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Research Article

Monotonicity of Harnack Inequality for Positive Invariant Harmonic Functions

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A monotonicity property and a refined estimate of Harnack inequality are derived for positive solutions of the Weinstein equation.

1. Introduction

Let $B^n = \{x \in \mathbb{R}^n : |x| < 1\}$, $n \geq 2$, be the unit ball in $\mathbb{R}^n$, $S^{n-1} = \partial B^n$. Consider the Weinstein equation in B^n of the form

$$\Delta_\lambda u = (1 - |x|^2)\left\{\frac{1 - |x|^2}{4}\sum_j \frac{\partial^2 u}{\partial x_j^2} + \lambda \sum_j x_j \frac{\partial u}{\partial x_j} + \lambda\left(\frac{n}{2} - 1 - \lambda\right)u\right\} = 0, \tag{1.1}$$

where $u = u(x)$, $x \in B^n$, $\lambda \in \mathbb{R}$. In this paper, we prove a monotonicity property and a refined estimate of Harnack inequality for positive solutions of (1.1).

The differential operator Δ_λ in (1.1) is a natural extension of the Laplacian operator ($\lambda = 0$). If T is a Möbius transformation from B^n onto B^n and $T'(x)$ denotes the Jacobian matrix, then for every solution u of (1.1) in B^n, the function

$$|\det T'(x)|^{(n-2-2\lambda)/2n} u(T(x)) \tag{1.2}$$

is also a solution of (1.1), as proved by Akın and Leutwiler [1]. More precisely,

$$\Delta_\lambda\left\{|\det T'(x)|^{(n-2-2\lambda)/2n} u(T(x))\right\} = |\det T'(x)|^{(n-2-2\lambda)/2n} \Delta_\lambda u(T(x)) \tag{1.3}$$

for twice differentiable functions $u(x)$ in B^n [2]. Therefore, Δ_λ is also called invariant Laplacian and the solutions of (1.1) are called invariant harmonic functions. The Dirichlet problem for (1.1) and its half-space counterpart are challenging and interesting, as summarized in Liu and Peng [2], where the authors also pointed out that invariant harmonic functions do not possess good boundary regularity in general. The classical Harnack inequality gives scale invariant bounds for nonnegative (nonpositive) harmonic functions in the plane. Harnack-type inequalities have been an important tool in the general theory of harmonic functions and partial differential equations, on which Kassmann [3] provided a through introduction and survey of the development and applications. In this paper, we give an estimate of Harnack inequality bounds for any two points in B^n based on a monotonicity property of positive invariant harmonic functions.

In this section, we state the main results. The proofs are provided in the subsequent sections. For positive solutions of (1.1), Theorem 1.1 describes a monotonicity property, Theorem 1.2 gives bounds for a Harnack-type inequality for two points on the same ray, and Theorem 1.3 extends the estimates for the Harnack-type inequality to any two points in B^n. Two interesting special cases—one on harmonic functions and another on the Laplace-Beltrami operator associated with the Poincaré metric—are stated as corollaries.

To study the properties of solutions of $\Delta_\lambda u = 0$, one often needs to distinguish the cases of $\lambda \geq -1/2$ and $\lambda < -1/2$. Throughout this paper, we denote

$$\delta = \delta(\lambda) = \frac{1}{2}\left(1+2\lambda - |1+2\lambda|\right) = \begin{cases} 0, & \lambda \geq -\dfrac{1}{2} \\ 1+2\lambda, & \lambda < -\dfrac{1}{2} \end{cases} \quad \text{for } \lambda \in \mathbb{R}. \tag{1.4}$$

Theorem 1.1. *Let $u(x)$, $x \in B^n$, be a positive solution of $\Delta_\lambda u = 0$, $\lambda \in \mathbb{R}$. Then for $\zeta \in S^{n-1}$, the function*

$$\frac{(1-r)^{n-1-\delta}}{(1+r)^{1+2\lambda-\delta}} u(r\zeta) \tag{1.5}$$

is decreasing for $0 \leq r < 1$ and the function

$$\frac{(1+r)^{n-1-\delta}}{(1-r)^{1+2\lambda-\delta}} u(r\zeta) \tag{1.6}$$

is increasing for $0 \leq r < 1$.

Theorem 1.2. *Let $u(x)$, $x \in B^n$, be a positive solution of $\Delta_\lambda u = 0$, $\lambda \in \mathbb{R}$. Then for $\zeta \in S^{n-1}$ and $0 \leq r' \leq r < 1$,*

$$\left(\frac{1-r}{1-r'}\right)^{2\lambda+1-\delta}\left(\frac{1+r'}{1+r}\right)^{n-1-\delta} u(r'\zeta) \leq u(r\zeta) \leq \left(\frac{1+r}{1+r'}\right)^{2\lambda+1-\delta}\left(\frac{1-r'}{1-r}\right)^{n-1-\delta} u(r'\zeta). \tag{1.7}$$

Notice that case $\lambda = 0$ ($\delta = 0$) gives the classical Harnack Inequality.

Theorem 1.3. *Let $u(x)$, $x \in B^n$, be a positive solution of $\Delta_\lambda u = 0$, $\lambda \in \mathbb{R}$. Let $\xi_1, \xi_2 \in S^{n-1}$ and $0 \le r_1 \le r_2 < 1$. Then*

$$f_\lambda(-r_1, -r_2)\exp\{-g_\lambda(r_1,\xi_1,\xi_2)\} \le \frac{u(r_2\xi_2)}{u(r_1\xi_1)} \le f_\lambda(r_1,r_2)\exp\{g_\lambda(r_1,\xi_1,\xi_2)\} \tag{1.8}$$

with

$$\begin{aligned} f_\lambda(r_1,r_2) &= \left(\frac{1+r_2}{1+r_1}\right)^{2\lambda+1-\delta}\left(\frac{1-r_1}{1-r_2}\right)^{n-1-\delta}, \\ g_\lambda(r_1,\xi_1,\xi_2) &= \frac{\pi}{2}\,|\xi_2-\xi_1|\,\frac{|n+2\lambda-2\delta|r_1}{(1-r_1)^2}. \end{aligned} \tag{1.9}$$

Remarks 1.4. The sharpness of the bounds in the above theorems are discussed in the proofs of Lemma 2.4, Theorems 1.2 and 1.3. The equality case can be achieved in Theorem 1.2. Only trivial equalities are attained in Theorem 1.3.

Case $\lambda = 0$ corresponds to harmonic functions, wherein Theorem 1.3 can be stated as the following.

Corollary 1.5. *Let $u(x)$ be a positive harmonic function in B^n. $\xi_1, \xi_2 \in S^{n-1}$, $0 \le r_1 \le r_2 < 1$. Then*

$$f(-r_1,-r_2)\exp\{-g(r_1)\} \le \frac{u(r_2\xi_2)}{u(r_1\xi_1)} \le f(r_1,r_2)\exp\{g(r_1)\}, \tag{1.10}$$

where

$$\begin{aligned} f(r_1,r_2) &= \left(\frac{1+r_2}{1+r_1}\right)\left(\frac{1-r_1}{1-r_2}\right)^{n-1}, \\ g(r_1) = g(r_1,\xi_1,\xi_2) &= \frac{\pi}{2}\,|\xi_2-\xi_1|\,\frac{nr_1}{(1-r_1)^2}. \end{aligned} \tag{1.11}$$

Notice that when $r_2 = r = |x|$, $r_1 = 0$, (1.8) becomes

$$\frac{1-r}{(1+r)^{n-1}} \le \frac{u(x)}{u(0)} \le \frac{1+r}{(1-r)^{n-1}}, \tag{1.12}$$

the classical Harnack inequality in B^n.

Case $\lambda = n/2-1$ corresponds to the Laplace-Beltrami operator $\Delta_{n/2-1}$ associated with the Poincaré metric. In this case, Theorem 1.3 has the following form.

Corollary 1.6. *Let $u(x)$, $x \in B^n$, be a positive solution of $\Delta_{n/2-1}u = 0$. Let $\xi_1, \xi_2 \in S^{n-1}$ and $0 \le r_1 \le r_2 < 1$. Then*

$$\frac{1}{C} \le \frac{u(r_2\xi_2)}{u(r_1\xi_1)} \le C, \tag{1.13}$$

where

$$C = C(r_1,r_2,\xi_1,\xi_2) = \left(\frac{1+r_2}{1+r_1}\cdot\frac{1-r_1}{1-r_2}\right)^{n-1}\exp\left\{\pi\,|\xi_2-\xi_1|\,\frac{(n-1)r_1}{(1-r_1)^2}\right\}. \tag{1.14}$$

2. Proof of Theorem 1.1

Our proofs depend on an integral representation for positive solutions of Weinstein equations in B^n derived by Leutwiler [4] based on earlier work of Huber [5] and Brelot-Collin and Brelot [6]. We state Leutwiler's representation as a theorem below (with a slightly different parametrization).

Theorem 2.1 (The representation theorem, Leutwiler [4, Theorem 3.2]). *Each positive solution of (1.1) admits the unique representation*

$$u(x)=\int_{S^{n-1}}\frac{\left(1-|x|^2\right)^{1+2\lambda-\delta}}{|x-\eta|^{n+2\lambda-2\delta}}\,d\mu(\eta), \tag{2.1}$$

where μ is a positive measure on S^{n-1}.

According to the representation theorem, every positive solution of $\Delta_\lambda u=0$ can be identified with its integral representation (2.1) and the corresponding positive measure μ. Notice that when $\lambda\geq -1/2$, the integrand of (2.1) is the Poisson kernel

$$P_\lambda(x,\eta)=\frac{\left(1-|x|^2\right)^{1+2\lambda}}{|x-\eta|^{n+2\lambda}}. \tag{2.2}$$

In the sequel, we will use the representation formula (2.1) in terms of the Poisson kernel for positive solutions of (1.1) for the case $\lambda\geq -1/2$. The solutions of (1.1) with $\lambda<-1/2$ is related to that of $\lambda>-1/2$ by a corresponding principle, also proved by Leutwiler [4]. We state a special case of the correspondence principle as a lemma.

Lemma 2.2 (The correspondence principle, Leutwiler [4, Lemma 3.4]). *If $u(x)$, $x\in B^n$, is a solution of $\Delta_\lambda u=0$, $\lambda<-1/2$, then*

$$\Delta_{\tilde{\lambda}}\tilde{u}=0 \quad \textit{with } \tilde{u}(x)=\left(1-|x|^2\right)^{-(1+2\lambda)}u(x),\ \tilde{\lambda}=-(1+\lambda)>-\frac{1}{2}. \tag{2.3}$$

We need the following two lemmas for the proof of Theorem 1.1.

Lemma 2.3. *Let $x\in\mathbb{R}^n$, $|x|=r$, $\zeta\in S^{n-1}$. If $\lambda\geq -1/2$, then*

$$\begin{aligned}
&-\frac{\left(n+2\lambda-(n-2\lambda-2)r\right)\left(1-r^2\right)^{2\lambda}}{|x-\zeta|^{n+2\lambda}}\\
&\qquad\leq\frac{\partial}{\partial r}\frac{\left(1-r^2\right)^{1+2\lambda}}{|x-\zeta|^{n+2\lambda}}\leq\frac{\left(n+2\lambda+(n-2\lambda-2)r\right)\left(1-r^2\right)^{2\lambda}}{|x-\zeta|^{n+2\lambda}}.
\end{aligned} \tag{2.4}$$

Proof. Write $x=|x|\eta=r\eta$, $\eta\cdot\zeta=\sum_{i=1}^{n}\eta_i\zeta_i$. Since

$$\begin{aligned}
\frac{\partial}{\partial r}|x-\zeta|^2&=\frac{\partial}{\partial r}\left(|x|^2-2r\eta\cdot\zeta+1\right)=2(r-\eta\cdot\zeta),\\
\frac{\partial}{\partial r}|x-\zeta|^{n+2\lambda}&=\frac{\partial}{\partial r}\left(|x-\zeta|^2\right)^{(n+2\lambda)/2}=(n+2\lambda)|x-\zeta|^{n+2\lambda-2}(r-\eta\cdot\zeta),
\end{aligned} \tag{2.5}$$

we have

$$\begin{aligned}
&\frac{\partial}{\partial r}\frac{\left(1-r^2\right)^{1+2\lambda}}{|x-\zeta|^{n+2\lambda}}\\
&\quad=\frac{(1+2\lambda)\left(1-r^2\right)^{2\lambda}(-2r)|x-\zeta|^{n+2\lambda}-\left(1-r^2\right)^{1+2\lambda}(\partial/\partial r)|x-\zeta|^{n+2\lambda}}{|x-\zeta|^{2(n+2\lambda)}}\\
&\quad=\frac{-2(1+2\lambda)\left(1-r^2\right)^{2\lambda}r|x-\zeta|^{n+2\lambda}-\left(1-r^2\right)^{1+2\lambda}(n+2\lambda)|x-\zeta|^{n+2\lambda-2}(r-\eta\cdot\zeta)}{|x-\zeta|^{2(n+2\lambda)}}\\
&\quad=\frac{-2(1+2\lambda)\left(1-r^2\right)^{2\lambda}r|x-\zeta|^{2}-\left(1-r^2\right)^{1+2\lambda}(n+2\lambda)(r-\eta\cdot\zeta)}{|x-\zeta|^{n+2\lambda+2}}.
\end{aligned} \tag{2.6}$$

To prove the right-side inequality in Lemma 2.3, it suffices to show

$$-2(1+2\lambda)r|x-\zeta|^2-\left(1-r^2\right)(n+2\lambda)(r-\eta\cdot\zeta)\leq\left(n+2\lambda+(n-2\lambda-2)r\right)|x-\zeta|^2, \tag{2.7}$$

which is equivalent to

$$-(n+2\lambda)\left(1-r^2\right)(r-\eta\cdot\zeta)\leq(n+2\lambda)(1+r)|x-\zeta|^2. \tag{2.8}$$

Since $\lambda\geq-1/2$, $n+2\lambda>0$, the above becomes

$$-\left(1-r^2\right)(r-\eta\cdot\zeta)\leq(1+r)|x-\zeta|^2, \tag{2.9}$$

or

$$-(1-r)(r-\eta\cdot\zeta)\leq r^2-2r\eta\cdot\zeta+1, \tag{2.10}$$

which, after a simplification, is equivalent to

$$\eta\cdot\zeta\leq 1. \tag{2.11}$$

The inequality is true since $\zeta,\eta\in S^{n-1}$. To prove the left-side inequality in Lemma 2.3, it suffices to show that

$$-2(1+2\lambda)r|x-\zeta|^2-\left(1-r^2\right)(n+2\lambda)(r-\eta\cdot\zeta)\geq-\left(n+2\lambda-(n-2\lambda-2)r\right)|x-\zeta|^2, \tag{2.12}$$

which is equivalent to

$$(n+2\lambda)\left(1-r^2\right)(r-\eta\cdot\zeta)\leq(n+2\lambda)(1-r)|x-\zeta|^2, \tag{2.13}$$

which is, after a simplification,

$$-\eta\cdot\zeta\leq 1, \tag{2.14}$$

true since $\zeta,\eta\in S^{n-1}$. This completes the proof of Lemma 2.3. □

Lemma 2.4. *Let $u(x)$ be a positive solution of $\Delta_\lambda u = 0$, $\lambda \geq -1/2$, $|x| = r$. Then*

$$-\frac{(n+2\lambda-(n-2\lambda-2)r)}{1-r^2}u(x) \leq \frac{\partial u(x)}{\partial r} \leq \frac{(n+2\lambda+(n-2\lambda-2)r)}{1-r^2}u(x). \tag{2.15}$$

Proof. By the representation theorem, u has the integral representation (2.1) of the Poisson kernel with a positive measure μ on S^{n-1},

$$u(x) = \int_{S^{n-1}} \frac{\left(1-|x|^2\right)^{1+2\lambda}}{|x-\zeta|^{n+2\lambda}} d\mu(\zeta). \tag{2.16}$$

Applying Lemma 2.3, we have

$$\begin{aligned}
&\int_{S^{n-1}} \frac{\partial}{\partial r}\left(\frac{\left(1-|x|^2\right)^{1+2\lambda}}{|x-\zeta|^{n+2\lambda}}\right) d\mu(\zeta) \\
&\quad \leq \int_{S^{n-1}} \frac{(n+2\lambda+(n-2\lambda-2)r)\left(1-r^2\right)^{2\lambda}}{|x-\zeta|^{n+2\lambda}} d\mu(\zeta) \\
&\quad = \frac{(n+2\lambda+(n-2\lambda-2)r)}{1-r^2} \int_{S^{n-1}} \frac{\left(1-|x|^2\right)^{1+2\lambda}}{|x-\zeta|^{n+2\lambda}} d\mu(\zeta) \\
&\quad = \frac{(n+2\lambda+(n-2\lambda-2)r)}{1-r^2} u(x).
\end{aligned} \tag{2.17}$$

It follows that

$$\frac{\partial u(x)}{\partial r} = \int_{S^{n-1}} \frac{\partial}{\partial r}\left(\frac{\left(1-|x|^2\right)^{1+2\lambda}}{|x-\zeta|^{n+2\lambda}}\right) d\mu(\zeta) \leq \frac{(n+2\lambda+(n-2\lambda-2)r)}{1-r^2} u(x). \tag{2.18}$$

The left-side inequality in Lemma 2.4 can be proved in the same manner. For the equality case, consider $u_y(x) = \left(1-|x|^2\right)^{1+2\lambda}/|x-y|^{n+2\lambda}$ which is invariant harmonic in $\mathbb{R}^n \setminus \{y\}$ for $y \in S^{n-1}$. A simple calculation shows that the equalities hold for $u_y(x)$ when $x = |x|y$ and $x = -|x|y$, respectively. This completes the proof of Lemma 2.4. □

Now we prove Theorem 1.1.

Proof. First, consider the case $\lambda \geq -1/2$ ($\delta = 0$). Define

$$\varphi(r) = \frac{(1-r)^{n-1}}{(1+r)^{1+2\lambda}}, \quad \psi(r) = \frac{(1+r)^{n-1}}{(1-r)^{1+2\lambda}}, \quad \text{for } 0 \leq r < 1, \tag{2.19}$$

then

$$\frac{\varphi'(r)}{\varphi(r)} = -\frac{(n+2\lambda+(n-2\lambda-2)r)}{1-r^2}, \qquad \frac{\psi'(r)}{\psi(r)} = \frac{(n+2\lambda-(n-2\lambda-2)r)}{1-r^2}. \tag{2.20}$$

For $x = r\zeta$, $|x| = r$, $\zeta \in S^{n-1}$, denote

$$I(r,\zeta) = \varphi(r)u(r\zeta), \qquad J(r,\zeta) = \psi(r)u(r\zeta). \tag{2.21}$$

To prove Theorem 1.1 for $\lambda \geq -1/2$, we need to show that $I(r,\zeta)$ is decreasing and $J(r,\zeta)$ is increasing in r. By Lemma 4.1,

$$\begin{aligned}\frac{d}{dr}(\log I(r,\zeta)) &= \frac{\varphi'(r)}{\varphi(r)} + \frac{1}{u(x)}\frac{\partial u(x)}{\partial r} \\ &\leq -\frac{(n+2\lambda+(n-2\lambda-2)r)}{1-r^2} + \frac{(n+2\lambda+(n-2\lambda-2)r)}{1-r^2} = 0.\end{aligned} \tag{2.22}$$

Therefore, $\log I(r,\zeta)$ is decreasing in r, and so is $I(r,\zeta)$. Similarly,

$$\begin{aligned}\frac{d}{dr}(\log J(r,\zeta)) &= \frac{\psi'(r)}{\psi(r)} + \frac{1}{u(x)}\frac{\partial u(x)}{\partial r} \\ &\geq \frac{(n+2\lambda-(n-2\lambda-2)r)}{1-r^2} - \frac{(n+2\lambda-(n-2\lambda-2)r)}{1-r^2} = 0.\end{aligned} \tag{2.23}$$

Hence, $J(r,\zeta)$ is increasing in r. We have proved Theorem 1.1 for the case $\lambda \geq -1/2$.

If $\Delta_\lambda u = 0$, $\lambda < -1/2$, then

$$\tilde{u}(x) = (1-|x|^2)^{-(1+2\lambda)}u(x) \tag{2.24}$$

satisfies

$$\Delta_{\tilde{\lambda}}\tilde{u} = 0 \quad \text{with } \tilde{\lambda} = -(1+\lambda) > -\frac{1}{2} \tag{2.25}$$

by the correspondence principle (Lemma 2.2). From the above results for $\lambda > -1/2$,

$$\frac{(1-r)^{n-1}}{(1+r)^{1+2\tilde{\lambda}}}\tilde{u}(r\zeta) = \frac{(1-r)^{n-1}}{(1+r)^{-(1+2\lambda)}}(1-r^2)^{-(1+2\lambda)}u(r\zeta) = (1-r)^{n-2-2\lambda}u(r\zeta) \tag{2.26}$$

is decreasing in r, and

$$\frac{(1+r)^{n-1}}{(1-r)^{1+2\tilde{\lambda}}}\tilde{u}(r\zeta) = \frac{(1+r)^{n-1}}{(1-r)^{-(1+2\lambda)}}(1-r^2)^{-(1+2\lambda)}u(r\zeta) = (1+r)^{n-2-2\lambda}u(r\zeta) \tag{2.27}$$

is increasing in r. Recall that $\delta = 0$ for $\lambda \geq -1/2$ and $\delta = 1+2\lambda$ for $\lambda < -1/2$, we have shown that

$$\frac{(1-r)^{n-1-\delta}}{(1+r)^{1+2\lambda-\delta}}u(r\zeta) \tag{2.28}$$

is decreasing in r and

$$\frac{(1+r)^{n-1-\delta}}{(1-r)^{1+2\lambda-\delta}}u(r\zeta) \tag{2.29}$$

is increasing in r for any $\lambda \in \mathbb{R}$. This completes the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

The proof for Theorem 1.2 is based on Theorem 1.1 and the following lemma.

Lemma 3.1. *Let $f(r)$ be a positive function on $r \in [0,1)$. If for $a,b \in \mathbb{R}$,*

$$-\frac{a+br}{1-r^2} f(r) \le f'(r) \le \frac{a-br}{1-r^2} f(r), \tag{3.1}$$

then for $0 \le r' \le r < 1$,

$$\left(\frac{1+r}{1+r'}\right)^{-a} \left(\frac{1-r^2}{1-r'^2}\right)^{(b+a)/2} f(r') \le f(r) \le \left(\frac{1+r}{1+r'}\right)^{a} \left(\frac{1-r^2}{1-r'^2}\right)^{(b-a)/2} f(r'). \tag{3.2}$$

Proof. The integral

$$\int \frac{a-br}{1-r^2} dr = a\ln(1+r) + \frac{1}{2}(b-a)\ln(1-r^2) + C. \tag{3.3}$$

Thus for $0 \le r' \le r'' < 1$, by (3.1),

$$\ln f(r'') - \ln f(r') = \int_{r'}^{r''} \frac{f'(r)}{f(r)} dr \le \int_{r'}^{r''} \frac{a-br}{1-r^2} dr \le \ln\left(\frac{1+r''}{1+r'}\right)^{a} \left(\frac{1-r''^2}{1-r'^2}\right)^{(b-a)/2}, \tag{3.4}$$

that is, the right-side inequality in (3.2) holds. Similarly, by the left-side of (3.1),

$$\ln f(r'') - \ln f(r') \ge \int_{r'}^{r''} -\frac{a+br}{1-r^2} dr \ge \ln\left(\frac{1+r''}{1+r'}\right)^{-a} \left(\frac{1-r''^2}{1-r'^2}\right)^{(b+a)/2}, \tag{3.5}$$

hence the left-side inequality in (3.2) holds. □

Now we prove Theorem 1.2.

Proof. For $\lambda \ge -1/2$, $u(r\zeta)$ satisfies Lemma 2.4. Therefore, (3.1) holds with $f(r) = u(r\zeta)$, $a = n+2\lambda$, $b = -n+2\lambda+2$. Let $0 \le r' \le r < 1$. Inequality (3.2) in Lemma 3.1 implies

$$\left(\frac{1+r}{1+r'}\right)^{-n-2\lambda} \left(\frac{1-r^2}{1-r'^2}\right)^{2\lambda+1} u(r'\zeta) \le u(r\zeta) \le \left(\frac{1+r}{1+r'}\right)^{n+2\lambda} \left(\frac{1-r^2}{1-r'^2}\right)^{-n+1} u(r'\zeta), \tag{3.6}$$

which is equivalent to

$$\left(\frac{1-r}{1-r'}\right)^{2\lambda+1} \left(\frac{1+r'}{1+r}\right)^{n-1} u(r'\zeta) \le u(r\zeta) \le \left(\frac{1+r}{1+r'}\right)^{2\lambda+1} \left(\frac{1-r'}{1-r}\right)^{n-1} u(r'\zeta). \tag{3.7}$$

If $\Delta_\lambda u = 0$, $\lambda < -1/2$, then $\widetilde{u}(x) = (1-|x|^2)^{-(1+2\lambda)} u(x)$ satisfies $\Delta_{\widetilde{\lambda}} \widetilde{u} = 0$ with $\widetilde{\lambda} = -(1+\lambda) > -1/2$ according to the correspondence principle (Lemma 2.2). From the above proof for $\lambda > -1/2$,

$$\left(\frac{1-r}{1-r'}\right)^{2\widetilde{\lambda}+1}\left(\frac{1+r'}{1+r}\right)^{n-1}\widetilde{u}(r'\zeta) \le \widetilde{u}(r\zeta) \le \left(\frac{1+r}{1+r'}\right)^{2\widetilde{\lambda}+1}\left(\frac{1-r'}{1-r}\right)^{n-1}\widetilde{u}(r'\zeta), \tag{3.8}$$

that is,

$$\left(\frac{1+r'}{1+r}\right)^{n-2-2\lambda} u(r'\zeta) \le u(r\zeta) \le \left(\frac{1-r'}{1-r}\right)^{n-2-2\lambda} u(r'\zeta). \tag{3.9}$$

Combining the cases $\lambda \geq -1/2$ and $\lambda < -1/2$ completes the proof of Theorem 1.2. The equality cases are achieved by the function $u_y(x) = (1-|x|^2)^{1+2\lambda-\delta}/|x-y|^{n+2\lambda-2\delta}$ at $x = |x|y$ and $x = -|x|y$, respectively. □

4. Proof of Theorem 1.3

The proof of Theorem 1.3 is based on the following three lemmas.

Lemma 4.1. *Let $u(x)$, $x \in B^n$, be a positive solution of $\Delta_\lambda u = 0$, $\lambda \in \mathbb{R}$. Let $\varphi(t)$, $t \in [0,1]$ be the shortest* arc *on the great circle connecting ξ_1 and ξ_2. Then*

$$\left|\frac{d}{dt}u(r\varphi(t))\right| \le \frac{r\left|(n+2\lambda-2\delta)\varphi'(t)\right|}{(1-r)^2}u(r\varphi(t)), \quad r \in [0,1]. \tag{4.1}$$

Proof. Direct calculation shows that

$$\frac{d}{dt}\frac{1}{\left|r\varphi(t)-\eta\right|^{n+2\lambda-2\delta}} = \frac{(n+2\lambda-2\delta)r\varphi'(t)\cdot\eta}{\left|r\varphi(t)-\eta\right|^{n+2\lambda-2\delta+2}}. \tag{4.2}$$

Applying $|r\varphi(t)-\eta| \geq |1-r\phi(t)\cdot\eta| \geq 1-r$, we have

$$\begin{aligned}
\int_{S^{n-1}}&\left|\frac{d}{dt}\frac{1}{\left|r\varphi(t)-\eta\right|^{n+2\lambda-2\delta}}\right|d\mu(\eta)\\
&= \int_{S^{n-1}}\frac{\left|(n+2\lambda-2\delta)r\varphi'(t)\cdot\eta\right|}{\left|r\varphi(t)-\eta\right|^{n+2\lambda-2\delta+2}}d\mu(\eta)\\
&\le |n+2\lambda-2\delta|r\int_{S^{n-1}}\frac{\left|\varphi'(t)\right|\left|\eta\right|}{\left|r\varphi(t)-\eta\right|^{n+2\lambda-2\delta}(1-r)^2}d\mu(\eta)\\
&= \frac{r\left|(n+2\lambda-2\delta)\varphi'(t)\right|}{(1-r)^2(1-r^2)^{1+2\lambda-\delta}}\int_{S^{n-1}}\frac{(1-r^2)^{1+2\lambda-\delta}}{\left|r\varphi(t)-\eta\right|^{n+2\lambda-2\delta}}d\mu(\eta)\\
&= \frac{r\left|(n+2\lambda-2\delta)\varphi'(t)\right|}{(1-r)^2(1-r^2)^{1+2\lambda-\delta}}u(r\varphi(t)).
\end{aligned} \tag{4.3}$$

By Lebesgue's dominant convergence theorem,

$$\begin{aligned}\left|\frac{d}{dt}u(r\varphi(t))\right| &= \left|\frac{d}{dt}\int_{S^{n-1}}\frac{\left(1-|r\varphi(t)|^2\right)^{1+2\lambda-\delta}}{|r\varphi(t)-\eta|^{n+2\lambda-2\delta}}d\mu(\eta)\right| \\ &= \left(1-r^2\right)^{1+2\lambda-\delta}\left|\int_{S^{n-1}}\frac{d}{dt}\frac{1}{|r\varphi(t)-\eta|^{n+2\lambda-2\delta}}d\mu(\eta)\right| \\ &\leq \frac{r\left|(n+2\lambda-2\delta)\varphi'(t)\right|}{(1-r)^2}u(r\varphi(t)).\end{aligned} \tag{4.4}$$

□

Lemma 4.2. *Let*

$$\begin{aligned}&\xi_i = (0,\dots,0,r\cos\theta_i,r\sin\theta_i)\in S^{n-1},\quad i=1,2,\qquad |\theta_2-\theta_1|\leq\pi,\\ &\varphi(t) = (0,\dots,0,r\cos\theta_t,r\sin\theta_t),\quad \theta_t = t\theta_2+(1-t)\theta_1,\ t\in[0,1].\end{aligned} \tag{4.5}$$

Then

$$|\varphi'(t)| \leq \frac{\pi}{2}|\xi_2-\xi_1|. \tag{4.6}$$

Proof. It suffices to prove for $n=2$. For computation convenience, we use the complex plane notations in $\mathbb{R}^2$. Denote $\xi_i = e^{1\theta_i}$;

$$\begin{aligned}\xi_2-\xi_1 &= e^{i\theta_2}-e^{i\theta_1}\\ &= \exp\left(i\frac{\theta_2+\theta_1}{2}\right)\left(\exp\left(i\frac{\theta_2-\theta_1}{2}\right)-\exp\left(-i\frac{\theta_2-\theta_1}{2}\right)\right)\\ &= \exp\left(i\frac{\theta_2+\theta_1}{2}\right)(2i)\sin\frac{\theta_2-\theta_1}{2}.\end{aligned} \tag{4.7}$$

Notice that

$$\left|\sin\frac{x}{2}\right| \geq \left|\frac{x}{\pi}\right| \quad \text{for } |x|\leq\pi, \tag{4.8}$$

so $|\theta_2-\theta_1|\leq\pi$ implies

$$|e^{i\theta_2}-e^{i\theta_1}| = 2\left|\sin\left(\frac{\theta_2-\theta_1}{2}\right)\right| \geq \frac{2}{\pi}|\theta_2-\theta_2|. \tag{4.9}$$

Furthermore,

$$\varphi'(t) = \frac{d}{dt}\left(e^{it\theta_2+i(1-t)\theta_1}\right) = \varphi(t)i(\theta_2-\theta_1). \tag{4.10}$$

Thus

$$|\varphi'(t)| = |\theta_2 - \theta_1| \leq \frac{\pi}{2}|e^{i\theta_2} - e^{i\theta_1}| = \frac{\pi}{2}|\xi_2 - \xi_1|. \tag{4.11}$$

□

LEMMA 4.3. *Let $u(x)$, $x \in B^n$, be a positive solution of $\Delta_\lambda u = 0$, $\lambda \in \mathbb{R}$. Let $\xi_1, \xi_2 \in S^{n-1}$. Then for $r \in [0,1)$,*

$$\exp\left\{-\frac{\pi}{2}|\xi_2 - \xi_1|\frac{|n+2\lambda-2\delta|r}{(1-r)^2}\right\} \leq \frac{u(r\xi_2)}{u(r\xi_1)} \leq \exp\left\{\frac{\pi}{2}|\xi_2 - \xi_1|\frac{|n+2\lambda-2\delta|r}{(1-r)^2}\right\}. \tag{4.12}$$

Proof. For $\xi_1, \xi_2 \in S^{n-1}$, there exists a Möbius transformation T in $\mathbb{R}^n$ such that $T(S^{n-1}) = S^{n-1}$, and for all $r \in [0,1]$,

$$T(r\xi_i) = (0,\dots,0,r\cos\theta_i, r\sin\theta_i) \quad \text{for } i = 1,2, \qquad |\theta_2 - \theta_1| \leq \pi \tag{4.13}$$

with

$$|\det T'(x)| = 1, \qquad |T(r\xi_2) - T(r\xi_1)| = |r\xi_2 - r\xi_1|, \tag{4.14}$$

and $u(T(x))$ is also a positive solution of (1.1) with respect to the measure $\mu(T^{-1}(x))$. Since Δ_λ is invariant under such orthogonal transformation, we may assume without loss of generality that

$$\xi_i = (0,\dots,0,\cos\theta_i, \sin\theta_i) \quad \text{for } i = 1,2, \qquad |\theta_2 - \theta_1| \leq \pi. \tag{4.15}$$

Let

$$\varphi(t) = (0,\dots,0,r\cos\theta_t, r\sin\theta_t), \quad \theta_t = t\theta_2 + (1-t)\theta_1,\ t \in [0,1]. \tag{4.16}$$

Then $\varphi(0) = \xi_1$, $\varphi(1) = \xi_2$. By (4.1) in Lemma 4.1 and (4.6) in Lemma 4.2,

$$\begin{aligned}\left|\int_0^1 \frac{(d/dt)u(r\varphi(t))}{u(r\varphi(t))}dt\right| &\leq \int_0^1 \left|\frac{(d/dt)u(r\varphi(t))}{u(r\varphi(t))}\right|dt \\ &\leq \frac{|n+2\lambda-2\delta|r}{(1-r)^2}\int_0^1 |\varphi'(t)|\,dt \leq \frac{\pi}{2}|\zeta_2 - \zeta_1|\frac{|n+2\lambda-2\delta|r}{(1-r)^2}.\end{aligned} \tag{4.17}$$

Since

$$\ln\frac{u(r\xi_2)}{u(r\xi_1)} = \ln\frac{u(r\varphi(1))}{u(r\varphi(0))} = \int_0^1 \frac{(d/dt)u(r\varphi(t))}{u(r\varphi(t))}dt, \tag{4.18}$$

we have

$$\left|\ln\frac{u(r\xi_2)}{u(r\xi_1)}\right| \leq \frac{\pi}{2}|\xi_2 - \xi_1|\frac{|n+2\lambda-2\delta|r}{(1-r)^2}. \tag{4.19}$$

Therefore,

$$-\frac{\pi}{2}\left|\xi_2-\xi_1\right|\frac{|n+2\lambda-2\delta|r}{(1-r)^2}\leq \ln\frac{u(r\xi_2)}{u(r\xi_1)}\leq \frac{\pi}{2}\left|\xi_2-\xi_1\right|\frac{|n+2\lambda-2\delta|r}{(1-r)^2}. \tag{4.20}$$

This completes the proof of Lemma 4.3. □

The proof of Theorem 1.3 follows immediately.

Proof. Write

$$\frac{u(r_2\xi_2)}{u(r_1\xi_1)}=\frac{u(r_2\xi_2)}{u(r_1\xi_2)}\frac{u(r_1\xi_2)}{u(r_1\xi_1)}. \tag{4.21}$$

Applying Lemma 4.3 and Theorem 1.2, Theorem 1.3 follows. The function $u_y(x)=(1-|x|^2)^{1+2\lambda-\delta}/|x-y|^{n+2\lambda-2\delta}$ at $x=|x|y$ and $x=-|x|y$ gives the trivial equalities achieved in Theorem 1.2. □

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