\phi(s)\|\}$, for $\phi \in \mathrm{PC}$.

In this paper, we will consider the following nonlinear impulsive delay differential equations:

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + g(t,x_t), \quad t \ne t_k, \\ \Delta x(t_k) &:= x(t_k) - x(t_k^-) = I_k(x(t_k^-)), \quad k \in \mathbb{N}, \end{aligned} \tag{2.2}$$

where $f \in C[\mathbb{R}^n,\mathbb{R}^n]$, $g \in C[[t_{k-1},t_k)\times \mathrm{PC},\mathbb{R}^n]$ and the limit $\lim_{(t,\phi)\to(t_k^-,\varphi)} g(t,\phi) = g(t_k^-,\varphi)$ exists, $I_k \in C[\mathbb{R}^n,\mathbb{R}^n]$, $x_t \in \mathrm{PC}$ is defined by $x_t(s) = x(t+s)$, $s \in [-\tau,0]$, $\dot{x}(t)$ denotes the right-hand derivative of $x(t)$.

Definition 2.1. A function $x : [t_0-\tau,\infty) \to \mathbb{R}^n$ is called a solution of (2.2) through (t_0,ϕ) if $x \in \mathrm{PC}[[t_0,\infty),\mathbb{R}^n]$ as $t \ge t_0$, and satisfies (2.2) with the initial condition

$$x(t_0+s) = \phi(s), \quad \phi \in \mathrm{PC},\ s \in [-\tau,0]. \tag{2.3}$$

Throughout the paper, we always assume that for any $\phi \in \mathrm{PC}$, system (2.2) has at least one solution through (t_0,ϕ), denoted by $x(t,t_0,\phi)$ or $x_t(t_0,\phi)$, where $x_t(t_0,\phi)(s) = x(t+s,t_0,\phi)$, $s \in [-\tau,0]$. Clearly, $x_t(t_0,\phi) \in \mathrm{PC}$ for $t \geq t_0$. For more details on the existence of solutions of impulsive delay differential equations, one refers to Liu and Ballinger [11], Baĭnov and Stamova [2].

Definition 2.2. A set $S \subset \mathrm{PC}$ is called an attracting set of (2.2) and $D \subset \mathrm{PC}$ is called an attracting basin of S if for any initial value $\phi \in D$, the solution $x_t(t_0,\phi)$ converges to S as $t \to +\infty$. That is,

$$\operatorname{dist}(x_t(t_0,\phi),S) \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty, \tag{2.4}$$

where $\operatorname{dist}(\varphi,S) = \inf_{\psi\in S} \operatorname{dist}(\varphi,\psi)$, $\operatorname{dist}(\varphi,\psi) = \sup_{s\in[-\tau,0]} \|\varphi(s)-\psi(s)\|$, for $\varphi \in \mathrm{PC}$.

Especially, the set S is called a global attracting set of (2.2) if $D = \mathrm{PC}$. The set D is called a domain of attraction if $x = 0$ is a solution of (2.2) and the zero solution (i.e., $S = \{0\}$) attracts solutions $x(t,t_0,\phi)$ for all $\phi \in D$. Moreover, if the zero solution is stable, we call D an asymptotically stable region of (2.2).

In order to introduce the concept of the nonlinear measure, we recall the matrix norms $\|A\|$ and the matrix measure $\mu(A)$ introduced by the vector norm $\|\cdot\|$ as follows:

$$\|A\| = \sup_{x\neq 0, x\in\mathbb{R}^n} \frac{\|Ax\|}{\|x\|}, \qquad \mu(A) = \lim_{s\to 0^+} \frac{\|I+sA\|-1}{s}, \quad A \in \mathbb{R}^{n\times n}. \tag{2.5}$$

Now, based on (2.5), we define the nonlinear measure as follows.

Definition 2.3. For a function $F : \mathbb{R}^n \to \mathbb{R}^n$, call

$$\mu(F) = \sup_{x\neq 0, x\in\mathbb{R}^n} \limsup_{s\to 0^+} \frac{\big|\|x+sF(x)\|\big| - \|x\|}{s\|x\|} \tag{2.6}$$

the nonlinear measure of F.

Obviously, $-\infty < \mu(F) \leq +\infty$. Especially, if $F(x) = Ax$, $A \in \mathbb{R}^{n\times n}$, then $\mu(F) = \mu(A)$, where $\mu(A)$ is the matrix measure. Therefore, the concept of the nonlinear measure actually is an extension of the matrix measure (see also, Kolmanovskii and Myshkis [4], Qiao et al. [14]). According to the definition, we easily verify the following.

Lemma 2.4. *Let $\mu(F),\mu(G) < +\infty$. Then,*

(i) $\mu(\lambda F) = \lambda\mu(F)$, *where* $\lambda \geq 0$;
(ii) $\mu(F+G) \leq \mu(F)+\mu(G)$;
(iii) $-L(F) \leq \mu(F) \leq L(F)$, *where the constant* $L(F) = \sup_{x\neq 0, x\in\mathbb{R}^n}(\|F(x)\|/\|x\|)$.

The following result on the impulsive delay differential inequality is an extension of the continuous case of Lakshmikantham and Leela [8, Theorem 6.9.1], and will play an important role in the qualitative analysis of impulsive delay differential equations in Section 3.

Lemma 2.5. *Let $F : [t_0 - \tau, \infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be such that $F(t,x,\cdot)$ is nondecreasing for each (t,x) and let $I_k : \mathbb{R} \to \mathbb{R}$ be nondecreasing. Suppose $u,v \in \mathrm{PC}[[t_0 - \tau, \infty), \mathbb{R}]$ satisfy*

$$\begin{gathered} D^+u(t) \le F\big(t,u(t),[u(t)]_\tau\big), \quad t \ge t_0, \\ u(t_k) \le I_k\big(u(t_k^-)\big), \quad k \in \mathbb{N}, \\ D^+v(t) > F\big(t,v(t),[v(t)]_\tau\big), \quad t \ge t_0, \\ v(t_k) \ge I_k\big(v(t_k^-)\big), \quad k \in \mathbb{N}. \end{gathered} \tag{2.7}$$

Then $u(t) \le v(t)$, for $t_0 - \tau \le t \le t_0$ implies that $u(t) \le v(t)$, for $t \ge t_0$.

Proof. We will first prove that

$$u(t) \le v(t), \quad t \in [t_0, t_1). \tag{2.8}$$

If the assertion (2.8) is false, by using the continuity of $u(t)$, $v(t)$ for $t \in [t_0, t_1)$ and $u(t) \le v(t)$ for $t \in [t_0 - \tau, t_0]$, then there must exist a number $t^* \in [t_0, t_1)$ such that

$$u(t^*) = v(t^*), \quad u(t) \le v(t), \quad t \le t^*, \tag{2.9}$$

$$D^+u(t^*) \ge D^+v(t^*). \tag{2.10}$$

In view of (2.9) and the monotonic character of F, we have

$$\begin{aligned} D^+u(t^*) &\le F\big(t^*,u(t^*),[u(t^*)]_\tau\big) \\ &\le F\big(t^*,u(t^*),[v(t^*)]_\tau\big) \\ &= F\big(t^*,v(t^*),[v(t^*)]_\tau\big) < D^+v(t^*). \end{aligned} \tag{2.11}$$

This contradicts the inequality (2.10), and so (2.8) holds. Suppose that for $k = 1,2,\ldots,m$

$$u(t) \le v(t), \quad t \in [t_{k-1}, t_k). \tag{2.12}$$

Then

$$u(t) \le v(t), \quad t_m - \tau \le t < t_m, \quad u(t_m^-) \le v(t_m^-). \tag{2.13}$$

It is clear from the monotonicity of I_m that

$$u(t_m) \le I_m\big(u(t_m^-)\big) \le I_m\big(v(t_m^-)\big) \le v(t_m). \tag{2.14}$$

Thus, $u(t) \le v(t)$ for $t_m - \tau \le t \le t_m$. Employing the similar process of the proof of (2.8), we have $u(t) \le v(t)$, for $t \in [t_m, t_{m+1})$. By the induction, the conclusion holds and the proof is complete. □

3. Main results

In this paper, we always suppose the following.

(A_1) $\mu(f) < +\infty$ and $\|g(t,\varphi)\| \le p(\|\varphi\|_\tau)$ for any $\varphi \in \mathrm{PC}$, $t \ge t_0$, where $p : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and monotonically nondecreasing.

(A_2) $\|x + I_k(x)\| \le \alpha\|x\|$ for any $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$, where the constant $\alpha > 0$.

Theorem 3.1. *Let $t_k - t_{k-1} \le \varrho$ for $k \in \mathbb{N}$. Assume that (A_1) and (A_2) with $0 < \alpha < 1$ hold. If there exist two nonnegative constants $z_1 < z_2$ such that for any $z \in [z_1, z_2]$*

$$\Psi_1(z) := \frac{p(z)}{\alpha} + \left[\mu(f) + \frac{\ln\alpha}{\varrho}\right] z < 0, \tag{3.1}$$

then $S_1 = \{\phi \in \mathrm{PC} \mid \|\phi\|_\tau \le z_1\}$ is an attracting set of (2.2) and $D_1 = \{\phi \in \mathrm{PC} \mid \|\phi\|_\tau < \alpha z_2\}$ is an attracting basin of S_1.

Proof. For any $\phi \in D_1 \subset \mathrm{PC}$, let $x(t) = (t, t_0, \phi)$ be the solution of (2.2) through (t_0, ϕ). By (A_1), we calculate the right upper derivative along the solution (2.2):

$$\begin{aligned}
D^+\|x(t)\| &= \limsup_{s\to 0^+} \frac{\|x(t+s)\| - \|x(t)\|}{s} \\
&\le \limsup_{s\to 0^+} \frac{\|x(t) + sf(x(t))\| - \|x(t)\|}{s} \\
&\quad + \limsup_{s\to 0^+} \frac{\|x(t+s)\| - \|x(t) + sf(x(t))\|}{s} \\
&\le \mu(f)\|x(t)\| + \limsup_{s\to 0^+} \left\| \frac{x(t+s) - x(t)}{s} - f(x(t)) \right\| \\
&\le \mu(f)\|x(t)\| + \|g(t, x_t)\| \\
&\le \mu(f)\|x(t)\| + p([\|x(t)\|]_\tau), \quad t \ge t_0.
\end{aligned} \tag{3.2}$$

On the other hand, by (A_2),

$$\|x(t_k^+)\| = \|x(t_k^-) + I_k(x(t_k^-))\| \le \alpha\|x(t_k^-)\|, \quad k \in \mathbb{N}. \tag{3.3}$$

From (3.1), the continuity of $p(z)$ and $\mu(f) + \ln\alpha/\varrho < 0$, there must exist an $\epsilon > 0$ such that

$$h(z) := z + \frac{p(z) + \epsilon}{\alpha}\left[\mu(f) + \frac{\ln\alpha}{\varrho}\right]^{-1} > 0 \quad \forall z \in [z_1, z_2]. \tag{3.4}$$

Let $v(t)$ be a solution of

$$\begin{aligned}
\dot v(t) &= \mu(f)v(t) + p([v(t)]_\tau) + \epsilon, \quad t \ne t_k,\ t \ge t_0, \\
v(t_k) &= \alpha v(t_k^-), \quad k \in \mathbb{N}, \\
v(t) &= \|\phi(t - t_0)\|, \quad t_0 - \tau \le t \le t_0.
\end{aligned} \tag{3.5}$$

Taking $F(t,x,y) = \mu(f)x + p(y)$ and $I_k(x) = \alpha x$, from the monotonicity of $p(\cdot)$ and $\alpha > 0$, then $F(t,x,y)$ is nondecreasing in y for each (t,x) and $I_k(x)$ is nondecreasing in x. By (3.2), (3.3), (3.5), and Lemma 2.5 with $u(t) = \|x(t)\|$, we have

$$0 \le ||x(t)|| \le v(t), \quad t \ge t_0. \tag{3.6}$$

By the formula for the variation of parameters, we have

$$v(t) = W(t,t_0)v(t_0) + \int_{t_0}^{t} W(t,s)[p([v(s)]_\tau) + \epsilon]ds, \quad t \ge t_0, \tag{3.7}$$

where $W(t,s)$, $t,s \ge t_0$ is the Cauchy matrix of linear impulsive system (refer to [6]):

$$\begin{aligned} \dot{\omega}(t) &= \mu(f)\omega(t), \quad t \ne t_k, \\ \omega(t_k) &= \alpha\omega(t_k^-), \quad k \in \mathbb{N}. \end{aligned} \tag{3.8}$$

It is easily seen that

$$W(t,s) = e^{\mu(f)(t-s)} \prod_{s<t_k\le t} \alpha, \quad t \ge s \ge t_0. \tag{3.9}$$

Since $0 < \alpha < 1$ and $\varrho \ge t_k - t_{k-1}$, we have the following estimate:

$$\begin{aligned} W(t,s) &\le e^{\mu(f)(t-s)}\alpha^{[(t-s)/\varrho - 1]} \\ &= \frac{1}{\alpha} e^{[\mu(f) + \ln\alpha/\varrho](t-s)}, \quad t \ge s \ge t_0. \end{aligned} \tag{3.10}$$

Combining with (3.7), we get

$$v(t) \le \frac{||\phi(0)||}{\alpha} e^{[\mu(f)+\ln\alpha/\varrho](t-t_0)} + \int_{t_0}^{t} \frac{p([v(s)]_\tau)+\epsilon}{\alpha} e^{[\mu(f)\ln\alpha/\varrho](t-s)}ds. \tag{3.11}$$

Since $\phi \in D_1$, we have

$$v(t) = ||\phi(t-t_0)|| < \alpha z_2 \le z_2, \quad t_0 - \tau \le t \le t_0. \tag{3.12}$$

In the following, we will prove that

$$v(t) < z_2, \quad t \ge t_0. \tag{3.13}$$

If this is not true, then by the estimate (3.12) and the piecewise continuity of $v(t)$, there exists a $t^* > t_0$ satisfying

$$v(t^*) \ge z_2, \quad v(t) < z_2, \quad \text{for } t < t^*. \tag{3.14}$$

By (3.11), (3.12), and (3.14), we have

$$\begin{aligned}
v(t^*) &\leq \frac{\|\phi(0)\|}{\alpha} e^{[\mu(f)+\ln\alpha/\varrho](t^*-t_0)} + \int_{t_0}^{t^*} \frac{p([v(s)]_\tau)+\epsilon}{\alpha} e^{[\mu(f)+\ln\alpha/\varrho](t^*-s)} ds \\
&\leq e^{[\mu(f)+\ln\alpha/\varrho](t^*-t_0)} z_2 + \frac{p(z_2)+\epsilon}{\alpha} \int_{t_0}^{t^*} e^{[\mu(f)+\ln\alpha/\varrho](t^*-s)} ds \\
&= e^{[\mu(f)+\ln\alpha/\varrho](t^*-t_0)} z_2 - \frac{p(z_2)+\epsilon}{\alpha} \left[\mu(f)+\frac{\ln\alpha}{\varrho}\right]^{-1} \left[1-e^{[\mu(f)+\ln\alpha/\varrho](t^*-t_0)}\right] \\
&= e^{[\mu(f)+\ln\alpha/\varrho](t^*-t_0)} \left\{ z_2 + \frac{p(z_2)+\epsilon}{\alpha} \left[\mu(f)+\frac{\ln\alpha}{\varrho}\right]^{-1} \right\} - \frac{p(z_2)+\epsilon}{\alpha} \left[\mu(f)+\frac{\ln\alpha}{\varrho}\right]^{-1}.
\end{aligned} \tag{3.15}$$

In term of (3.4), $\mu(f)+\ln\alpha/\varrho < 0$ and $t^* > t_0$, then

$$v(t^*) < z_2 + \frac{p(z_2)+\epsilon}{\alpha} \left[\mu(f)+\frac{\ln\alpha}{\varrho}\right]^{-1} - \frac{p(z_2)+\epsilon}{\alpha} \left[\mu(f)+\frac{\ln\alpha}{\varrho}\right]^{-1} = z_2. \tag{3.16}$$

This contradicts the first inequality in (3.14), and so the estimate (3.13) holds. Thus,

$$\eta := \limsup_{t\to+\infty} v(t) \leq z_2. \tag{3.17}$$

Since $\mu(f)+\ln\alpha/\varrho < 0$, for any given $\delta > 0$, there must be $T > 0$ such that

$$\int_T^{+\infty} e^{[\mu(f)+\ln\alpha/\varrho]s} ds < \delta. \tag{3.18}$$

Furthermore, for the above positive number δ, there is a $T' > t_0$ such that

$$v(t) \leq \eta+\delta, \quad \text{for } t > T'. \tag{3.19}$$

Employing (3.11), (3.13), (3.18), (3.19), and the monotonicity of $p(\cdot)$, when $t > T+T'+\tau$, we have

$$\begin{aligned}
v(t) &\leq \frac{\|\phi(0)\|}{\alpha} e^{[\mu(f)+\ln\alpha/\varrho](t-t_0)} + \int_{t_0}^{t} \frac{p([v(s)]_\tau)+\epsilon}{\alpha} e^{[\mu(f)+\ln\alpha/\varrho](t-s)} ds \\
&\leq e^{[\mu(f)+\ln\alpha/\varrho](t-t_0)} z_2 + \int_{t_0}^{t-T} \frac{p([v(s)]_\tau)+\epsilon}{\alpha} e^{[\mu(f)+\ln\alpha/\varrho](t-s)} ds \\
&\quad + \int_{t-T}^{t} \frac{p([v(s)]_\tau)+\epsilon}{\alpha} e^{[\mu(f)+\ln\alpha/\varrho](t-s)} ds \\
&\leq e^{[\mu(f)+\ln\alpha/\varrho](t-t_0)} z_2 + \frac{p(z_2)+\epsilon}{\alpha} \int_T^{+\infty} e^{[\mu(f)+\ln\alpha/\varrho]s} ds \\
&\quad + \frac{p(\eta+\delta)+\epsilon}{\alpha} \int_{t-T}^{t} e^{[\mu(f)+\ln\alpha/\varrho](t-s)} ds \\
&\leq e^{[\mu(f)+\ln\alpha/\varrho](t-t_0)} z_2 + \frac{p(z_2)+\epsilon}{\alpha}\delta - \frac{p(\eta+\delta)+\epsilon}{\alpha} \left[\mu(f)+\frac{\ln\alpha}{\varrho}\right]^{-1} \left[1-e^{[\mu(f)+\ln\alpha/\varrho]T}\right].
\end{aligned} \tag{3.20}$$

Then, by $\mu(f)+\ln\alpha/\varrho<0$,

$$\eta=\limsup_{t\to+\infty}v(t)\le\frac{p(z_2)+\epsilon}{\alpha}\delta-\frac{p(\eta+\delta)+\epsilon}{\alpha}\left[\mu(f)+\frac{\ln\alpha}{\varrho}\right]^{-1}. \tag{3.21}$$

Letting $\delta\to 0^+$, we have

$$h(\eta)=\eta+\frac{p(\eta)+\epsilon}{\alpha}\left[\mu(f)+\frac{\ln\alpha}{\varrho}\right]^{-1}\le 0. \tag{3.22}$$

Combining with $\eta\le z_2$ and $h(z)>0$ for any $z\in[z_1,z_2]$, then $\eta<z_1$. From (3.6),

$$\limsup_{t\to+\infty}||x(t)||\le\limsup_{t\to+\infty}v(t)\le z_1. \tag{3.23}$$

The conclusion holds and the proof is complete. □

Remark 3.2. The above conclusion remains valid even when the inequality (3.1) holds for $z\in(z_1,z_2)$. In fact, for an enough small $\epsilon>0$, the inequality (3.1) holds when $z\in[z_1+\epsilon,z_2-\epsilon]$. According to Theorem 3.1, $S_1'=\{\phi\in\mathrm{PC}\mid\|\phi\|_\tau\le z_1+\epsilon\}$ is an attracting set of (2.2) and $D_1'=\{\phi\in\mathrm{PC}\mid\|\phi\|_\tau<\alpha(z_2-\epsilon)\}$ is an attracting basin of S_1'. Letting $\epsilon\to 0^+$, we can obtain the conclusion.

According to Theorem 3.1 and Remark 3.2, we have the following corollaries.

Corollary 3.3. *Let $t_k-t_{k-1}\le\varrho$ for $k\in\mathbb{N}$. Suppose that (A_1) and (A_2) with $0<\alpha<1$ and $p(z)=\beta z+\gamma$ hold, where $\beta,\gamma\ge 0$. If $\beta/\alpha+\ln\alpha/\varrho+\mu(f)<0$, then $S_1=\{\phi\in\mathrm{PC}\mid\|\phi\|_\tau\le z_1=-(\gamma/\alpha)[\beta/\alpha+\ln\alpha/\varrho+\mu(f)]^{-1}\}$ is a globally attracting set of (2.2).*

Proof. It is obvious that

$$\Psi_1(z)=\left[\frac{\beta}{\alpha}+\frac{\ln\alpha}{\varrho}+\mu(f)\right]z+\frac{\gamma}{\alpha}<0,\quad z\in(z_1,+\infty). \tag{3.24}$$

According to the above results, $S_1=\{\phi\in\mathrm{PC}\mid\|\phi\|_\tau\le z_1\}$ is a globally attracting set of (2.2). □

Corollary 3.4. *Let $x=0$ be a solution of (2.2). If all the conditions in Theorem 3.1 are satisfied except that the inequality (3.1) holds for $z\in(0,z_2)$, then the zero solution is asymptotically stable and $D_1=\{\phi\in\mathrm{PC}\mid\|\phi\|_\tau<\alpha z_2\}$ is an asymptotically stable region of (2.2).*

Proof. According to Theorem 3.1 and Remark 3.2, the zero solution attracts solutions $x(t,t_0,\phi)$ for all $\phi\in D_1$ and D_1 is a domain of attraction. Furthermore, for any given $z_\epsilon\in(0,z_2]$ and $\phi\in D_\epsilon=\{\phi\in\mathrm{PC}\mid\|\phi\|_\tau<\alpha z_\epsilon\}$, we can refer to the proof of (3.13) and obtain that

$$||x(t,t_0,\phi)||<z_\epsilon,\quad t\ge t_0. \tag{3.25}$$

This implies that the zero solution is stable. Thus, the zero solution is asymptotically stable and $D_1=\{\phi\in\mathrm{PC}\mid\|\phi\|_\tau<\alpha z_2\}$ is an asymptotically stable region of (2.2). The proof is complete. □

Similarly, we can obtain the following results for the case with $\alpha \geq 1$.

Theorem 3.5. *Let $0 < \theta \leq t_k - t_{k-1}$ for $k \in \mathbb{N}$. Assume that (A_1) and (A_2) with $\alpha \geq 1$ hold. If there exist two nonnegative constants $z_1 < z_2$ such that for any $z \in (z_1, z_2)$*

$$\Psi_2(z) := \alpha p(z) + \left[\mu(f) + \frac{\ln \alpha}{\theta}\right] z < 0, \tag{3.26}$$

then $S_2 = \{\phi \in \mathrm{PC} \mid \|\phi\|_\tau \leq z_1\}$ is an attracting set of (2.2) and $D_2 = \{\phi \in \mathrm{PC} \mid \|\phi\|_\tau < z_2/\alpha\}$ is an attracting basin of S_2.

Proof. Since $0 < \theta \leq t_k - t_{k-1}$ and $\alpha \geq 1$, we replace the estimate (3.10) in the proof of Theorem 3.1 with

$$W(t,s) \leq \alpha e^{[\mu(f) + \ln \alpha/\theta](t-s)}, \quad t \geq s \geq t_0. \tag{3.27}$$

The rest of the proof is similar to one of the proof in Theorem 3.1 and we omit it here. □

According to Theorem 3.5, we have the following.

Corollary 3.6. *Let $t_k - t_{k-1} \geq \theta > 0$ for $k \in \mathbb{N}$. Suppose that (A_1) and (A_2) with $\alpha \geq 1$ and $p(z) = \beta z + \gamma$ hold, where $\beta, \gamma \geq 0$. If $\alpha\beta + \ln \alpha/\theta + \mu(f) < 0$, then $S_1 = \{\phi \in \mathrm{PC} \mid \|\phi\|_\tau \leq -\alpha\gamma[\alpha\beta + \ln \alpha/\theta + \mu(f)]^{-1}\}$ is a globally attracting set of (2.2).*

Corollary 3.7. *Let $x = 0$ be a solution of (2.2). If all the conditions in Theorem 3.5 are satisfied except that the inequality (3.26) holds for $z \in (0, z_2)$, then the zero solution is asymptotically stable and $D_2 = \{\phi \in \mathrm{PC} \mid \|\phi\|_\tau < z_2/\alpha\}$ is an asymptotically stable region.*

4. Illustrative examples

Example 4.1. Consider a scalar nonlinear impulsive delay system:

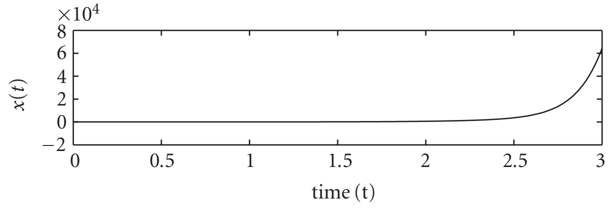
$$\begin{aligned} \dot{x}(t) &= ax(t) + bx^2(t-\tau) + c, \quad t \neq t_k,\ t \geq t_0 = 0, \\ \Delta x(t_k) &:= x(t_k) - x(t_k^-) = I_k(x(t_k^-)), \quad k \in \mathbb{N}, \end{aligned} \tag{4.1}$$

where $b \neq 0$, $0 < \theta \leq t_k - t_{k-1} \leq \varrho < +\infty$, $|x + I_k(x)| \leq \alpha|x|$. From (A_1) and (A_2), $\mu(f) = a$, $p(z) = |b|z^2 + |c|$. We discuss the attractiveness of (4.1) for the following cases.

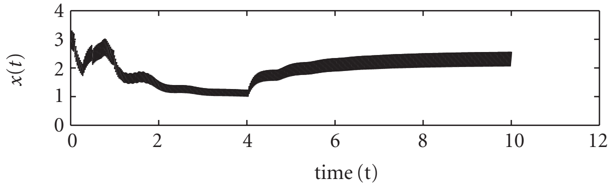
Case 1 $(0 < \alpha < 1)$. Clearly, $\Psi_1(z) = (|b|/\alpha)z^2 + (a + \ln \alpha/\varrho)z + |c|/\alpha$. If $a + \ln \alpha/\varrho < -2\sqrt{|bc|}/\alpha$, then the algebraic equation $\Psi_1(z) = 0$ has two different nonnegative roots:

$$z_{1,2} = \frac{-(a + \ln \alpha/\varrho) \mp \sqrt{(a + \ln \alpha/\varrho)^2 - 4|bc|/\alpha^2}}{2|b|/\alpha}, \tag{4.2}$$

and so $\Psi_1(z) < 0$ for $z \in (z_1, z_2)$. According to Theorem 3.1 and Remark 3.2, $S_1 = \{\phi \in \mathrm{PC} \mid \|\phi\|_\tau \leq z_1\}$ is an attracting set of (4.1) and $D_1 = \{\phi \in \mathrm{PC} \mid \|\phi\|_\tau < \alpha z_2\}$ is an attracting basin of S_1. Especially, when $c = 0$ and $a + \ln \alpha/\varrho < 0$, it follows from Corollary 3.4 that the zero solution of (4.1) is asymptotically stable and an asymptotically stable region $D_1 = \{\phi \in \mathrm{PC} \mid \|\phi\|_\tau < -(\alpha^2/|b|)(a + \ln \alpha/\varrho)\}$.



(a)



(b)

Figure 4.1. The trajectory of (4.1): (a) no impulses; (b) impulsive effects.

Case 2 ($\alpha \geq 1$). Clearly, $\Psi_2(z) = \alpha|b|z^2 + (a+\ln\alpha/\theta)z + \alpha|c|$. If $a+\ln\alpha/\theta < -2\alpha\sqrt{|bc|}$, then the algebraic equation $\Psi_2(z) = 0$ has two different nonnegative roots:

$$z_{1,2} = \frac{-(a+\ln\alpha/\theta) \mp \sqrt{(a+\ln\alpha/\theta)^2 - 4\alpha^2|bc|}}{2\alpha|b|}, \tag{4.3}$$

and so $\Psi_2(z) < 0$ for $z \in (z_1, z_2)$. According to Theorem 3.5, $S_2 = \{\phi \in \mathrm{PC} \mid \|\phi\|_\tau \leq z_1\}$ is an attracting set of (4.1) and $D_2 = \{\phi \in \mathrm{PC} \mid \|\phi\|_\tau < z_2/\alpha\}$ is an attracting basin of S_2. Especially, when $c = 0$ and $a + \ln\alpha/\theta < 0$, it follows from Corollary 3.7 that the zero solution of (4.1) is asymptotically stable and an asymptotically stable region $D_2 = \{\phi \in \mathrm{PC} \mid \|\phi\|_\tau < -(1/\alpha^2|b|)(a+\ln\alpha/\theta)\}$.

Take $a = b = 1$, $c = 10$, $\tau = 1$, $I_k(x) = -0.2x$, and $t_k = t_{k-1} + 0.025$. From Case 1 with $\alpha = 0.8$ and $\varrho = 0.025$, we have $z_1 = 2.9449$, $z_2 = 3.3956$, and so $S_1 = \{\phi \in \mathrm{PC} \mid \|\phi\|_\tau \leq 2.9449\}$ is an attracting set of the impulsive system (4.1), $D_1 = \{\phi \in \mathrm{PC} \mid \|\phi\|_\tau < \alpha z_2 = 2.7165\}$ is an attracting basin of S_1. However, any solution of the corresponding continuous delay system (i.e., $\Delta x(t_k) = 0$ in (4.1)) is unbounded. Figure 4.1 shows the different asymptotical behaviors between the impulsive system and the corresponding continuous system under the initial condition $x(t) = 2.7$, $t \in [-1, 0]$.

Example 4.2. Consider a two dimensional impulsive delay system

$$\begin{aligned}
\dot{x}_1(t) &= x_1 \sin x_1(t) + x_2(t-1)\cos\left(x_1(t-1)\right) + 0.5\sin e^t, \quad t \geq t_0 = 0,\\
\dot{x}_2(t) &= x_2 \cos x_2(t) - x_1(t-1)\sin\left(x_2(t-1)\right) + 0.5\cos e^t, \quad t \neq t_k,\\
\Delta x_1 &:= x_1\left(t_k^+\right) - x_1\left(t_k^-\right) = -0.4x_1\left(t_k^-\right), \quad t_k = t_{k-1} + 0.15,\\
\Delta x_2 &:= x_2\left(t_k^+\right) - x_2\left(t_k^-\right) = -0.4x_2\left(t_k^-\right), \quad k \in \mathbb{N}.
\end{aligned} \tag{4.4}$$

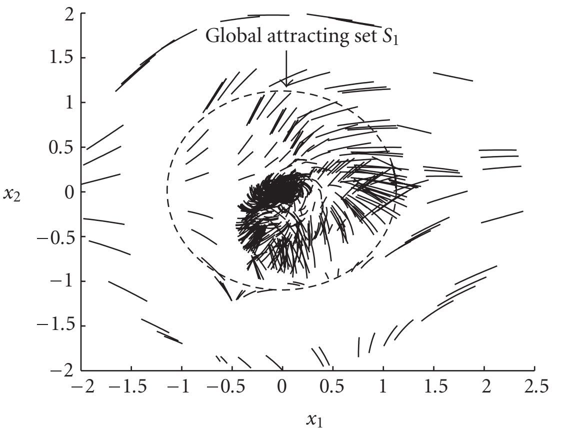


Figure 4.2. The global attracting set of the impulsive delay system (4.4).

Letting $f(x) = (x_1 \sin x_1, x_2 \cos x_2)^T, g(t,x) = (x_2 \cos x_1 + 0.5 \sin e^t, -x_1 \sin x_2 + 0.5 \cos e^t)^T$, then $\| f(x)\| \le \|x\|$, $\|g(t,x)\| \le \|x\| + 0.5$, where $\| \cdot \|$ is the 2-norm of the vector. From Lemma 2.4, we can calculate the condition parameters in Theorem 3.1:

$$\begin{gathered} \mu(f) \le 1, \qquad p(z) = z + 0.5, \qquad \alpha = 0.6, \qquad \varrho = 0.15, \\ \Psi_1(z) \le -0.7388z + 0.8333 < 0, \quad z \in (1.1279, +\infty). \end{gathered} \tag{4.5}$$

It follows from Theorem 3.1 (or Corollary 3.3) that $S_1 = \{\phi \mid \|\phi\|_\tau \le 1.1279\}$ is a global attracting set of (4.4). Figure 4.2 shows the attractivity of the impulsive delay system (4.4) under the different initial conditions.

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