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Research Article

Hölder Continuity up to the Boundary of Minimizers for Some Integral Functionals with Degenerate Integrands

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We study qualitative properties of minimizers for a class of integral functionals, defined in a weighted space. In particular we obtain Hölder regularity up to the boundary for the minimizers of an integral functional of high order by using an interior local regularity result and a modified Moser method with special test function.

1. Introduction

In this paper, we will study regularity properties of minimizers for integral functionals of the form

$$I(u) = \int_{\Omega} \{A(x, \nabla_2 u) + A_0(x,u)\}\, dx, \tag{1.1}$$

defined in a suitable weighted Banach space; Ω is an open and bounded set of $\mathbb{R}^n$ and $\nabla_2 u = \{D^{\alpha}u : |\alpha| = 1,2\}$.

We note that in this paper we obtain our regularity result directly working with the functional $I(u)$ instead of working with its Euler equation. In fact, we will not suppose any differentiability of $A(x,\xi)$, principal part of integrand of the functional $I(u)$, but only that it is a Caratheodory function, convex with respect to ξ, satisfying the following growth condition: for almost every $x \in \Omega$ and for every $\xi = \{\xi_{\alpha} : |\alpha| = 1,2\}$,

$$\begin{aligned} c_1\Big\{\sum_{|\alpha|=1} \nu(x)\,|\xi_{\alpha}|^{q} + \sum_{|\alpha|=2} \mu(x)\,|\xi_{\alpha}|^{p}\Big\} - f(x) &\le A(x,\xi) \\ &\le c_2\Big\{\sum_{|\alpha|=1} \nu(x)\,|\xi_{\alpha}|^{q} + \sum_{|\alpha|=2} \mu(x)\,|\xi_{\alpha}|^{p}\Big\} + f(x), \end{aligned} \tag{1.2}$$

where c_1, c_2 are positive constants, $f(x)$ is a nonnegative function, belonging to a suitable Lebesgue space, and $\nu(x)$, $\mu(x)$ are positive measurable functions that we will specify later.

This kind of condition, introduced by Skrypnik in [1], is stronger than the one that is usually considered (see, e.g., [2, 3]), but the usual growth condition in general cannot give even the boundedness of the minima of $I(u)$ (see [4]).

In [5], boundedness and Hölder continuity for minimizers of the same functional $I(u)$ in the interior of Ω were already established. Now the aim of this paper is to establish Hölder continuity up to the boundary of any minimizer $u(x)$.

Under some hypotheses on weighted functions in order to guarantee embedding between Banach spaces and under some hypotheses of regularity of the boundary $\partial\Omega$, using the convexity properties of the functions $A(x,\xi)$ and $A_0(x,\eta)$ and the above growth conditions, we obtain an integral estimate of the gradient of the minimizers. Then the iterative Moser method (see [6]) opportunely modified permits us to estimate the oscillation of $u(x)$ near the boundary of Ω. So with the interior regularity result of [5], we obtain our goal.

In the nondegenerate case, the problem of regularity of minimizers of integral functionals was studied, for example, by [4, 7–9]. Among recent researches, we recall [10–12].

Note that in the case of $2p < q < n$, some results on Hölder continuity of solutions of equations and variational inequalities with degenerate nonlinear high-order operators have been obtained in [1, 13–16].

2. Hypotheses and statement of main results

In this section, we give hypotheses concerning weighted functions in order to define our weighted Banach spaces, and to guarantee some embedding results, we give hypotheses on the integrand functions and state the main result.

Let Ω be a bounded open set of $\mathbb{R}^n$. Let $p \geq 2$, q be two real numbers such that $2p < q < n$.

Hypothesis 2.1. Let $\nu(x) : \Omega \to \mathbb{R}^+$ be a measurable function such that

$$\nu \in \mathrm{L}^1_{\mathrm{loc}}(\Omega), \qquad \left(\frac{1}{\nu}\right)^{1/(q-1)} \in \mathrm{L}^1_{\mathrm{loc}}(\Omega). \tag{2.1}$$

$W^{1,q}(\nu,\Omega)$ is the space of all functions $u \in L^q(\Omega)$ such that their derivatives, in the sense of distribution, $D^\alpha u$, $|\alpha| = 1$, are functions for which the following properties hold: $\nu^{1/q} D^\alpha u \in L^q(\Omega)$ if $|\alpha| = 1$; $W^{1,q}(\nu,\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{1,q,\nu} = \left(\int_\Omega |u|^q\, dx + \sum_{|\alpha|=1} \int_\Omega \nu(x)\, |D^\alpha u|^q\, dx\right)^{1/q}. \tag{2.2}$$

$\overset{\circ}{W}{}^{1,q}(\Omega, \nu)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,q}(\nu,\Omega)$.

Hypothesis 2.2. Let $\mu(x) : \Omega \to \mathbb{R}^+$ be a measurable function such that

$$\mu \in \mathrm{L}^1_{\mathrm{loc}}(\Omega), \qquad \left(\frac{1}{\mu}\right)^{1/(p-1)} \in \mathrm{L}^1_{\mathrm{loc}}(\Omega). \tag{2.3}$$

$W^{1,q}_{2,p}(\Omega,\nu,\mu)$ is the space of all functions $u \in W^{1,q}(\Omega,\nu)$ such that their derivatives, in the sense of distribution, $D^\alpha u, |\alpha| = 2$, are functions for which the following properties hold: $\mu^{1/p} D^\alpha u \in L^p(\Omega)$ if $|\alpha| = 2$; $W^{1,q}_{2,p}(\nu,\mu,\Omega)$ is a Banach space with respect to the norm

$$\|u\| = \|u\|_{1,q,\nu} + \left(\sum_{|\alpha|=2} \int_\Omega \mu(x) \left| D^\alpha u \right|^p dx \right)^{1/p}. \tag{2.4}$$

$\overset{\circ}{W}{}^{1,q}_{2,p}(\Omega,\nu,\mu)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,q}_{2,p}(\Omega,\nu,\mu)$.

Hypothesis 2.3. We assume the function $1/\nu \in L^t(\Omega)$ with $t > n/q$.

We put $\tilde{q} = nqt/(n(1+t) - qt)$. We can easily prove that a constant $c_0 > 0$ exists such that if $u \in \overset{\circ}{W}{}^{1,q}(\Omega,\nu)$, the following inequality holds:

$$\int_\Omega |u|^{\tilde{q}}\, dx \le c_0 \left(\int_\Omega \left[\frac{1}{\nu(x)} \right]^t dx \right)^{\tilde{q}/qt} \left(\sum_{|\alpha|=1} \int_\Omega \nu(x) \left| D^\alpha u \right|^q dx \right)^{\tilde{q}/q}. \tag{2.5}$$

We set $\tilde{\nu}(x) = \mu(x)^{q/(q-2p)} [1/\nu(x)]^{2p/(q-2p)}$.

Hypothesis 2.4. There exists $t^* > nt/(qt - n)$ such that $\nu, \tilde{\nu} \in \mathrm{L}^{t^*}(\Omega)$.

For every $y \in \mathbb{R}^n$ and $\rho > 0$, we denote

$$B(y, \rho) = \{ x \in \mathbb{R}^n : |x - y| < \rho \}. \tag{2.6}$$

Hypothesis 2.5. There exists a constant $c' > 0$ such that for every $y \in \Omega$ and $\rho > 0$, with $\overline{B(y, \rho)} \subset \Omega$, we have

$$\left(\rho^{-n} \int_{B(y,\rho)} \left[\frac{1}{\nu(x)} \right]^t dx \right)^{1/t} \left(\rho^{-n} \int_{B(y,\rho)} [\nu(x)]^{t^*} dx \right)^{1/t^*} \le c'. \tag{2.7}$$

We need these previous hypotheses in order to ensure the regularity of minimizers of our functional in the interior of Ω. To have the regularity to the boundary, we need the following further hypotheses concerning the boundary of Ω and the extension of weights on the boundary.

Hypothesis 2.6. There exist c^*, ρ^* such that for every $y \in \partial\Omega$ and $\rho \in]0, \rho^*[$, we have

$$\operatorname{meas}(B(y,\rho) \setminus \Omega) \ge c^* \operatorname{meas}(B(y,\rho)). \tag{2.8}$$

Consequently, Ω belongs to the class S (see, e.g., [17]).

Let us put

$$\hat{\Omega} = \{ x \in \mathbb{R}^n : \operatorname{dist}(x, \Omega) < \rho^* \}. \tag{2.9}$$

Hypothesis 2.7. There exist a positive measurable function $\hat{\nu}(x) : \hat{\Omega} \to \mathbb{R}$ and a real positive number c'' such that $\hat{\nu}(x) = \nu(x)$ in Ω and

(i) $1/\hat{\nu} \in L^t(\hat{\Omega}), \hat{\nu} \in L^{t^*}(\hat{\Omega})$

(ii) for all $y \in \partial\Omega$ and $\rho \in]0, \rho^*[$,

$$\left(\rho^{-n}\int_{B(y,\rho)}\left[\frac{1}{\hat{\nu}(x)}\right]^t dx\right)^{1/t}\left(\rho^{-n}\int_{B(y,\rho)}\left[\hat{\nu}(x)\right]^{t^*} dx\right)^{1/t^*} \leq c''. \tag{2.10}$$

We denote by $\mathbb{R}^{n,2}$ the space of all sets $\xi = \{\xi_\alpha \in \mathbb{R} : |\alpha| = 1,2\}$ of real numbers.

Hypothesis 2.8. We suppose that $A(x,\xi) : \Omega \times \mathbb{R}^{n,2} \to \mathbb{R}$ and $A_0(x,\eta) : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions. Moreover, functions $A(x,\cdot)$, $A_0(x,\cdot)$ are convex in $\mathbb{R}^{n,2}$ and $\mathbb{R}$, respectively, for almost all $x \in \Omega$.

Hypothesis 2.9. There exist $c_1, c_2 > 0$ and a nonnegative function $f \in L^{t^*}(\Omega)$ such that for almost $x \in \Omega$ and for every $\xi \in \mathbb{R}^{n,2}$, the inequality (1.2) holds.

Hypothesis 2.10. There exist $c_3 > 0, c_4 \in [0, c_1/c_0[$, and $f_0 \in L^{t^*}(\Omega)$ such that, almost everywhere in Ω and for all $\eta \in \mathbb{R}$, the following inequality holds:

$$-c_4|\eta|^q - f_0(x) \leq A_0(x,\eta) \leq c_3|\eta|^q + f_0(x). \tag{2.11}$$

Let $I : \overset{\circ}{W}{}^{1,q}_{2,p}(\nu,\mu,\Omega) \to \mathbb{R}$ be the functional of the form

$$I(u) = \int_\Omega \{A(x,\nabla_2 u) + A_0(x,u)\}\, dx. \tag{2.12}$$

From the theory of monotone and coercive operators, it is well known that under the previous hypotheses there exists $u(x)$ minimizer of I in $\overset{\circ}{W}{}^{1,q}_{2,p}(\Omega,\nu,\mu)$. Moreover, $u(x)$ is essentially bounded in Ω and Hölder continuous in every compact subset of Ω (see [5]).

Now, we can formulate our regularity result more precisely.

THEOREM 2.11. *Let $u(x)$ be a minimizer of $I(u)$ in $\overset{\circ}{W}{}^{1,q}_{2,p}(\nu,\mu,\Omega)$, then there exists $\overline{u}(x) : \overline{\Omega} \to \mathbb{R}$ such that $\overline{u}(x) = u(x)$ a.e. in Ω, and for every $x, y \in \overline{\Omega}$, we have*

$$|\overline{u}(x) - \overline{u}(y)| \leq C|x-y|^\gamma, \tag{2.13}$$

where positive constants C and γ depend only on known values and on $\|u\|_{L^{\tilde{q}}(\Omega)}$.

3. Proof of Theorem 2.11

In this section, we give a proof of Theorem 2.11.

We set

$$m_1 = \frac{q^2}{q-2p}, \qquad \sigma = \frac{1}{2m_1}\left(q - \frac{n}{t} - \frac{n}{t^*}\right). \tag{3.1}$$

Let $u(x):\Omega\to\mathbb{R}$ be a minimizer of I in $\overset{\circ}{W}{}^{1,q}_{2,p}(\nu,\mu,\Omega)$. Let us put

$$\overline{u}(x)=\begin{cases}u(x), & \text{if } x\in\Omega,\\ 0, & \text{if } x\in\mathbb{R}^n\setminus\Omega.\end{cases} \tag{3.2}$$

Let us fix $y\in\Omega$ and $\rho\in\,]0,(1/2)\rho^*[$, and let us put

$$\omega_1=\operatorname*{ess\,inf}_{B(y,2\rho)}\overline{u},\qquad \omega_2=\operatorname*{ess\,sup}_{B(y,2\rho)}\overline{u},\qquad \omega=\omega_2-\omega_1. \tag{3.3}$$

It is simple to prove that $\omega_1\le 0$ and $\omega_2\ge 0$.

By Ladyzhenskaya's lemma (see [8, Lemma 4.8]), it is sufficient to prove that

$$\operatorname{osc}\{\overline{u},B(y,\rho)\}\le c_5\,\omega+\rho^\sigma,\quad \text{with } c_5\in\,]0,1[. \tag{3.4}$$

Here and in the sequel, with c_i, $i=5,6,\dots$, we intend positive constants depending only on n, p, q, $\tilde{q}$, c_0, c_1, c_2, c_3, c_4, c', c'', t, t^*, ρ^*, c^*, $\operatorname{diam}\Omega$, on the norms of $1/\hat{\nu}(x)$ in $L^t(\hat{\Omega})$ and $f(x)$ in $\mathrm{L}^{t^*}(\Omega)$, and on the norm of $u(x)$ in $\mathrm{L}^{\tilde{q}}(\Omega)$.

We will assume that

$$\omega\ge\rho^\sigma \text{ (otherwise it is clear that (3.4) is true)} \tag{3.5}$$

$$\omega_2\ge\frac{\omega}{2}. \tag{3.6}$$

It is known that there exists a set $E\subset\Omega\cap B(y,2\rho)$ such that $\operatorname{meas}E=0$, and for all $x\in(\Omega\cap B(y,2\rho)\setminus E)$, we have

$$\omega_1\le u(x)\le\omega_2. \tag{3.7}$$

We introduce now the following function:

$$F(x)=\begin{cases}\dfrac{1}{\omega_2-u(x)+\rho^\sigma}, & \text{if } x\in(\Omega\cap B(y,2\rho))\setminus E,\\[2mm] \dfrac{1}{2\omega}, & \text{if } x\in(\Omega\setminus B(y,2\rho))\cup E,\end{cases} \tag{3.8}$$

and the cutoff function $\varphi \in C^\infty(\Omega)$, $0 \le \varphi \le 1$ in Ω, defined by

$$\varphi(x) = \begin{cases} 1, & \text{if } x \in \Omega \cap B\left(y, \dfrac{3}{2}\rho\right), \\ 0, & \text{if } x \in \Omega \setminus B(y, 2\rho). \end{cases} \tag{3.9}$$

Moreover, we can choose $\varphi(x)$ satisfying $|D^\alpha \varphi| \le c_6 \rho^{-|\alpha|}$, $|\alpha| = 1, 2$.

We observe that if (3.6) does not hold, it is possible to repeat all considerations substituting $u(x) - \omega_1 + \rho^\sigma$ to $\omega_2 - u(x) + \rho^\sigma$ in the definition of function $F(x)$.

Let us fix $s > m_1$ and define

$$v(x) = \left[[F(x)]^{q-1} - \frac{1}{(\omega_2 + \rho^\sigma)^{q-1}} \right] \varphi^s(x). \tag{3.10}$$

It is useful to note that due to (3.6) and (3.7)

$$\left| [F(x)]^{q-1} - \frac{1}{(\omega_2 - \rho^\sigma)^{q-1}} \right| \le 2^{q-1} [F(x)]^{q-1}. \tag{3.11}$$

Thanks to Hypotheses 2.1, 2.2, 2.4 and (3.5), (3.7), (3.11), we have $v \in \overset{\circ}{W}{}^{1,q}_{2,p}(\Omega, \nu, \mu)$ and

$$|D^\alpha v - (q-1)\varphi^s(x) F^q(x) D^\alpha u| \le c_7 s [\varphi(x)]^{s-1} [F(x)]^{q-1} \rho^{-1}, \quad \text{for } |\alpha| = 1, \tag{3.12}$$

$$\begin{aligned} &|D^\alpha v - (q-1)\varphi^s(x) F^q(x) D^\alpha u| \\ &\quad \le c_8 s^2 [\varphi(x)]^{s-2} [F(x)]^{q-1} \left\{ \sum_{|\beta|=1} \frac{|D^\beta u|^2}{(\omega_2 - u + \rho^\sigma)^2} + \rho^{-2} \right\} \quad \text{for } |\alpha| = 2. \end{aligned} \tag{3.13}$$

Next, if we put

$$\lambda = \frac{\rho^{\sigma q}}{q}, \qquad z(x) = (q-1)\varphi^s(x) F^q(x), \tag{3.14}$$

it follows that $0 \le \lambda z(x) \le 1$ in Ω.

$u(x)$ being a minimizer for our functional, we have

$$I(u) \le I(u - \lambda v), \tag{3.15}$$

or

$$\int_\Omega A(x, \nabla_2 u)\, dx \le \int_\Omega A(x, \nabla_2 u - \lambda \nabla_2 v)\, dx + \int_\Omega A_0(x, u - \lambda v)\, dx - \int_\Omega A_0(x, u)\, dx. \tag{3.16}$$

Since $A(x,\xi)$ is convex, the first term on the right-hand side can be evaluated in such a way that

$$A(x,\nabla_2 u-\lambda\nabla_2 v)\ \le (1-\lambda z)A(x,\nabla_2 u)+\lambda z A\Big(x,-\frac{H}{z}\Big), \tag{3.17}$$

where $H(x)=\nabla_2 v(x)-z(x)\nabla_2 u(x)$.

From (3.12) and (3.13), using Young's inequality, we obtain

$$\begin{aligned} A(x,\nabla_2 u-\lambda\nabla_2 v) \le{}& (1-\lambda z)A(x,\nabla_2 u)+\lambda c_9 s^q\varepsilon \sum_{|\alpha|=1} \nu(x)\,|D^\alpha u|^q F^q(x)\varphi^s(x)\\ &+\lambda c_{10}s^q\varepsilon^{-m_1}\big[\rho^{-q}\nu+\rho^{-\sigma m_1}\big(f(x)+\widetilde{\nu}(x)\big)\big]\varphi^{s-m_1}. \end{aligned} \tag{3.18}$$

Let us evaluate now the term

$$A_0(x,u-\lambda v)\le A_0(x,u)+c_{11}\frac{\lambda(q-1)\varphi^s}{\rho^{\sigma q}}\big(1+f_0(x)\big). \tag{3.19}$$

So using Hypothesis 2.9,

$$\begin{aligned} &\int_\Omega\Big\{\sum_{|\alpha|=1}\nu(x)\,|D^\alpha u|^q\Big\}(q-1)F^q(x)\varphi^s(x)dx\\ &\quad\le c_{12}s^q\varepsilon^{-m_1}\int_\Omega\Big[\rho^{-q}\nu(x)+\rho^{-\sigma m_1}\big(f(x)+\widetilde{\nu}(x)+f_0(x)+1\big)\Big][\varphi(x)]^{s-m_1}dx\\ &\qquad+c_9 s^q\varepsilon\int_\Omega\Big\{\sum_{|\alpha|=1}\nu(x)\,|D^\alpha u|^q\Big\}F^q(x)\varphi^s(x)dx, \end{aligned} \tag{3.20}$$

from which, choosing ε in a suitable way, we obtain

$$\begin{aligned} &\int_{\Omega\cap B(y,2\rho)}\Big\{\sum_{|\alpha|=1}\nu(x)\,|D^\alpha u|^q\Big\}F^q(x)\varphi^s(x)dx\\ &\quad\le c_{13}s^{(m_1+1)q}\int_\Omega\big[\rho^{-q}\nu(x)+\rho^{-\sigma m_1}\big(\widetilde{\nu}(x)+f(x)+f_0(x)+1\big)\big][\varphi(x)]^{s-m_1}dx. \end{aligned} \tag{3.21}$$

Then, definition of $\varphi(x)$ gives

$$\int_{\Omega\cap B(y,(2/3)\rho)}\Big\{\sum_{|\alpha|=1}\nu(x)\,|D^\alpha u|^q\Big\}F^q(x)\,dx\le c_{14}\int_{\Omega\cap B(y,2\rho)}\phi(x)\,dx, \tag{3.22}$$

where $\phi(x)=\rho^{-q}\nu(x)+\rho^{-\sigma m_1}(\widetilde{\nu}(x)+f(x)+f_0(x)+1)$.

Let us introduce now the function $\varphi_1 \in C^\infty(\Omega)$, defined by

$$\varphi_1(x) = \begin{cases} 1, & \text{if } x \in \Omega \cap B(y,\rho), \\ 0, & \text{if } x \in \Omega \setminus B\left(y, \frac{3}{2}\rho\right), \end{cases} \tag{3.23}$$

with $|D^\alpha \varphi_1| \le c_{15}\rho^{-|\alpha|}, |\alpha| = 1,2$.

Let us put $\chi = 2pq/(q-2p)$, and let us fix $r > 0$, $s > m_1$. We define

$$\begin{aligned} G(x) &= \max\left\{[F(x)]^{(q-1)/\chi} - \frac{1}{(\omega_2+\rho^\sigma)^{(q-1)/\chi}}, 0\right\}, \\ w(x) &= G^\chi(x)\left[\log(2\omega e F(x))\right]^r \varphi_1^s, \\ \tilde{w}(x) &= (q-1)[G(x)]^{\chi-1}[F(x)]^{(q-1)/\chi+1}\left[\log(2\omega e F(x))\right]^r \\ &\quad + r[G(x)]^\chi F(x)\left[\log(2\omega e F(x))\right]^{r-1}. \end{aligned} \tag{3.24}$$

Thanks to Hypotheses 2.1, 2.2, 2.4 and (3.5), (3.7), we can prove that $w(x)$ belongs to $\overset{\circ}{W}{}^{1,q}_{2,p}(\Omega,\nu,\mu)$ and

$$\begin{aligned} \left|D^\alpha w - \tilde{w}\varphi_1^s(x) D^\alpha u\right| &\le c_{16} s G^\chi(x)\left[\log(2\omega e F(x))\right]^r \rho^{-1}[\varphi_1(x)]^{s-1}, \quad \text{if } |\alpha| = 1, \\ \left|D^\alpha w - \tilde{w}\varphi_1^s(x) D^\alpha u\right| &\le c_{17} s^2 (r+1)^2 [G(x)]^{\chi-2}[F(x)]^{2((q-1)/\chi)}\left[\log(2\omega e F(x))\right]^r [\varphi_1(x)]^{s-2} \\ &\quad \times \left\{\left(\sum_{|\beta|=1} \frac{D^\beta u}{\omega_2 - u + \rho^\sigma}\right)^2 + \rho^{-2}\right\}, \quad \text{if } |\alpha| = 2. \end{aligned} \tag{3.25}$$

Let us put now

$$\lambda_1 = \frac{\rho^{\sigma q}}{(q+r)\left[\log(2\omega e/\rho^\sigma)\right]^r}, \qquad H_1(x) = \nabla_2 w(x) - \tilde{w}(x)\varphi_1^s(x)\nabla_2 u(x), \tag{3.26}$$

and we introduce $E_1 = \{x \in \Omega : \tilde{w}(x)\varphi_1^s(x) \neq 0\}$.

It is easy to prove that

$$0 \le \lambda_1 \tilde{w}(x)\varphi_1^s(x) \le 1, \quad \forall x \in \Omega. \tag{3.27}$$

Taking into account that

$$I(u) \le I(u - \lambda_1 w), \tag{3.28}$$

we obtain

$$\int_\Omega A(x,\nabla_2 u)\,dx \le \int_\Omega A(x,\nabla_2 u - \lambda_1 \nabla_2 w)\,dx + \int_\Omega A_0(x,u-\lambda_1 w)\,dx - \int_\Omega A_0(x,u)\,dx. \tag{3.29}$$

We can write for the first term in the right-hand side

$$A(x,\nabla_2 u-\lambda_1\nabla_2 w)\le(1-\lambda_1\widetilde{w}\varphi_1^s(x))A(x,\nabla_2 u)+\lambda_1\widetilde{w}\varphi_1^s(x)A\left(x,-\frac{H_1}{\widetilde{w}\varphi_1^s(x)}\right),\tag{3.30}$$

and using Hypothesis 2.9 and Young's inequality, we derive

$$\begin{aligned}
&A(x,\nabla_2 u-\lambda_1\nabla_2 w)\\
&\quad\le(1-\lambda_1\widetilde{w}\varphi_1^s(x))A(x,\nabla_2 u)+c_{18}\lambda_1\varepsilon(r+s)^{q+1}\\
&\qquad\times\left\{\sum_{|\beta|=1}\nu(x)\,|D^\beta u|^q\right\}[G(x)]^{\chi-1}[F(x)]^{(q-1)/\chi+1}\\
&\qquad\times[\log(2\omega eF(x))]^r\varphi_1^s(x)+c_{19}\lambda_1\varepsilon^{-m_1}(r+s)^{m_1+1}\\
&\qquad\times[\rho^{-q}\nu(x)+\rho^{-2\sigma m_1}(\widetilde{\nu}(x)+f(x))][\log(2\omega eF(x))]^r[\varphi_1(x)]^{s-m_1}.
\end{aligned}\tag{3.31}$$

We evaluate now

$$A_0(x,u-\lambda_1 w)\le A_0(x,u)+\lambda_1\varphi_1^s(r+s)[\log(2\omega eF(x))]^r\rho^{-\sigma q}(f_0(x)+1).\tag{3.32}$$

Then, from (3.31) and (3.32), we have

$$\begin{aligned}
&\int_\Omega\left\{\sum_{|\alpha|=1}\nu(x)\,|D^\alpha u|^q\right\}[G(x)]^{\chi-1}[F(x)]^{(q-1)/\chi+1}[\log(2\omega eF(x))]^r\varphi_1^s(x)dx\\
&\quad\le c_{18}\int_\Omega\varepsilon(r+s)^{q+1}\left\{\sum_{|\beta|=1}\nu(x)\,|D^\beta u|^q\right\}[G(x)]^{\chi-1}[F(x)]^{(q-1)/\chi+1}[\log(2\omega eF(x))]^r\varphi_1^s(x)dx\\
&\qquad+\int_\Omega\varepsilon^{-m_1}c_{19}(r+s)^{m_1+1}[\rho^{-q}\nu(x)+\rho^{-2\sigma m_1}(\widetilde{\nu}(x)+f(x)+f_0(x)+1)]\\
&\qquad\times[\log(2\omega eF(x))]^r[\varphi_1(x)]^{s-m_1}dx.
\end{aligned}\tag{3.33}$$

And so choosing ε in a suitable way, we obtain

$$\begin{aligned}
&\int_\Omega\left\{\sum_{|\alpha|=1}\nu(x)\,|D^\alpha u|^q\right\}[G(x)]^{\chi-1}[F(x)]^{(q-1)/\chi+1}[\log(2\omega eF(x))]^r\varphi_1^s(x)dx\\
&\quad\le c_{20}\int_\Omega(r+s)^{m_2+1}\phi_1(x)[\log(2\omega eF(x))]^r[\varphi_1(x)]^{s-m_1}dx,
\end{aligned}\tag{3.34}$$

where $m_2=m_1(q+2)$ and $\phi_1(x)=[\rho^{-q}\nu(x)+\rho^{-2\sigma m_1}(\widetilde{\nu}(x)+f(x)+f_0(x)+1)]$.

We define

$$E_0(\rho)=\left\{x\in B\left(y,\frac{3}{2}\rho\right)\cap\Omega:F(x)\ge\frac{2^{\chi/(q-1)}}{\omega_2+\rho^\sigma}\right\}.\tag{3.35}$$

We can suppose that

$$\operatorname{meas}(E_0(\rho)) \neq 0. \tag{3.36}$$

In fact, if (3.36) is not true, without any difficulties we obtain by the use of (3.6) the same inequality (3.4). We have

$$G(x) \geq \frac{1}{2}[F(x)]^{(q-1)/\chi}, \quad \forall x \in E_0(\rho). \tag{3.37}$$

Using this fact, inequality (3.34) gives

$$\begin{aligned} &\int_{E_0(\rho)} \left\{ \sum_{|\alpha|=1} \nu(x) |D^\alpha u|^q \right\} F^q(x) [\log(2\omega e F(x))]^r \varphi_1^s(x) dx \\ &\leq c_{21}(r+s)^{m_2+1} \int_\Omega \phi_1(x) [\log(2\omega e F(x))]^r [\varphi_1(x)]^{s-m_1} dx. \end{aligned} \tag{3.38}$$

Therefore, for every $r > 0$ and $s > m_1$, using Hölder's inequality,

$$\begin{aligned} &\int_{E_0(\rho)} \left\{ \sum_{|\alpha|=1} \nu(x) |D^\alpha u|^q \right\} F^q(x) [\log(2\omega e F(x))]^r \varphi_1^s(x) dx \\ &\leq c_{21}(r+s)^{c_{22}} \left(\int_{\Omega \cap B(y,2\rho)} \phi_1^{t^*}(x) dx \right)^{1/t^*} \\ &\quad \times \left(\int_{\Omega \cap B(y,2\rho)} [\log(2\omega e F(x))]^{rt^*/(t^*-1)} [\varphi_1(x)]^{(s-m_1)t^*/(t^*-1)} dx \right)^{(t^*-1)/t^*}. \end{aligned} \tag{3.39}$$

Let us put $\vartheta = (\widetilde{q}/q)((t^*-1)/t^*)$, $m^* = (m_1 t^*/(t^*-1))$, $J = \log[(2^{\chi/(q-1)+1} e\omega)/(\omega_2 + \rho^\sigma)]$, and for any r, $s > 0$, we define

$$H(r,s) = \int_{\Omega \cap B(y,2\rho)} [\log(2\omega e F(x))]^r \varphi_1^s(x)\, dx + J^r \rho^n. \tag{3.40}$$

We introduce a new cutoff function $\varphi_2(x) : B(y,2\rho) \to \mathbb{R}$, $\varphi_2 \in C^\infty(B(y,2\rho))$, such that

$$\varphi_2(x) = \begin{cases} 1, & \text{if } x \in B(y,\rho), \\ 0, & \text{if } x \in B(y,2\rho) \setminus B\left(y, \frac{3}{2}\rho\right). \end{cases} \tag{3.41}$$

Let us observe that $\varphi_2(x) = \varphi_1(x)$ in $B(y,2\rho) \cap \Omega$.

We define the following function:

$$\overline{F}(x) = \begin{cases} \dfrac{1}{\omega_2 - \overline{u} + \rho^\sigma}, & \text{if } x \in \Omega \cap B(y,2\rho) \setminus E, \\ \dfrac{1}{2\omega}, & \text{if } x \in E, \end{cases} \tag{3.42}$$

and we set

$$\overline{v}(x)=\Big[\max\{[\log(2\omega e\overline{F}(x))]^r,J^r\}\Big]^{1/\tilde{q}}\varphi_2^{s/\tilde{q}}(x). \tag{3.43}$$

We have

$$\int_{\Omega\cap B(y,2\rho)}[\log(2\omega e\overline{F}(x))]^r\varphi_1^s(x)\,dx\le\int_{\Omega\cap B(y,2\rho)}\overline{v}^{\tilde{q}}\,dx, \tag{3.44}$$

and by Hypothesis 2.7, due to $\overline{v}\in\overset{\circ}{W}{}^{1,q}(B(y,2\rho),\hat{\nu})$,

$$\int_{B(y,2\rho)}|\overline{v}|^{\tilde{q}}dx\le\tilde{c}\left(\int_{B(y,2\rho)}\left[\frac{1}{\hat{\nu}(x)}\right]^t dx\right)^{\tilde{q}/qt}\left(\sum_{|\alpha|=1}\int_{B(y,2\rho)}\hat{\nu}(x)\,|D^\alpha\overline{v}|^q\,dx\right)^{\tilde{q}/q}. \tag{3.45}$$

From the definition of the function $\overline{v}(x)$ and (3.39), we have

$$\begin{aligned}&\left(\int_{B(y,2\rho)}\sum_{|\alpha|=1}\hat{\nu}(x)\,|D^\alpha\overline{v}|^q dx\right)^{\tilde{q}/q}\\&\quad\le c_{23}(r+s)^{c_{24}}\left(\int_{B(y,2\rho)}\phi_1^{t^*}(x)dx\right)^{\tilde{q}/qt^*}\left(\int_{B(y,2\rho)}[\log(2\omega eF(x))]^{r/\vartheta}[\varphi_2(x)]^{(s/\vartheta)-m^*}dx\right)^{\vartheta}\\&\qquad+c_{25}J^r s^{\tilde{q}}\left(\rho^{-q}\int_{B(y,2\rho)}\hat{\nu}(x)dx\right)^{\tilde{q}/q}.\end{aligned} \tag{3.46}$$

From (3.45) and (3.46), using Hypothesis 2.7 and (3.40), we deduce that

$$\int_{B(y,2\rho)}\overline{v}^{\tilde{q}}\,dx\le c_{26}(r+s)^{c_{27}}\rho^{n(1-\vartheta)}\left[H\left(\frac{r}{\vartheta},\frac{s}{\vartheta}-m^*\right)\right]^{\vartheta}, \tag{3.47}$$

and taking into account inequality (3.44), finally we obtain

$$H(r,s)\le c_{28}(r+s)^{c_{29}}\rho^{n(1-\vartheta)}\left[H\left(\frac{r}{\vartheta},\frac{s}{\vartheta}-m^*\right)\right]^{\vartheta}, \tag{3.48}$$

for all $r>0$ and $s>m_1(\tilde{q}/q)$.

Now, we can organize the iterative Moser method (see [6]). We introduce for $i=0,1,2,\dots,$

$$r_i=\frac{tq}{t+1}\vartheta^i,\qquad s_i=\frac{m^*\vartheta}{\vartheta-1}(\vartheta^{i+1}-1). \tag{3.49}$$

Then, (3.48) written with $r=r_i$ and $s=s_i$ gives us

$$H(r_i,s_i)\le c_{30}\rho^{n(1-\vartheta)}\vartheta^{ic_{29}}[H(r_{i-1},s_{i-1})]^{\vartheta}. \tag{3.50}$$

Using this recurrent relation, we obtain, for any integer i,

$$H(r_i,s_i) \leq c_{31}\left[\rho^{-n}H(r_0,s_0)\right]^{\vartheta^i}. \tag{3.51}$$

Our aim now is to get a suitable estimate of the following integral:

$$\int_{\Omega\cap B(y,(3/2)\rho)} \left[\log\left(\frac{2\omega e}{\omega_2 - u + \rho^\sigma}\right)\right]^{tq/(t+1)} dx. \tag{3.52}$$

We put

$$v_0(x) = \left[\log\left(2\omega e\overline{F}(x)\right)\right]^{tq/(t+1)} \quad \forall x \in B\left(y,\frac{3}{2}\rho\right). \tag{3.53}$$

It is not difficult to prove that $v_0 \in W^{1,1}(B(y,(3/2)\rho))$. Moreover, by Hypothesis 2.6,

$$\operatorname{meas}\left(B\left(y,\frac{3}{2}\rho\right)\setminus\Omega\right) \geq c_{32}\rho^n \tag{3.54}$$

and since $\omega_2 \geq \omega/2$ and $\omega \geq \rho^\sigma$,

$$v_0(x) \leq c_{33} \quad \forall x \in B\left(y,\frac{3}{2}\rho\right)\setminus\Omega. \tag{3.55}$$

So, by [18, Lemma 4], we have

$$\int_{B(y,(3/2)\rho)} v_0\, dx \leq c_{34}\rho^n + c_{34}\rho\int_{B(y,(3/2)\rho)} \left|D^\alpha v_0\right|^q dx. \tag{3.56}$$

By applying Young's inequality, we obtain

$$\int_{\Omega\cap B(y,(3/2)\rho)} v_0\, dx \leq c_{35}\rho^n + c_{35}\rho^{tq/(t+1)}\int_{B(y,(3/2)\rho)} \left[\overline{F}(x)\right]^{tq/(t+1)}\left[\sum_{|\alpha|=1}\left|D^\alpha\overline{u}\right|\right]^{tq/(t+1)} dx. \tag{3.57}$$

But using Hölder inequality, (3.22), Hypothesis 2.7, and definition of σ, we find

$$\begin{aligned}
&\int_{B(y,(3/2)\rho)} \left[\overline{F}(x)\right]^{tq/(t+1)}\left[\sum_{|\alpha|=1}\left|D^\alpha\overline{u}\right|\right]^{tq/(t+1)} dx \\
&\quad\leq c_{36}\rho^{nt/t^{*'}(t+1)}\left[\int_{\Omega\cap B(y,2\rho)}\phi^{t^*}(x)dx\right]^{t/t^*(t+1)}\left[\int_{\Omega\cap B(y,2\rho)}\left[\frac{1}{\nu(x)}\right]^t dx\right]^{t/t(t+1)} \\
&\quad\leq c_{37}\rho^{(n/t)+(n/t^*)-q}.
\end{aligned} \tag{3.58}$$

Hence, from (3.57) and (3.58), we establish

$$\int_{B(y,(3/2)\rho)} \left[\log\left(2\omega e\overline{F}(x)\right)\right]^{tq/(t+1)} dx \leq c_{38}\rho^n. \tag{3.59}$$

Finally, from the last inequality and (3.51), we obtain

$$\int_{B(y,\rho)} [\log(2\omega e F(x))]^{r_i} \le c_{39}^{r_i}. \tag{3.60}$$

Therefore,

$$\operatorname*{ess\,sup}_{B(y,\rho)\cap\Omega} \log(2\omega e F(x)) \le c_{40}. \tag{3.61}$$

From this assertion, we deduce the inequality (3.4). Now, using [8, Lemma 4.8] and the interior regularity result of [5], we get the conclusion of Theorem 2.11.

4. Examples

Now, we describe a situation where hypotheses stated in Section 2 are satisfied.

First of all, we consider an example of the integrand:

$$A(x,\xi) + A_0(x,\eta), \tag{4.1}$$

satisfying conditions in Hypotheses 2.9 and 2.10.

For every n-dimensional multi-index α and every $x \in \Omega$, $\xi \in \mathbb{R}^{n,2}$, and $\eta \in \mathbb{R}$,

$$A(x,\xi) + A_0(x,\eta) = \nu(x) \sum_{|\alpha|=1} |\xi_\alpha|^{q-1}\xi_\alpha + \mu(x) \sum_{|\alpha|=2} |\xi_\alpha|^{p-1}\xi_\alpha + |\eta|^{q-1}\eta. \tag{4.2}$$

Let us choose

$$\mu(x) = |x - x_0|^{\alpha_p}, \qquad \nu(x) = |x - x_0|^{\alpha_q}, \tag{4.3}$$

where x_0 is an arbitrary point in $B(1)$. By this choice, the conditions of Hypotheses 2.1 and 2.2 are satisfied if we assume

$$-n < \alpha_q < n(q-1), \qquad -n < \alpha_p < n(p-1). \tag{4.4}$$

Finally, choosing $0 < \alpha_q < q$ and $q\alpha_p - 2p\alpha_q > 0$ and taking the number t, t^* such that

$$\frac{n}{q} < t < \frac{n}{\alpha_q}, \qquad t^* > \frac{nt}{qt-n}, \tag{4.5}$$

then Hypotheses 2.3–2.7 also hold.

References

[1] I. V. Skrypnik, "Higher order quasilinear elliptic equations with continuous generalized solutions," *Differential Equations*, vol. 14, no. 6, pp. 786–795, 1978.

[2] X. G. Liu, "Partial regularity of multiple variational integrals of any order," *Proceedings of the Royal Society of Edinburgh: Section A*, vol. 114, no. 3-4, pp. 279–290, 1990.

[3] N. G. Meyers, "Quasi-convexity and lower semi-continuity of multiple variational integrals of any order," *Transactions of the American Mathematical Society*, vol. 119, no. 1, pp. 125–149, 1965.

[4] E. De Giorgi, "Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari," *Memorie della Accademia delle Scienze di Torino, Classe di Scienze Fisiche, Matematiche, Naturali, 3*, vol. 3, pp. 25–43, 1957.

[5] V. Cataldo, S. D'Asero, and F. Nicolosi, "Regularity of minimizers of some integral functionals with degenerate integrands," to appear in *Nonlinear Analysis: Theory, Methods & Applications*.

[6] J. Moser, "A new proof of de Giorgi's theorem concerning the regularity problem for elliptic differential equations," *Communications on Pure and Applied Mathematics*, vol. 13, pp. 457–468, 1960.

[7] M. Giaquinta and E. Giusti, "On the regularity of the minima of variational integrals," *Acta Mathematica*, vol. 148, pp. 31–46, 1982.

[8] O. A. Ladyzhenskaya and N. N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, NY, USA, 1968.

[9] C. B. Morrey Jr., *Multiple Integrals in the Calculus of Variations*, vol. 130 of *Die Grundlehren der mathematischen Wissenschaften*, Springer, New York, NY, USA, 1966.

[10] D. Giachetti and M. M. Porzio, "Local regularity results for minima of functionals of the calculus of variation," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 39, no. 4, pp. 463–482, 2000.

[11] P. Marcellini, "Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions," *Archive for Rational Mechanics and Analysis*, vol. 105, no. 3, pp. 267–284, 1989.

[12] G. Moscariello and L. Nania, "Hölder continuity of minimizers of functionals with nonstandard growth conditions," *Ricerche di Matematica*, vol. 40, no. 2, pp. 259–273, 1991.

[13] S. Bonafede and S. D'Asero, "Hölder continuity of solutions for a class of nonlinear elliptic variational inequalities of high order," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 44, no. 5, pp. 657–667, 2001.

[14] A. Kovalevsky and F. Nicolosi, "On Hölder continuity of solutions of equations and variational inequalities with degenerate nonlinear elliptic high order operators," in *Problemi Attuali dell'Analisi e della Fisica Matematica*, pp. 205–220, Aracne, Rome, Italy, 2000.

[15] A. Kovalevsky and F. Nicolosi, "On regularity up to the boundary of solutions to degenerate nonlinear elliptic high-order equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 40, no. 1, pp. 365–379, 2000.

[16] F. Nicolosi and I. V. Skrypnik, "Nirenberg-Gagliardo interpolation inequality and regularity of solutions of nonlinear higher order equations," *Topological Methods in Nonlinear Analysis*, vol. 7, no. 2, pp. 327–347, 1996.

[17] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, vol. 88 of *Pure and Applied Mathematics*, Academic Press, New York, NY, USA, 1980.

[18] I. V. Skrypnik, *Nonlinear Elliptic Equations of Higher Order*, Naukova Dumka, Kiev, Ukraine, 1973.

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