

*Research Article*

## Hereditary Portfolio Optimization with Taxes and Fixed Plus Proportional Transaction Costs—Part II

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This paper is the continuation of the paper entitled “Hereditary portfolio optimization with taxes and fixed plus proportional transaction costs I” that treats an infinite-time horizon hereditary portfolio optimization problem in a market that consists of one *savings* account and one *stock* account. Within the *solvency* region, the investor is allowed to consume from the *savings* account and can make transactions between the two assets subject to paying capital-gain taxes as well as a fixed plus proportional transaction cost. The *investor* is to seek an optimal consumption-trading strategy in order to maximize the expected utility from the total discounted consumption. The portfolio optimization problem is formulated as an infinite dimensional stochastic classical impulse control problem due to the hereditary nature of the stock price dynamics and inventories. This paper contains the verification theorem for the optimal strategy. It also proves that the value function is a viscosity solution of the QVHJBI.

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### 1. Introduction and summary of results in [1]

This is the second of the two companion papers (see [1] for the first paper) that treat an infinite time horizon hereditary portfolio optimization problem in a financial market that consists of one *savings* account and one *stock* account. It is assumed that the *savings* account compounds continuously with a constant interest rate  $r > 0$  and the unit price process,  $\{S(t), t \geq 0\}$ , of the underlying *stock* follows a nonlinear stochastic hereditary differential equation (see (1.23)). The main purpose of the *stock* account is to keep track of the inventories (i.e., the time instants and the base prices at which shares were purchased or sold) for the purpose of calculating the capital-gain taxes, and so forth. In the stock price dynamics, we assume that both  $f(S_t)$  (the mean rate of return) and  $g(S_t)$  (the

volatility coefficient) depend on the entire history of stock prices  $S_t$  over the time interval  $(-\infty, t]$  instead of just the current stock price  $S(t)$  at time  $t \geq 0$  alone. Within the solvency region  $\mathcal{S}_\kappa$  (to be defined in (1.9)) and under the requirements of paying a fixed plus proportional transaction costs and capital-gain taxes, the *investor* is allowed to consume from his *savings* account in accordance with a consumption rate process  $C = \{C(t), t \geq 0\}$  and can make transactions between his *savings* and *stock* accounts according to a trading strategy  $\mathcal{T} = \{(\tau(i), \zeta(i)), i = 1, 2, \dots\}$ , where  $\tau(i), i = 0, 1, 2, \dots$  denote the sequence of transaction times and  $\xi(i)$  stands for the quantities of transactions at time  $\tau(i)$  (see Definition 1.10).

The *investor* will follow the following set of consumption, transaction, and taxation rules (Rules 1.1–1.6). Note that an action of the *investor* in the market is called a transaction if it involves trading of shares of the *stock* such as buying and selling.

*Rule 1.1.* At the time of each transaction, the *investor* has to pay a transaction cost that consists of a fixed cost  $\kappa > 0$  and a proportional transaction cost with the cost rate of  $\mu \geq 0$  for both selling and buying shares of the *stock*. All the purchases and sales of any number of stock shares will be considered one transaction if they are executed at the same time instant and therefore incur only one fixed fee  $\kappa > 0$  (in addition to a proportional transaction cost).

*Rule 1.2.* Within the solvency region  $\mathcal{S}_\kappa$ , the *investor* is allowed to consume and to borrow money from his *savings* account for *stock* purchases. He can also sell and/or buy-back at the current price shares of the *stock* he bought and/or short sold at a previous time.

*Rule 1.3.* The proceeds for the sales of the *stock* minus the transaction costs and capital-gain taxes will be deposited in his *savings* account and the purchases of stock shares together with the associated transaction costs and capital-gain taxes (if short shares of the *stock* are bought back at a profit) will be financed from his *savings* account.

*Rule 1.4.* Without loss of generality, it is assumed that the interest income in the *savings* account is tax free by using the effective interest rate  $r > 0$ , where the effective interest rate equals the interest rate paid by the bank minus the tax rate for the interest income.

*Rule 1.5.* At the time of a transaction (say  $t \geq 0$ ), the *investor* is required to pay a capital-gain tax (resp., be paid a capital-loss credit) in the amount that is proportional to the amount of profit (resp., loss). A sale of stock shares is said to result in a profit if the current stock price  $S(t)$  is higher than the base price  $B(t)$  of the stock and it is a loss otherwise. The base price  $B(t)$  is defined to be the price at which the stock shares were previously bought or short sold, that is,  $B(t) = S(t - \tau(t))$  where  $\tau(t) > 0$  is the time duration for which those shares (long or short) have been held at time  $t$ . The *investor* will also pay capital-gain taxes (resp., be paid capital-loss credits) for the amount of profit (resp., loss) by short-selling shares of the *stock* and then buying back the shares at a lower (resp., higher) price at a later time. The tax will be paid (or the credit will be given) at the buying-back time. Throughout the end, a negative amount of tax will be interpreted as a capital-loss credit. The capital-gain tax and capital-loss credit rates are assumed to be the same as  $\beta > 0$  for simplicity. Therefore, if  $|m|$  ( $m > 0$  stands for buying and  $m < 0$  stands for selling) shares of the stock are traded at the current price  $S(t)$  at the base  $B(t) = S(t - \tau(t))$ , then the

amount of tax due at the transaction time is given by

$$|m|\beta(S(t) - S(t - \tau(t))). \quad (1.1)$$

*Rule 1.6.* The tax and/or credit will not exceed all other gross proceeds and/or total costs of the stock shares, that is,

$$\begin{aligned} m(1 - \mu)S(t) &\geq \beta m |S(t) - S(t - \tau(t))| && \text{if } m \geq 0, \\ m(1 + \mu)S(t) &\leq \beta m |S(t) - S(t - \tau(t))| && \text{if } m < 0, \end{aligned} \quad (1.2)$$

where  $m \in \mathfrak{R}$  denotes the number of shares of the stock traded with  $m \geq 0$  being the number of shares purchased and  $m < 0$  being the number of shares sold.

*Convention 1.7.* Throughout the end, we assume that  $\mu + \beta < 1$ .

Under the above assumptions and Rules 1.1–1.6, the *investor's* objective is to seek an optimal consumption-trading strategy  $(C^*, \mathcal{T}^*)$  in order to maximize

$$E \left[ \int_0^\infty e^{-\delta t} \frac{C^\gamma(t)}{\gamma} dt \right], \quad (1.3)$$

the expected utility from the total discounted consumption over the infinite time horizon, where  $\delta > 0$  represents the discount rate and  $0 < \gamma < 1$  represents the *investor's* risk aversion factor.

Due to the fixed plus proportional transaction costs and the hereditary nature of the stock dynamics and inventories, the problem will be formulated as a combination of a classical control (for consumptions) and an impulse control (for the transactions) problem in infinite dimensions. In the first paper [1], a quasivariational Hamilton-Jacobi-Bellman inequality (QVHJBI) for the value function together with its boundary conditions are derived. This paper establishes the verification theorem for the optimal investment-trading strategy. It is also shown here that the value function is a viscosity solution of the QVHJBI (see QVHJBI(\*) in Section 2). Due to the complexity of the analysis involved, the uniqueness result and finite dimensional approximations for the viscosity solution of QVHJBI(\*) will be treated separately in a future paper.

In this and the previous paper, the state space will be  $\mathbf{S} = \mathfrak{R} \times \mathbf{N} \times \mathfrak{R} \times L_\rho^2$ . In the above,

- (i) the stock inventory space,  $\mathbf{N}$ , is the space of bounded measurable functions  $\xi : (-\infty, 0] \rightarrow \mathfrak{R}$  of the following form:

$$\xi(\theta) = \sum_{k=0}^{\infty} n(-k) \mathbf{1}_{\{\tau(-k)\}}(\theta), \quad \theta \in (-\infty, 0], \quad (1.4)$$

where  $\{n(-k), k = 0, 1, 2, \dots\}$  is a sequence in  $\mathfrak{R}$  with  $n(-k) = 0$  for all but finitely many  $k$ ,

$$-\infty < \dots < \tau(-k) < \dots < \tau(-1) < \tau(0) = 0, \quad (1.5)$$

and  $\mathbf{1}_{\{\tau(-k)\}}$  is the indicator function at  $\tau(-k)$ .

Let  $\|\cdot\|_N$  (the norm of the space  $\mathbf{N}$ ) be defined by

$$\|\xi\|_N = \sup_{\theta \in (-\infty, 0]} |\xi(\theta)| \quad \forall \xi \in \mathbf{N}; \tag{1.6}$$

(ii) the historical stock price space is  $\mathfrak{X} \times L^2_\rho$  with  $L^2_\rho$  being the  $\rho$ -weighted Hilbert space of functions  $\phi : (-\infty, 0] \rightarrow \mathfrak{X}$  with

$$\int_{-\infty}^0 |\phi(\theta)|^2 \rho(\theta) d\theta < \infty. \tag{1.7}$$

Throughout the end of this paper, let  $\rho : (-\infty, 0] \rightarrow [0, \infty)$  be the *influence function with relaxation property* that satisfies the following conditions.

*Condition 1.8.*  $\rho$  is summable on  $(-\infty, 0]$ , that is,  $0 < \int_{-\infty}^0 \rho(\theta) d\theta < \infty$ .

*Condition 1.9.* For every  $\lambda \leq 0$  one has

$$\overline{K}(\lambda) = \operatorname{ess\,sup}_{\theta \in (-\infty, 0]} \frac{\rho(\theta + \lambda)}{\rho(\theta)} \leq \overline{K} < \infty, \quad \underline{K}(\lambda) = \operatorname{ess\,sup}_{\theta \in (-\infty, 0]} \frac{\rho(\theta)}{\rho(\theta + \lambda)} < \infty. \tag{1.8}$$

An element  $(x, \xi, \psi(0), \psi) \in \mathbf{S}$  will be referred to as a portfolio, where  $x \in \mathfrak{X}$  represents the investor's holding in his savings account,  $\xi \in \mathbf{N}$  represents his stock inventory, and  $(\psi(0), \psi) \in \mathfrak{X} \times L^2_\rho$  stands for a profile of historical stock prices.

The solvency region  $\mathcal{S}_\kappa$  of the portfolio optimization problem is defined as

$$\mathcal{S}_\kappa = \{(x, \xi, \psi(0), \psi) \in \mathbf{S} \mid G_\kappa(x, \xi, \psi(0), \psi) \geq 0\} \cup \mathbf{S}_+, \tag{1.9}$$

where  $G_\kappa : \mathbf{S} \rightarrow \mathfrak{X}$  is the liquidating function defined by

$$\begin{aligned} G_\kappa(x, \xi, \psi(0), \psi) &= x - \kappa + \sum_{k=0}^{\infty} \left[ \min \{(1 - \mu)n(-k), (1 + \mu)n(-k)\} \psi(0) - n(-k)\beta(\psi(0) - \psi(\tau(-k))) \right], \end{aligned} \tag{1.10}$$

and  $\mathbf{S}_+ = \mathfrak{X}_+ \times \mathbf{N}_+ \times \mathfrak{X}_+ \times L^2_{\rho,+}$  is the positive cone of the state space  $\mathbf{S}$ .

Let  $(X(0-), N_{0-}, S(0), S_0) = (x, \xi, \psi(0), \psi) \in \mathfrak{X} \times \mathbf{N} \times \mathfrak{X}_+ \times L^2_{\rho,+}$  be the *investor's* initial portfolio immediately prior to  $t = 0$ , that is, the investor starts with  $x \in \mathfrak{X}$  dollars in his *savings* account, the initial stock inventory

$$\xi(\theta) = \sum_{k=0}^{\infty} n(-k) \mathbf{1}_{\{\tau(-k)\}}(\theta), \quad \theta \in (-\infty, 0), \tag{1.11}$$

and the initial profile of historical stock prices  $(\psi(0), \psi) \in \mathfrak{X}_+ \times L^2_{\rho,+}$ , where  $n(-k) > 0$  (resp.,  $n(-k) < 0$ ) represents an open long (resp., short) position at  $\tau(-k)$ . Within the *solvency region*  $\mathcal{S}_\kappa$ , the *investor* is allowed to consume from his *savings* account and can make transactions between his *savings* and *stock* accounts under Rules 1.1–1.6 and according to a consumption-trading strategy  $\pi = (C, \mathcal{T})$  defined below.

*Definition 1.10.* The pair  $\pi = (C, \mathcal{T})$  is said to be a consumption-trading strategy if

- (i) the consumption rate process  $C = \{C(t), t \geq 0\}$  is a nonnegative  $\mathbf{G}$ -progressively measurable process such that

$$\int_0^T C(t)dt < \infty, \quad P\text{-a.s.}, \quad \forall T > 0; \quad (1.12)$$

- (ii)  $\mathcal{T} = \{(\tau(i), \zeta(i)), i = 1, 2, \dots\}$  is a trading strategy with  $\tau(i), i = 1, 2, \dots$ , being a sequence of trading times that are  $\mathbf{G}$ -stopping times such that

$$0 = \tau(0) \leq \tau(1) < \dots < \tau(i) < \dots, \quad \lim_{i \rightarrow \infty} \tau(i) = \infty \text{ a.s.}, \quad (1.13)$$

and for each  $i = 0, 1, \dots$ ,

$$\zeta(i) = (\dots, m(i-k), \dots, m(i-2), m(i-1), m(i)) \quad (1.14)$$

is an  $\mathbf{N}$ -valued  $\mathcal{G}(\tau(i))$ -measurable random vector (instead of a random variable in  $\mathfrak{K}$ ) that represents the trading quantity at the trading time  $\tau(i)$ . In the above,  $m(i) > 0$  (resp.,  $m(i) < 0$ ) is the number of stock shares newly purchased (resp., short-sold) at the current time  $\tau(i)$  and at the current price of  $S(\tau(i))$  and, for  $k = 1, 2, \dots$ ,  $m(i-k) > 0$  (resp.,  $m(i-k) < 0$ ) is the number of stock shares bought back (resp., sold) at the current time  $\tau(i)$  and the current price of  $S(\tau(i))$  in his open short (resp., long) position at the previous time  $\tau(i-k)$  and the base price of  $S(\tau(i-k))$ .

Note that  $\mathbf{G} = \{\mathcal{G}(t), t \geq 0\}$  is the filtration generated by  $\{S(t), t \geq 0\}$ , that is,

$$\mathcal{G}(t) = \sigma(S(s), 0 \leq s \leq t) (= \sigma((S(s), S_s), 0 \leq s \leq t)), \quad \forall t \geq 0. \quad (1.15)$$

For each stock inventory  $\xi$  of the form expressed (1.4), Rules 1.1–1.6 also dictate that the investor can purchase or short sell new shares and/or buy back (resp., sell) all or part of what he owes (resp., owns). Therefore, the trading quantity  $\{m(-k), k = 0, 1, \dots\}$  must satisfy the constraint set  $\mathcal{R}(\xi) \subset \mathbf{N}$  defined by

$$\begin{aligned} \mathcal{R}(\xi) = \left\{ \zeta \in \mathbf{N} \mid \zeta = \sum_{k=0}^{\infty} m(-k) \mathbf{1}_{\{\tau(-k)\}}, \quad -\infty < m(0) < \infty, \right. \\ \text{either } n(-k) > 0, \quad m(-k) \leq 0 \text{ \& } n(-k) + m(-k) \geq 0 \\ \left. \text{or } n(-k) < 0, \quad m(-k) \geq 0 \text{ \& } n(-k) + m(-k) \leq 0 \text{ for } k \geq 1 \right\}. \end{aligned} \quad (1.16)$$

Given the initial portfolio

$$(X(0-), N_{0-}, S(0), S_0) = (x, \xi, \psi(0), \psi) \in \mathbf{S} \quad (1.17)$$

and applying a consumption-trading strategy  $\pi = (C, \mathcal{T})$  (see Definition 1.10), the portfolio dynamics of  $\{Z(t) = (X(t), N_t, S(t), S_t), t \geq 0\}$  can then be described as follows.

Firstly, the *savings* account holdings  $\{X(t), t \geq 0\}$  satisfy the following differential equation between the trading times:

$$dX(t) = [rX(t) - C(t)]dt, \quad \tau(i) \leq t < \tau(i+1), \quad i = 0, 1, 2, \dots, \quad (1.18)$$

and the following jumped quantity at the trading time:

$$\begin{aligned} X(\tau(i)) &= X(\tau(i)-) - \kappa - \sum_{k=0}^{\infty} m(i-k) \left[ (1-\mu)S(\tau(i)) - \beta(S(\tau(i)) - S(\tau(i-k))) \right] \\ &\quad \times \mathbf{1}_{\{n(i-k)>0, -n(i-k) \leq m(i-k) \leq 0\}} \\ &\quad - \sum_{k=0}^{\infty} m(i-k) \left[ (1+\mu)S(\tau(i)) - \beta(S(\tau(i)) - S(\tau(i-k))) \right] \\ &\quad \times \mathbf{1}_{\{n(i-k)<0, 0 \leq m(i-k) \leq -n(i-k)\}}. \end{aligned} \quad (1.19)$$

As a reminder,  $m(i) > 0$  (resp.,  $m(i) < 0$ ) means buying (resp., selling) new stock shares at  $\tau(i)$  and  $m(i-k) > 0$  (resp.,  $m(i-k) < 0$ ) means buying back (resp., selling) some or all of what he owed (resp., owned).

Secondly, the inventory of the *investor's* stock account at time  $t \geq 0$ ,  $N_t \in \mathbf{N}$  does not change between the trading times and can be expressed as the following equation:

$$N_t = N_{\tau(i)} = \sum_{k=-\infty}^{Q(t)} n(k) \mathbf{1}_{\tau(k)} \quad \text{if } \tau(i) \leq t < \tau(i+1), \quad i = 0, 1, 2, \dots, \quad (1.20)$$

where  $Q(t) = \sup\{k \geq 0 \mid \tau(k) \leq t\}$ . It has the following jumped quantity at the trading time  $\tau(i)$ :

$$N_{\tau(i)} = N_{\tau(i)-} \oplus \zeta(i), \quad (1.21)$$

where  $N_{\tau(i)-} \oplus \zeta(i) : (-\infty, 0] \rightarrow \mathbf{N}$  is defined by

$$\begin{aligned} (N_{\tau(i)-} \oplus \zeta(i))(\theta) &= \sum_{k=0}^{\infty} \hat{n}(i-k) \mathbf{1}_{\{\tau(i-k)\}}(\tau(i) + \theta) \\ &= m(i) \mathbf{1}_{\{\tau(i)\}}(\tau(i) + \theta) \\ &\quad + \sum_{k=1}^{\infty} [n(i-k) + m(i-k) (\mathbf{1}_{\{n(i-k)<0, 0 \leq m(i-k) \leq -n(i-k)\}} \\ &\quad \quad \quad + \mathbf{1}_{\{n(i-k)>0, -n(i-k) \leq m(i-k) \leq 0\}})] \mathbf{1}_{\{\tau(i-k)\}}(\tau(i) + \theta), \quad \theta \in (-\infty, 0]. \end{aligned} \quad (1.22)$$

Thirdly, since the *investor* is small, the unit stock price process  $\{S(t), t \geq 0\}$  will not be in anyway affected by the *investor's* action in the market and is assumed to satisfy the following nonlinear stochastic hereditary differential equation:

$$dS(t) = S(t)[f(S_t)dt + g(S_t)dW(t)], \quad t \geq 0, \quad (1.23)$$

with the initial historical price function  $(S(0), S_0) = (\psi(0), \psi) \in \mathfrak{X}_+ \times L_{\rho,+}^2$ . Note that  $f(S_t)$  and  $g(S_t)$  in (1.4) represent, respectively, the *mean growth rate* and the *volatility rate* of the stock price at time  $t \geq 0$  and that they are dependent on the entire history of stock prices  $S_t$  ( $S_t(\theta)$ ,  $\theta \in (-\infty, 0]$ ) over the time interval  $(-\infty, t]$ .

Under the Lipschitz and linear growth conditions (see [1, Assumptions 2.4–2.6]) of the functions  $(\phi(0), \phi) \mapsto \phi(0)f(\phi)$  and  $(\phi(0), \phi) \mapsto \phi(0)g(\phi)$  on the space  $\mathfrak{X} \times L_{\rho}^2$ , it can be shown that (1.23) (see [2, 1, 3–5]) has a unique strong solution  $\{S(t), t \in (-\infty, \infty)\}$  and that the  $\mathfrak{X} \times L_{\rho}^2$ -valued process  $\{(S(t), S_t), t \geq 0\}$  is a strong Markovian with respect to the filtration  $\mathbf{G}$ .

*Definition 1.11.* If the investor starts with an initial portfolio

$$(X(0-), N_{0-}, S(0), S_0) = (x, \xi, \psi(0), \psi) \in \mathcal{S}_{\kappa}. \quad (1.24)$$

The consumption-trading strategy  $\pi = (C, \mathcal{T})$  defined in Definition 1.10 is said to be *admissible* at  $(x, \xi, \psi(0), \psi)$  if

$$\zeta(i) \in \mathcal{R}(N_{\tau(i)-}) \quad \forall i = 1, 2, \dots, \quad (X(t), N_t, S(t), S_t) \in \mathcal{S}_{\kappa}, \quad \forall t \geq 0. \quad (1.25)$$

The class of consumption-investment strategies admissible at  $(x, \xi, \psi(0), \psi) \in \mathcal{S}_{\kappa}$  will be denoted by  $\mathcal{U}_{\kappa}(x, \xi, \psi(0), \psi)$ .

The investor's objective is to find an admissible consumption-trading strategy  $\pi^* \in \mathcal{U}_{\kappa}(x, \xi, \psi(0), \psi)$  that maximizes the following expected utility from the total discounted consumption:

$$J_{\kappa}(x, \xi, \psi(0), \psi; \pi) = E^{x, \xi, \psi(0), \psi; \pi} \left[ \int_0^{\infty} e^{-\delta t} \frac{C^{\gamma}(t)}{\gamma} dt \right] \quad (1.26)$$

among the class of admissible consumption-trading strategies  $\mathcal{U}_{\kappa}(x, \xi, \psi(0), \psi)$ , where  $E^{x, \xi, \psi(0), \psi; \pi}[\dots]$  is the expectation with respect to  $P^{x, \xi, \psi(0), \psi; \pi} \{\dots\}$ , the probability measure induced by the controlled (by  $\pi$ ) state process  $\{(X(t), N_t, S(t), S_t), t \geq 0\}$  and conditioned on the initial state

$$(X(0-), N_{0-}, S(0), S_0) = (x, \xi, \psi(0), \psi). \quad (1.27)$$

In the above,  $\delta > 0$  denotes the discount factor, and  $0 < \gamma < 1$  indicates that the utility function  $U(c) = c^{\gamma}/\gamma$ , for  $c > 0$ , is a function of HARA (hyperbolic absolute risk aversion) type. The admissible (consumption-investment) strategy  $\pi^* \in \mathcal{U}_{\kappa}(x, \xi, \psi(0), \psi)$  that maximizes  $J_{\kappa}(x, \xi, \psi(0), \psi; \pi)$  is called an optimal (consumption-trading) strategy and the function  $V_{\kappa} : \mathcal{S}_{\kappa} \rightarrow \mathfrak{R}_+$  defined by

$$V_{\kappa}(x, \xi, \psi(0), \psi) = \sup_{\pi \in \mathcal{U}_{\kappa}(x, \xi, \psi(0), \psi)} J_{\kappa}(x, \xi, \psi(0), \psi; \pi) = J_{\kappa}(x, \xi, \psi(0), \psi; \pi^*) \quad (1.28)$$

is called the value function of the hereditary portfolio optimization problem.

The results obtained in [1] include derivations of the infinite dimensional quasivariational Hamilton-Jacobi-Bellman (GVHJB) together with its boundary conditions. The boundary conditions for the value function are given as follows.

Let  $\aleph \equiv \{0, 1, 2, \dots\}$ . The boundary  $\partial\mathcal{S}_\kappa$  of  $\mathcal{S}_\kappa$  can be decomposed as follows:

$$\partial\mathcal{S}_\kappa = \bigcup_{I \subset \aleph} (\partial_{-,I}\mathcal{S}_\kappa \cup \partial_{+,I}\mathcal{S}_\kappa), \tag{1.29}$$

where

$$\begin{aligned} \partial_{-,I}\mathcal{S}_\kappa &= \partial_{-,I,1}\mathcal{S}_\kappa \cup \partial_{-,I,2}\mathcal{S}_\kappa, \\ \partial_{+,I}\mathcal{S}_\kappa &= \partial_{+,I,1}\mathcal{S}_\kappa \cup \partial_{+,I,2}\mathcal{S}_\kappa, \\ \partial_{+,I,1}\mathcal{S}_\kappa &= \left\{ (x, \xi, \psi(0), \psi) \mid G_\kappa(x, \xi, \psi(0), \psi) = 0, x \geq 0, \right. \\ &\quad \left. n(-i) < 0 \ \forall i \in I \ \& \ n(-i) \geq 0 \ \forall i \notin I \right\}, \\ \partial_{+,I,2}\mathcal{S}_\kappa &= \left\{ (x, \xi, \psi(0), \psi) \mid G_\kappa(x, \xi, \psi(0), \psi) < 0, x \geq 0, \right. \\ &\quad \left. n(-i) = 0 \ \forall i \in I \ \& \ n(-i) \geq 0 \ \forall i \notin I \right\}, \\ \partial_{-,I,1}\mathcal{S}_\kappa &= \left\{ (x, \xi, \psi(0), \psi) \mid G_\kappa(x, \xi, \psi(0), \psi) = 0, x < 0, \right. \\ &\quad \left. n(-i) < 0 \ \forall i \in I \ \& \ n(-i) \geq 0 \ \forall i \notin I \right\}, \\ \partial_{-,I,2}\mathcal{S}_\kappa &= \left\{ (x, \xi, \psi(0), \psi) \mid G_\kappa(x, \xi, \psi(0), \psi) < 0, x = 0, \right. \\ &\quad \left. n(-i) = 0 \ \forall i \in I \ \& \ n(-i) \geq 0 \ \forall i \notin I \right\}. \end{aligned} \tag{1.30}$$

The interface (intersection) between  $\partial_{+,I,1}\mathcal{S}_\kappa$  and  $\partial_{+,I,2}\mathcal{S}_\kappa$  is denoted by

$$\begin{aligned} Q_{+,I} &= \left\{ (x, \xi, \psi(0), \psi) \mid G_\kappa(x, \xi, \psi(0), \psi) = 0, x \geq 0, \right. \\ &\quad \left. n(-i) = 0 \ \forall i \in I \ \& \ n(-i) \geq 0 \ \forall i \notin I \right\}. \end{aligned} \tag{1.31}$$

Whereas the interface between  $\partial_{-,I,1}\mathcal{S}_\kappa$  and  $\partial_{-,I,2}\mathcal{S}_\kappa$  is denoted by

$$\begin{aligned} Q_{-,I} &= \left\{ (0, \xi, \psi(0), \psi) \mid G_\kappa(0, \xi, \psi(0), \psi) = 0, x = 0, \right. \\ &\quad \left. n(-i) = 0 \ \forall i \in I \ \& \ n(-1) \geq 0 \ \forall i \notin I \right\}. \end{aligned} \tag{1.32}$$

The QVHJBI (together with the boundary conditions) is derived in [1] and restated as follows:

$$\text{QVHJBI}(\ast) = \begin{cases} \max \{ \mathcal{A}\Phi, \mathcal{M}_\kappa\Phi - \Phi \} = 0 & \text{on } \mathcal{S}_\kappa^\circ, \\ \mathcal{A}\Phi = 0 & \text{on } \bigcup_{I \subset \aleph} \partial_{+,I,2}\mathcal{S}_\kappa, \\ \mathcal{L}^0\Phi = 0 & \text{on } \bigcup_{I \subset \aleph} \partial_{-,I,2}\mathcal{S}_\kappa, \\ \mathcal{M}_\kappa\Phi - \Phi = 0 & \text{on } \bigcup_{I \subset \aleph} \partial_{I,1}\mathcal{S}_\kappa, \end{cases} \tag{1.33}$$



where

$$\begin{aligned} \mathcal{A}\Phi &= (\mathbf{A} + \Gamma + rx\partial_x - \delta)\Phi + \sup_{c \geq 0} \left( \frac{c^\gamma}{\gamma} - c\partial_x\Phi \right), \\ \mathbf{A}\Phi(\psi(0), \psi) &= \frac{1}{2}\partial_{\psi(0)}^2\Phi(\psi(0), \psi)\psi^2(0)g^2(\psi) + \partial_{\psi(0)}\Phi(\psi(0), \psi)\psi(0)f(\psi), \end{aligned} \quad (1.34)$$

$$\mathcal{L}^0\Phi = (\mathbf{A} + \Gamma + rx\partial_x - \delta)\Phi,$$

$$\Gamma(\Phi)(\phi(0), \phi) \equiv \lim_{t \downarrow 0} \frac{\Phi(\phi(0), \tilde{\phi}_t) - \Phi(\phi(0), \phi)}{t}, \quad (1.35)$$

with  $\tilde{\phi} : (-\infty, \infty) \rightarrow \mathfrak{X}$  being defined by

$$\tilde{\phi}(t) = \begin{cases} \phi(0) & \text{for } t \in [0, \infty), \\ \phi(t) & \text{for } t \in (-\infty, 0). \end{cases} \quad (1.36)$$

Then for each  $\theta \in (-\infty, 0]$  and  $t \in [0, \infty)$ ,

$$\tilde{\phi}_t(\theta) = \tilde{\phi}(t + \theta) = \begin{cases} \phi(0) & \text{for } t + \theta \geq 0, \\ \phi(t + \theta) & \text{for } t + \theta < 0. \end{cases} \quad (1.37)$$

$$\mathcal{L}^0\Phi = (\mathbf{A} + \Gamma + rx\partial_x - \delta)\Phi.$$

Furthermore,  $\mathcal{M}_\kappa\Phi$  is given by

$$\mathcal{M}_\kappa\Phi(x, \xi, \psi(0), \psi) = \sup \left\{ \Phi(\hat{x}, \hat{\xi}, \hat{\psi}(0), \hat{\psi}) \mid \zeta \in \mathcal{R}(\xi) - \{\mathbf{0}\}, (\hat{x}, \hat{\xi}, \hat{\psi}(0), \hat{\psi}) \in \mathcal{S}_\kappa \right\}, \quad (1.38)$$

where the new portfolio immediately after a transaction,  $(\hat{x}, \hat{\xi}, \hat{\psi}(0), \hat{\psi})$ , is as defined as follows:

$$\begin{aligned} \hat{x} &= x - \kappa - (m(0) + \mu |m(0)|)\psi(0) \\ &\quad - \sum_{k=1}^{\infty} \left[ (1 + \mu)m(-k)\psi(0) - \beta m(-k)(\psi(0) - \psi(\tau(-k))) \right] \\ &\quad \times \mathbf{1}_{\{n(-k) < 0, 0 \leq m(-k) \leq -n(-k)\}} \\ &\quad - \sum_{k=1}^{\infty} \left[ (1 - \mu)m(-k)\psi(0) - \beta m(-k)(\psi(0) - \psi(\tau(-k))) \right] \\ &\quad \cdot \mathbf{1}_{\{n(-k) > 0, -n(-k) \leq m(-k) \leq 0\}}, \end{aligned} \quad (1.39)$$

and for all  $\theta \in (-\infty, 0]$ ,

$$\begin{aligned} \widehat{\xi}(\theta) &= (\xi \oplus \zeta)(\theta) = m(0)\mathbf{1}_{\{\tau(0)\}}(\theta) \\ &\quad + \sum_{k=1}^{\infty} \left( n(-k) + m(-k) \right. \\ &\quad \left. \times [\mathbf{1}_{\{n(-k)<0, 0 \leq m(-k) \leq -n(-k)\}} + \mathbf{1}_{\{n(-k)>0, -n(-k) \leq m(-k) \leq 0\}}] \right) \mathbf{1}_{\{\tau(-k)\}}(\theta), \end{aligned} \tag{1.40}$$

and again

$$(\widehat{\psi}(0), \widehat{\psi}) = (\psi(0), \psi). \tag{1.41}$$

If  $(\widehat{x}, \widehat{\xi}, \widehat{\psi}(0), \widehat{\psi}) \notin \mathcal{S}_\kappa$  for all  $\zeta \in \mathcal{R}(\xi) - \{0\}$ , we set  $\mathcal{M}_\kappa \Phi(x, \xi, \psi(0), \psi) = 0$ .

In this paper, we obtain the verification theorem for the optimal consumption-trading strategy  $\pi^*$ . This result is contained in Section 2. In Section 3, we also prove that the value function  $V_\kappa : \mathcal{S}_\kappa \rightarrow \mathfrak{R}$  is a viscosity solution of QVHBJI(\*).

### 2. The verification theorem

Let

$$\widetilde{\mathcal{A}}\Phi = \begin{cases} \mathcal{A}\Phi & \text{on } \mathcal{S}_\kappa^\circ \cup \bigcup_{I \subset \mathbb{N}} \partial_{+, I, 2} \mathcal{S}_\kappa; \\ \mathcal{L}^0 \Phi & \text{on } \bigcup_{I \subset \mathbb{N}} \partial_{-, I, 2} \mathcal{S}_\kappa. \end{cases} \tag{2.1}$$

Let  $\mathcal{D}(\Gamma)$  be the domain of the operator  $\Gamma$  defined in (1.35), that is,  $\mathcal{D}(\Gamma)$  is the set of (Borel) measurable functions  $\Phi : \mathcal{S}_\kappa \rightarrow \mathfrak{R}$  such that the limit in (1.35) exists for each fixed  $(x, \xi, \psi(0), \psi) \in \mathcal{S}_\kappa$ . Let  $C_{\text{lip}}^{1,0,2,2}(\mathcal{S}_\kappa)$  be the collection of functions  $\Phi : \mathcal{S}_\kappa \rightarrow \mathfrak{R}$  that are continuously differentiable with respect to its first variable  $x$  and twice continuously differentiable and Fréchet differentiable with respect to its third variable  $\psi(0)$  and fourth variable  $\psi$  and the second-order Fréchet derivative  $D^2\Phi(x, \xi, \cdot, \cdot)$  is said to be globally Lipschitz on  $\mathfrak{R} \times L_\rho^2$  in operator norm  $\|\cdot\|^\dagger$ , that is, there exists a constant  $K > 0$  such that

$$\begin{aligned} &\|D^2\Phi(x, \xi, \phi(0), \phi) - D^2\Phi(x, \xi, \varphi(0), \varphi)\|^\dagger \\ &\leq K \|(\phi(0), \phi) - (\varphi(0), \varphi)\|, \quad \forall (\phi(0), \phi), (\varphi(0), \varphi) \in \mathfrak{R} \times L_\rho^2. \end{aligned} \tag{2.2}$$

We have the following verification theorem for the value function  $V_\kappa : \mathcal{S}_\kappa \rightarrow \mathfrak{R}$  for our hereditary portfolio optimization problem.

**THEOREM 2.1** (the verification theorem). (a) *Let  $U_\kappa = \mathcal{S}_\kappa - \bigcup_{I \subset \mathbb{N}} \partial_{I, 1} \mathcal{S}_\kappa$ . Suppose there exists a locally bounded nonnegative valued function  $\Phi \in C_{\text{lip}}^{1,0,2,2}(\mathcal{S}_\kappa) \cap \mathcal{D}(\Gamma)$  such that*

$$\widetilde{\mathcal{A}}\Phi \leq 0 \quad \text{on } U_\kappa, \quad \Phi \geq \mathcal{M}_\kappa \Phi \quad \text{on } U_\kappa. \tag{2.3}$$

*Then  $\Phi \geq V_\kappa$  on  $\mathcal{U}_\kappa$ .*

(b) Define  $D \equiv \{(x, \xi, \psi(0), \psi) \in U_\kappa \mid \Phi(x, \xi, \psi(0), \psi) > \mathcal{M}_\kappa \Phi(x, \xi, \psi(0), \psi)\}$ . Suppose

$$\tilde{\mathcal{A}}\Phi(x, \xi, \psi(0), \psi) = 0 \quad \text{on } D \quad (2.4)$$

and that  $\hat{\zeta}(x, \xi, \psi(0), \psi) = \hat{\zeta}_\Phi(x, \xi, \psi(0), \psi)$  exists for all  $(x, \xi, \psi(0), \psi) \in \mathcal{S}_\kappa$  by [1, Assumption 4.2]. Let

$$c^* = \begin{cases} (\partial_x \Phi)^{1/(\gamma-1)} & \text{on } \mathcal{S}_\kappa^\circ \cup \bigcup_{I \subset \mathbb{N}} \partial_{+, I, 2} \mathcal{S}_\kappa, \\ 0 & \text{on } \bigcup_{I \subset \mathbb{N}} \partial_{-, I, 2} \mathcal{S}_\kappa. \end{cases} \quad (2.5)$$

Define the impulse control  $\mathcal{T}^* = \{\tau^*(i), \zeta^*(i), i = 1, 2, \dots\}$  inductively as follows.

First put  $\tau^*(0) = 0$  and inductively

$$\tau^*(i+1) = \inf \{t > \tau^*(i) \mid (X^{(i)}(t), N_t^{(i)}, S(t), S_t) \notin D\}, \quad (2.6)$$

$$\zeta^*(i+1) = \hat{\zeta}(X^{(i)}(\tau^*(i+1)-), N_{\tau^*(i+1)-}^{(i)}, S(\tau^*(i+1)), S_{\tau^*(i+1)}), \quad (2.7)$$

$\{(X^{(i)}(t), N_t^{(i)}, S(t), S_t), t \geq 0\}$  is the controlled state process obtained by applying the combined control

$$\pi^*(i) = (c^*, (\tau^*(1), \tau^*(2), \dots, \tau^*(i); \zeta^*(1), \zeta^*(2), \dots, \zeta^*(i))), \quad i = 1, 2, \dots \quad (2.8)$$

Suppose  $\pi^* = (C^*, \mathcal{T}^*) \in \mathcal{U}_\kappa(x, \xi, \psi(0), \psi)$ ,

$$e^{-\delta t} \Phi(X^*(t), N_t^*, S(t), S_t) \rightarrow 0, \quad \text{as } t \rightarrow \infty \text{ a.s.} \quad (2.9)$$

and that the family

$$\{e^{-\delta \tau} \Phi(X^*(\tau), N_\tau^*, S(\tau), S_\tau) \mid \tau \text{ is a } \mathbf{G} - \text{stopping time}\} \quad (2.10)$$

is uniformly integrable. Then  $\Phi(x, \xi, \psi(0), \psi) = V_\kappa(x, \xi, \psi(0), \psi)$  and  $\pi^*$  obtained in (2.5)–(2.7) is optimal.

*Proof.* (a) Suppose  $\pi = (C, \mathcal{T}) \in \mathcal{U}_\kappa(x, \xi, \psi(0), \psi)$ , where  $C = \{C(t), t \geq 0\}$  is a consumption rate process and  $\mathcal{T} = \{\tau(i), \zeta(i), i = 1, 2, \dots\}$  is a trading strategy. Denote the controlled state processes (by  $\pi$ ) with the initial state by  $(x, \xi, \psi(0), \psi)$  by

$$\{Z(t) = (X(t), N_t, S(t), S_t), t \geq 0\}. \quad (2.11)$$

For  $R > 0$  put

$$T(R) = R \wedge \inf \{t > 0 \mid \|Z(t)\| \geq R\} \quad (2.12)$$

and set

$$\theta(i+1) = \theta(i+1; R) = \tau(i) \vee (\tau(i+1) \wedge T(R)), \quad (2.13)$$

where  $\|Z(t)\|$  is the norm of  $Z(t)$  in  $\mathfrak{R} \times \mathbf{N} \times \mathfrak{R} \times L^2_\rho$  in the product topology. Then by the generalized Dynkin's formula (see [1, Theorem 3.6]), we have

$$\begin{aligned} & E[e^{-\delta\theta(i+1)}\Phi(Z(\theta(i+1)-))] \\ &= E\left[e^{-\delta\tau(i)}\Phi(Z(\tau(i)))\right] + \int_{\tau(i)}^{\theta(i+1)-} e^{-\delta t}\mathcal{L}^{C(t)}\Phi(Z(t))dt \\ &\leq E\left[e^{-\delta\tau(i)}\Phi(Z(\tau(i)))\right] - E\left[\int_{\tau(i)}^{\theta(i+1)-} e^{-\delta t}\frac{C^\gamma(t)}{\gamma}dt\right], \quad \text{since } \tilde{\mathfrak{A}}\Phi \leq 0. \end{aligned} \tag{2.14}$$

Equivalently, we have

$$E\left[e^{-\delta\tau(i)}\Phi(Z(\tau(i)))\right] - E\left[e^{-\delta\theta(i+1)}\Phi(Z(\theta(i+1)-))\right] \geq E\left[\int_{\tau(i)}^{\theta(i+1)-} e^{-\delta t}\frac{C^\gamma(t)}{\gamma}dt\right]. \tag{2.15}$$

Letting  $R \rightarrow \infty$ , using the Fatou's lemma, and then summing from  $i = 0$  to  $i = k$  gives

$$\begin{aligned} & \Phi(x, \xi, \psi(0), \psi) + \sum_{i=1}^k E\left[e^{-\delta\tau(i)}(\Phi(Z(\tau(i))) - \Phi(Z(\tau(i)-)))\right] \\ & - E\left[e^{-\delta\tau(k+1)}\Phi(Z(\tau(k+1)-))\right] \geq E\left[\int_0^{\theta(k+1)} e^{-\delta t}\frac{C^\gamma(t)}{\gamma}dt\right]. \end{aligned} \tag{2.16}$$

Now

$$\Phi(Z(\tau(i))) \leq \mathcal{M}_\kappa\Phi(Z(\tau(i)-)) \quad \text{for } i = 1, 2, \dots \tag{2.17}$$

and therefore

$$\begin{aligned} & \Phi(x, \xi, \psi(0), \psi) + \sum_{i=1}^k E\left[e^{-\delta\tau(i)}(\mathcal{M}_\kappa\Phi(Z(\tau(i)-)) - \Phi(Z(\tau(i)-)))\right] \\ & \geq E\left[\int_0^{\theta(k+1)-} e^{-\delta t}\frac{C^\gamma(t)}{\gamma}dt + e^{-\delta\tau(k+1)}\Phi(Z(\tau(k+1)-))\right]. \end{aligned} \tag{2.18}$$

It is clear that

$$\mathcal{M}_\kappa\Phi(Z(\tau(i)-)) - \Phi(Z(\tau(i)-)) \leq 0 \tag{2.19}$$

and hence

$$\Phi(x, \xi, \psi(0), \psi) \geq E\left[\int_0^{\theta(k+1)-} e^{-\delta t}\frac{C^\gamma(t)}{\gamma}dt + e^{-\delta\tau(k+1)}\Phi(Z(\tau(k+1)-))\right]. \tag{2.20}$$

Letting  $k \rightarrow \infty$ , we get

$$\Phi(x, \xi, \psi(0), \psi) \geq E \left[ \int_0^\infty e^{-\delta t} \frac{C^\gamma(t)}{\gamma} dt \right], \quad (2.21)$$

since  $\Phi$  is a locally bounded nonnegative function.

Hence

$$\Phi(x, \xi, \psi(0), \psi) \geq J_\kappa(x, \xi, \psi(0), \psi; \pi) \quad \forall \pi \in \mathcal{U}_\kappa(x, \xi, \psi(0), \psi). \quad (2.22)$$

Therefore  $\Phi(x, \xi, \psi(0), \psi) \geq V_\kappa(x, \xi, \psi(0), \psi)$ .

(b) Next assume that (2.4) also holds. Define  $\pi^* = (C^*, \mathcal{T}^*)$ , where  $\mathcal{T}^* = \{(\tau^*(i), \zeta^*(i)), i = 1, 2, \dots\}$  by (2.5)–(2.7). Then repeat the argument in part (a) for  $\pi = \pi^*$ . By (2.10), the inequalities (2.20)–(2.22) become equalities. So we conclude that

$$\begin{aligned} & \Phi(x, \xi, \psi(0), \psi) \\ &= E \left[ \int_0^{\tau^*(k+1)} e^{-\delta t} \frac{C^\gamma(t)}{\gamma} dt + e^{-\delta \tau^*(k+1)} \Phi(Z(\tau^*(k+1) -)) \right] \quad \forall k = 1, 2, \dots \end{aligned} \quad (2.23)$$

Letting  $k \rightarrow \infty$  in (2.23), we get by (2.10)

$$\Phi(x, \xi, \psi(0), \psi) = J_\kappa(x, \xi, \psi(0), \psi; \pi^*). \quad (2.24)$$

Combining this with (2.22), we obtain

$$\begin{aligned} \Phi(x, \xi, \psi(0), \psi) &\geq \sup_{\pi \in \mathcal{U}_\kappa(x, \xi, \psi(0), \psi)} J_\kappa(x, \xi, \psi(0), \psi; \pi) \\ &\geq J_\kappa(x, \xi, \psi(0), \psi; \pi^*) = \Phi(x, \xi, \psi(0), \psi). \end{aligned} \quad (2.25)$$

Hence  $\Phi(x, \xi, \psi(0), \psi) = V_\kappa(x, \xi, \psi(0), \psi)$  and  $\pi^*$  is optimal. This proves the verification theorem.  $\square$

### 3. The viscosity solution

It is clear that the value function  $V_\kappa : \mathcal{I}_\kappa \rightarrow \mathfrak{R}_+$  has discontinuity on the interfaces  $Q_{I,+}$  and  $Q_{I,-}$  and hence it can not be a solution of QVHJBI(\*) in the classical sense. The main purpose of this section is to show that it is a viscosity solution of the QVHJBI(\*). See [6, 7] for connection of viscosity solutions of second-order elliptic equations with stochastic classical control and classical-impulse control problems.

To give a definition of a viscosity solution, we first define the upper and lower semicontinuity concept as follows.

Let  $\Xi$  be a metric space, and let  $\Phi : \Xi \rightarrow \mathfrak{R}$  be a Borel measurable function. Then the upper semicontinuous (USC) envelop  $\bar{\Phi} : \Xi \rightarrow \mathfrak{R}$  and the lower semicontinuous (LSC)

envelop  $\underline{\Phi} : \Xi \rightarrow \mathfrak{R}$  of  $\Phi$  are defined, respectively, by

$$\overline{\Phi}(\mathbf{x}) = \limsup_{\mathbf{y} \rightarrow \mathbf{x}, \mathbf{y} \in \Xi} \Phi(\mathbf{y}), \quad \underline{\Phi}(\mathbf{x}) = \liminf_{\mathbf{y} \rightarrow \mathbf{x}, \mathbf{y} \in \Xi} \Phi(\mathbf{y}). \quad (3.1)$$

We let USC( $\Xi$ ) and LSC( $\Xi$ ) denote the set of USC and LSC functions on  $\Xi$ , respectively.

Note that in general, one has

$$\underline{\Phi} \leq \Phi \leq \overline{\Phi}, \quad (3.2)$$

and that  $\Phi$  is USC if and only if  $\Phi = \overline{\Phi}$ ,  $\Phi$  is LSC if and only if  $\Phi = \underline{\Phi}$ . In particular,  $\Phi$  is continuous if and only if

$$\underline{\Phi} = \Phi = \overline{\Phi}. \quad (3.3)$$

Let  $\mathcal{L}(\mathfrak{X} \times L_\rho^2)$  and  $(\mathfrak{X} \times L_\rho^2)^\dagger$  be the space of bounded linear and bilinear functionals equipped with the usual operator norms  $\|\cdot\|^*$  and  $\|\cdot\|^\dagger$ , respectively.

To define a viscosity solution, let us consider the following equation:

$$F(\mathbf{A}, \Gamma, \partial_x, V_\kappa, (x, \xi, \psi(0), \psi)) = 0 \quad \forall (x, \xi, \psi(0), \psi) \in \mathcal{S}_\kappa, \quad (3.4)$$

where

$$F : (\mathfrak{X} \times L_\rho^2)^\dagger \times \mathcal{L}(\mathfrak{X} \times L_\rho^2) \times \mathfrak{X} \times \mathfrak{X}^{\mathcal{S}_\kappa} \times \mathcal{S}_\kappa \rightarrow \mathfrak{R} \quad (3.5)$$

is defined by

$$F = \begin{cases} \max \{ \Lambda(\mathbf{A}, \Gamma, \partial_x, \Phi, (x, \xi, \psi(0), \psi)), (\mathcal{M}_\kappa \Phi - \Phi)(x, \xi, \psi(0), \psi) \} & \text{on } \mathcal{S}_\kappa^\circ, \\ \Lambda(\mathbf{A}, \Gamma, \partial_x, \Phi, (x, \xi, \psi(0), \psi)) & \text{on } \bigcup_{I \subset \mathbb{N}} \partial_{+, I, 2} \mathcal{S}_\kappa, \\ \Lambda^0(\mathbf{A}, \Gamma, \partial_x, \Phi, (x, \xi, \psi(0), \psi)) & \text{on } \bigcup_{I \subset \mathbb{N}} \partial_{-, I, 2} \mathcal{S}_\kappa, \\ (\mathcal{M} \Phi - \Phi)((x, \xi, \psi(0), \psi)) & \text{on } \bigcup_{I \subset \mathbb{N}} \partial_{I, 1} \mathcal{S}_\kappa, \end{cases}$$

$$\begin{aligned} \Lambda(\mathbf{A}, \Gamma, \partial_x, \Phi, (x, \xi, \psi(0), \psi)) &= \mathcal{A} \Phi(x, \xi, \psi(0), \psi), \\ \Lambda^0(\mathbf{A}, \Gamma, \partial_x, \Phi, (x, \xi, \psi(0), \psi)) &= \mathcal{L}^0 \Phi(x, \xi, \psi(0), \psi). \end{aligned} \quad (3.6)$$

Note that

$$\begin{aligned} F(\mathbf{A}, \Gamma, \partial_x, \Phi, (x, \xi, \psi(0), \psi)) &= \text{QVHJBI}(*), \\ \overline{F}(\mathbf{A}, \Gamma, \partial_x, \Phi, (x, \xi, \psi(0), \psi)) &= \max \{ \Lambda(\mathbf{A}, \Gamma, \partial_x, \Phi, (x, \xi, \psi(0), \psi)), (\mathcal{M}_\kappa \Phi - \Phi)(x, \xi, \psi(0), \psi) \} \quad \forall (x, \xi, \psi(0), \psi) \in \mathcal{S}_\kappa, \\ \underline{F}(\mathbf{A}, \Gamma, \partial_x, \Phi, (x, \xi, \psi(0), \psi)) &= F(\mathbf{A}, \Gamma, \partial_x, \Phi, (x, \xi, \psi(0), \psi)). \end{aligned} \quad (3.7)$$

*Definition 3.1.* (i) A function  $\Phi \in \text{USC}(\mathcal{S}_\kappa)$  is said to be a viscosity subsolution of (3.4) if for every function  $\Psi \in C_{\text{lip}}^{1,0,2,2}(\mathcal{S}_\kappa) \cap \mathcal{D}(\Gamma)$  and for every  $(x, \xi, \psi(0), \psi) \in \mathcal{S}_\kappa$  such that  $\Psi \geq \Phi$  on  $\mathcal{S}_\kappa$  and  $\Psi(x, \xi, \psi(0), \psi) = \Phi(x, \xi, \psi(0), \psi)$ ,

$$\overline{F}(\mathbf{A}, \Gamma, \partial_x, \Psi, (x, \xi, \psi(0), \psi)) \geq 0. \quad (3.8)$$

(ii) A function  $\Phi \in \text{LSC}(\mathcal{S}_\kappa)$  is a viscosity supersolution of (3.4) if for every function  $\Psi \in C_{\text{lip}}^{1,0,2,2}(\mathcal{S}_\kappa) \cap \mathcal{D}(\Gamma)$  and for every  $(x, \xi, \psi(0), \psi) \in \mathcal{S}_\kappa$  such that  $\Psi \leq \Phi$  on  $\mathcal{S}_\kappa$  and  $\Psi(x, \xi, \psi(0), \psi) = \Phi(x, \xi, \psi(0), \psi)$ ,

$$\underline{F}(\mathbf{A}, \Gamma, \partial_x, \Psi, (x, \xi, \psi(0), \psi)) \leq 0. \quad (3.9)$$

(iii) A locally bounded function  $\Phi : \mathcal{S}_\kappa \rightarrow \mathfrak{R}$  is a viscosity solution of (3.4) if  $\overline{\Phi}$  is viscosity subsolution and  $\underline{\Phi}$  is a viscosity supersolution of (3.4).

The following properties of the *intervention operator*  $\mathcal{M}_\kappa$  can be established similar to [7, Lemmas 3.2, 3.3, and Corollary 3.4] with some modifications to fit our situation.

LEMMA 3.2. *The following statements hold true regarding  $\mathcal{M}_\kappa$  defined by (1.38).*

- (i) *If  $\Phi : \mathcal{S}_\kappa \rightarrow \mathfrak{R}$  is USC, then  $\mathcal{M}_\kappa \Phi$  is USC.*
- (ii) *If  $\Phi : \mathcal{S}_\kappa \rightarrow \mathfrak{R}$  is continuous, then  $\mathcal{M}_\kappa \Phi$  is continuous.*
- (iii) *Let  $\Phi : \mathcal{S}_\kappa \rightarrow \mathfrak{R}$ . Then  $\overline{\mathcal{M}_\kappa \Phi} \leq \mathcal{M}_\kappa \overline{\Phi}$ .*
- (iv) *Let  $\Phi : \mathcal{S}_\kappa \rightarrow \mathfrak{R}$  be such that  $\Phi \geq \mathcal{M}_\kappa \Phi$ . Then  $\underline{\Phi} \geq \mathcal{M}_\kappa \underline{\Phi}$ .*
- (v) *Suppose  $\Phi : \mathcal{S}_\kappa \rightarrow \mathfrak{R}$  is USC and  $\Phi(x, \xi, \psi(0), \psi) > \mathcal{M}_\kappa \Phi(x, \xi, \psi(0), \psi) + \epsilon$  for some  $(x, \xi, \psi(0), \psi) \in \mathcal{S}_\kappa$  and  $\epsilon > 0$ . Then*

$$\Phi(x, \xi, \psi(0), \psi) > \overline{\mathcal{M}_\kappa \Phi}(x, \xi, \psi(0), \psi) + \epsilon. \quad (3.10)$$

*Proof.* We only need to prove (i). The conclusions (ii)–(v) are consequences of (i). Their proofs are very similar to that of [7, Lemmas 3.2–3.3 and Corollary 3.4] and are therefore omitted here. Let  $(x, \xi, \psi(0), \psi) \in \mathcal{S}_\kappa$  with  $I = \{i \in \mathfrak{N} \mid n(-i) < 0\}$  and  $I^c = \mathfrak{N} - I = \{i \in \mathfrak{N} \mid n(-i) \geq 0\}$ . Define

$$\begin{aligned} \mathcal{P}(x, \xi, \psi(0), \psi) &= \{(\hat{x}, \hat{\xi}, \hat{\psi}(0), \hat{\psi}) \in \mathcal{S}_\kappa \mid \zeta \in \mathcal{R}(\xi) - \{\mathbf{0}\}\} \\ &= \mathcal{P}_+(x, \xi, \psi(0), \psi) \cup \mathcal{P}_-(x, \xi, \psi(0), \psi), \end{aligned} \quad (3.11)$$

where

$$\begin{aligned} \mathcal{P}_+(x, \xi, \psi(0), \psi) &= \{(\hat{x}, \hat{\xi}, \hat{\psi}(0), \hat{\psi}) \in \mathcal{S}_\kappa \mid m(0) \geq 0, \\ &\quad 0 \leq m(-i) \leq -n(-i) \text{ for } i \in I - \{0\}; \\ &\quad \& -n(-i) \leq m(-i) \leq 0 \text{ for } i \in I^c - \{0\}\}, \\ \mathcal{P}_-(x, \xi, \psi(0), \psi) &= \{(\hat{x}, \hat{\xi}, \hat{\psi}(0), \hat{\psi}) \in \mathcal{S}_\kappa \mid m(0) < 0, \\ &\quad 0 \leq m(-i) \leq -n(-i) \text{ for } i \in I - \{0\}; \\ &\quad \& -n(-i) \leq m(-i) \leq 0 \text{ for } i \in I^c - \{0\}\}, \end{aligned} \quad (3.12)$$

and  $\hat{x}$  and  $\hat{\xi}$  are as defined in (1.39)-(1.40), and  $(\hat{\psi}(0), \hat{\psi}) = (\psi(0), \psi)$  due to the continuity and the uncontrollability (by the *investor*) of the stock prices. We claim that for each  $(x, \xi, \psi(0), \psi) \in \mathcal{S}_\kappa^\circ$  both  $\mathcal{P}_+(x, \xi, \psi(0), \psi)$  and  $\mathcal{P}_-(x, \xi, \psi(0), \psi)$  are compact subsets of  $\mathcal{S}_\kappa$ . To see this, we consider the following two cases.

*Case 1.*  $G_\kappa(x, \xi, \psi(0), \psi) \geq 0$ .

In this case,  $\mathcal{P}_+(x, \xi, \psi(0), \psi)$  intersects with the hyperplane  $\partial_{I-\{0\},1}\mathcal{S}_\kappa$  (since  $m(0) > 0$ ). By the facts that  $0 \leq m(-i) \leq -n(-i)$  for  $i \in I - \{0\}$  and  $n(-i) = 0$  for all but finitely many  $i \in \mathfrak{N}$  as required in (1.4). Therefore,  $\mathcal{P}_+(x, \xi, \psi(0), \psi)$  is compact.

*Case 2.*  $G_\kappa(x, \xi, \psi(0), \psi) < 0$ .

In this case,  $\mathcal{P}_+(x, \xi, \psi(0), \psi)$  is bounded by the set

$$\{(x, \xi, \psi(0), \psi) \mid G_\kappa(x, \xi, \psi(0), \psi)\} = 0 \tag{3.13}$$

and the boundary of  $\mathfrak{R}_+ \times \mathbf{N}_+ \times \mathfrak{R} \times L_{\rho,+}^2$ .

From Cases 1 and 2,  $\mathcal{P}_+(x, \xi, \psi(0), \psi)$  is a compact subset of  $\mathcal{S}_\kappa$ . We can also prove the compactness of  $\mathcal{P}_-(x, \xi, \psi(0), \psi)$  in a similar manner. Since  $\Phi$  is USC on  $\mathcal{P}(x, \xi, \psi(0), \psi)$ , there exists

$$(x^*, \xi^*, \psi^*(0), \psi^*) \in \mathcal{P}(x, \xi, \psi(0), \psi) \tag{3.14}$$

such that

$$\mathcal{M}_\kappa\Phi(x, \xi, \psi(0), \psi) = \sup \{\Phi(\hat{x}, \hat{\xi}, \hat{\psi}(0), \hat{\psi}) \mid \zeta \in \mathcal{R}(\xi) - \{\mathbf{0}\}\} = \Phi(x^*, \xi^*, \psi^*(0), \psi^*). \tag{3.15}$$

Fix  $(x^{(0)}, \xi^{(0)}, \psi^{(0)}(0), \psi^{(0)}) \in \mathcal{S}_\kappa$  and let  $\{(x^{(n)}, \xi^{(n)}, \psi^{(n)}(0), \psi^{(n)})\}_{n=1}^\infty$  be a sequence in  $\mathcal{S}_\kappa$  such that

$$(x^{(n)}, \xi^{(n)}, \psi^{(n)}(0), \psi^{(n)}) \longrightarrow (x^{(0)}, \xi^{(0)}, \psi^{(0)}(0), \psi^{(0)}) \quad \text{as } n \longrightarrow \infty. \tag{3.16}$$

To show that  $\mathcal{M}_\kappa\Phi$  is USC, we must show that

$$\begin{aligned} \mathcal{M}_\kappa\Phi(x^{(0)}, \xi^{(0)}, \psi^{(0)}(0), \psi^{(0)}) &\geq \limsup_{n \rightarrow \infty} \mathcal{M}_\kappa\Phi(x^{(n)}, \xi^{(n)}, \psi^{(n)}(0), \psi^{(n)}) \\ &= \limsup_{n \rightarrow \infty} \Phi(x^{(n)*}, \xi^{(n)*}, \psi^{(n)*}(0), \psi^{(n)*}). \end{aligned} \tag{3.17}$$

Let  $(\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi})$  be a cluster point of

$$\left\{ \left( x^{(n)*}, \xi^{(n)*}, \psi^{(n)*}(0), \psi^{(n)*} \right) \right\}_{n=1}^\infty, \tag{3.18}$$

that is,  $(\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi})$  is the limit of some of convergent subsequence

$$\left\{ \left( x^{(n_k)*}, \xi^{(n_k)*}, \psi^{(n_k)*}(0), \psi^{(n_k)*} \right) \right\}_{k=1}^\infty \quad \text{of} \quad \left\{ \left( x^{(n)*}, \xi^{(n)*}, \psi^{(n)*}(0), \psi^{(n)*} \right) \right\}_{n=1}^\infty. \tag{3.19}$$



Since

$$\left(x^{(n)}, \xi^{(n)}, \psi^{(n)}(0), \psi^{(n)}\right) \longrightarrow \left(x^{(0)}, \xi^{(0)}, \psi^{(0)}(0), \psi^{(0)}\right), \quad (3.20)$$

we see that

$$\mathcal{P}\left(x^{(n)}, \xi^{(n)}, \psi^{(n)}(0), \psi^{(n)}\right) \longrightarrow \mathcal{P}\left(x^{(0)}, \xi^{(0)}, \psi^{(0)}(0), \psi^{(0)}\right) \quad (3.21)$$

in Hausdorff distance. Hence, since

$$\left(x^{(n_k)*}, \xi^{(n_k)*}, \psi^{(n_k)*}(0), \psi^{(n_k)*}\right) \in \mathcal{P}\left(x^{(n_k)}, \xi^{(n_k)}, \psi^{(n_k)}(0), \psi^{(n_k)}\right) \quad (3.22)$$

for all  $k$ , we conclude that

$$\left(\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi}\right) = \lim_{k \rightarrow \infty} \left(x^{(n_k)*}, \xi^{(n_k)*}, \psi^{(n_k)*}(0), \psi^{(n_k)*}\right) \in \mathcal{P}\left(x^{(0)}, \xi^{(0)}, \psi^{(0)}(0), \psi^{(0)}\right). \quad (3.23)$$

Therefore,

$$\begin{aligned} \mathcal{M}_\kappa \Phi\left(x^{(0)}, \xi^{(0)}, \psi^{(0)}(0), \psi^{(0)}\right) &\geq \Phi\left(\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi}\right) \\ &\geq \limsup_{k \rightarrow \infty} \Phi\left(x^{(n_k)*}, \xi^{(n_k)*}, \psi^{(n_k)*}(0), \psi^{(n_k)*}\right) \\ &= \limsup_{n \rightarrow \infty} \mathcal{M}_\kappa \Phi\left(x^{(n)}, \xi^{(n)}, \psi^{(n)}(0), \psi^{(n)}\right). \end{aligned} \quad (3.24)$$

**THEOREM 3.3.** *Suppose  $\delta > r\gamma$ . Then the value function  $V_\kappa : \mathcal{S}_\kappa \rightarrow \mathfrak{R}_+$  defined by (1.28) is a viscosity solution of the QVHJBI(\*).*

The theorem can be proved by verifying the following two propositions: the first of which shows that the value function is a viscosity supersolution of the QVHJBI(\*), and the second shows that the value function is a viscosity subsolution of the QVHJBI(\*).

**PROPOSITION 3.4.** *The lower semicontinuous envelop  $\underline{V}_\kappa : \mathcal{S}_\kappa \rightarrow \mathfrak{R}_+$  of the value function  $V_\kappa$  is a viscosity supersolution of the QVHJBI(\*).*  $\square$

*Proof.* Let  $\Phi : \mathcal{S}_\kappa \rightarrow \mathfrak{R}$  be any smooth function with  $\Phi \in C_{\text{lip}}^{1,0,2,2}(\mathbb{O}) \cap \mathcal{D}(\Gamma)$  on a neighborhood  $\mathbb{O}$  of  $\mathcal{S}_\kappa$  and let  $(x, \xi, \psi(0), \psi) \in \mathcal{S}_\kappa$  be such that  $\Phi \leq \underline{V}_\kappa$  on  $\mathcal{S}_\kappa$  and  $\Phi(x, \xi, \psi(0), \psi) = \underline{V}_\kappa(x, \xi, \psi(0), \psi)$ . We need to prove that

$$\underline{F}(\mathbf{A}, \Gamma, \partial_x, \Phi, (x, \xi, \psi(0), \psi)) = F(\mathbf{A}, \Gamma, \partial_x, \Phi, (x, \xi, \psi(0), \psi)) \leq 0. \quad (3.25)$$

Note that by Lemma 3.2(iv),

$$\mathcal{M}_\kappa V_\kappa \leq V_\kappa \implies \mathcal{M}_\kappa \underline{V}_\kappa = \mathcal{M}_\kappa \Phi \leq \underline{V}_\kappa = \Phi \quad \text{on } \mathcal{S}_\kappa. \tag{3.26}$$

In particular, this inequality holds on  $\bigcup_{I \subset \mathbb{N}} \partial_{I,1} \mathcal{S}_\kappa$ . Therefore, we only need to show that

$$\mathcal{A}\Phi \leq 0 \quad \text{on } \mathcal{S}_\kappa^\circ \cup \bigcup_{I \subset \mathbb{N}} \partial_{+,I,2} \mathcal{S}_\kappa, \quad \mathcal{L}^0 \Phi \leq 0 \quad \text{on } \bigcup_{I \subset \mathbb{N}} \partial_{-,I,2} \mathcal{S}_\kappa. \tag{3.27}$$

For  $\epsilon > 0$ , let  $B(\epsilon) = B(\epsilon; (x, \xi, \psi(0), \psi))$  be the open ball in  $\mathcal{S}_\kappa$  centered at  $(x, \xi, \psi(0), \psi)$  and with radius  $\epsilon > 0$ . Let

$$\pi(\epsilon) = (C^\epsilon, \mathcal{T}^\epsilon) \in \mathcal{U}_\kappa(x, \xi, \psi(0), \psi) \tag{3.28}$$

be the admissible strategy beginning with a constant consumption rate  $C(t) = c \geq 0$  and no transactions up to the first time  $\tau(\epsilon)$  at which the controlled state process  $\{(X(t), N_t, S(t), S_t), t \geq 0\}$  exits from the set  $B(\epsilon)$ . Note that  $\tau(\epsilon) > 0$   $P$ -a.s. since there is no transaction and the controlled state process  $\{(X(t), N_t, S(t), S_t), t \geq 0\}$  is continuous on  $B(\epsilon)$ .

Choose  $(x^{(n)}, \xi^{(n)}, \psi^{(n)}(0), \psi^{(n)}) \in B(\epsilon)$  such that

$$\begin{aligned} (x^{(n)}, \xi^{(n)}, \psi^{(n)}(0), \psi^{(n)}) &\longrightarrow (x, \xi, \psi(0), \psi), \\ V_\kappa(x^{(n)}, \xi^{(n)}, \psi^{(n)}(0), \psi^{(n)}) &\longrightarrow \underline{V}_\kappa(x, \xi, \psi(0), \psi) \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.29}$$

Then by the Bellman's dynamic programming principle (DPP) (see [1, Proposition 4.1]), we have for all  $n$ ,

$$\begin{aligned} &V_\kappa(x^{(n)}, \xi^{(n)}, \psi^{(n)}(0), \psi^{(n)}) \\ &\geq E^{(n)} \left[ \int_0^{\tau(\epsilon)} e^{-\delta t} \frac{C^\epsilon}{\gamma} dt + e^{-\delta \tau(\epsilon)} \underline{V}_\kappa(X(\tau(\epsilon)), N_{\tau(\epsilon)}, S(\tau(\epsilon)), S_{\tau(\epsilon)}) \right] \\ &\geq E^{(n)} \left[ \int_0^{\tau(\epsilon)} e^{-\delta t} \frac{C^\epsilon}{\gamma} dt + e^{-\delta \tau(\epsilon)} \Phi(X(\tau(\epsilon)), N_{\tau(\epsilon)}, S(\tau(\epsilon)), S_{\tau(\epsilon)}) \right], \end{aligned} \tag{3.30}$$

where  $E^{(n)}[\dots]$  is the short notation of  $E^{x^{(n)}, \xi^{(n)}, \psi^{(n)}(0), \psi^{(n)}, \pi(\epsilon)}[\dots]$ , the conditional expectation given the initial state  $(x^{(n)}, \xi^{(n)}, \psi^{(n)}(0), \psi^{(n)})$  and the strategy  $\pi(\epsilon)$ . In particular, for all  $0 \leq t \leq \tau(\epsilon)$ ,

$$\begin{aligned} 0 &= \underline{V}_\kappa(x, \xi, \psi(0), \psi) - \Phi(x, \xi, \psi(0), \psi) \\ &= \lim_{n \rightarrow \infty} \left[ V_\kappa(x^{(n)}, \xi^{(n)}, \psi^{(n)}(0), \psi^{(n)}) - \Phi(x^{(n)}, \xi^{(n)}, \psi^{(n)}(0), \psi^{(n)}) \right] \\ &\geq \lim_{n \rightarrow \infty} E^{(n)} \left[ \int_0^t e^{-\delta s} \frac{C^\epsilon}{\gamma} ds + e^{-\delta t} V_\kappa(X(t), N_t, S(t), S_t) - \Phi(x^{(n)}, \xi^{(n)}, \psi^{(n)}(0), \psi^{(n)}) \right] \\ &= E \left[ \int_0^t e^{-\delta s} \frac{C^\epsilon}{\gamma} ds + e^{-\delta t} \Phi(X(t), N_t, S(t), S_t) - \Phi(x, \xi, \psi(0), \psi) \right]. \end{aligned} \tag{3.31}$$

Dividing both sides of the inequality by  $t$  and letting  $t \rightarrow 0$ , we have from Dynkin's formula [1, Theorem 3.6] that

$$\begin{aligned}
0 &\geq \lim_{t \rightarrow 0} \frac{1}{t} E \left[ \int_0^t e^{-\delta s} \frac{c^\gamma}{\gamma} ds + e^{-\delta t} \Phi(X(t), N_t, S(t), S_t) - \Phi(x, \xi, \psi(0), \psi) \right] \\
&= \frac{c^\gamma}{\gamma} + \lim_{t \rightarrow 0} \frac{1}{t} E \left[ e^{-\delta t} \Phi(X(t), N_t, S(t), S_t) - \Phi(x, \xi, \psi(0), \psi) \right] \\
&= \frac{c^\gamma}{\gamma} + \mathcal{L}^c \Phi(x, \xi, \psi(0), \psi),
\end{aligned} \tag{3.32}$$

where

$$\mathcal{L}^c \Phi(x, \xi, \psi(0), \psi) = (\mathbf{A} + \Gamma - \delta + rx\partial_x - c)\Phi(x, \xi, \psi(0), \psi). \tag{3.33}$$

We conclude from the above that

$$\mathcal{L}^c \Phi(x, \xi, \psi(0), \psi) + \frac{c^\gamma}{\gamma} \leq 0 \tag{3.34}$$

for all  $c \geq 0$  such that  $\pi(\epsilon) \in \mathcal{U}_\kappa(x, \xi, \psi(0), \psi)$  for  $\epsilon > 0$  small enough. This implies that

$$\mathcal{A}\Phi(x, \xi, \psi(0), \psi) \equiv \sup_{c \geq 0} \left( \mathcal{L}^c \Phi(x, \xi, \psi(0), \psi) + \frac{c^\gamma}{\gamma} \right) \leq 0. \tag{3.35}$$

If  $(x, \xi, \psi(0), \psi) \in \mathcal{S}_\kappa^\circ \cup \bigcup_{I \subset \mathbb{N}} \partial_{+, I, 2} \mathcal{S}_\kappa$ , then this is clearly the case for all  $c \geq 0$ , and therefore QVHJBI(\*) implies that  $\mathcal{A}\Phi(x, \xi, \psi(0), \psi) \leq 0$ . If  $(x, \xi, \psi(0), \psi) \in \bigcup_{I \subset \mathbb{N}} \partial_{-, I, 2} \mathcal{S}_\kappa$ , then the only such admissible  $c$  is  $c = 0$ . Therefore, we get  $\mathcal{L}^0 \Phi(x, \xi, \psi(0), \psi) \leq 0$  as required. This proves the proposition.  $\square$

**PROPOSITION 3.5.** *Suppose  $\delta > r\gamma$ . Then upper semicontinuous envelop  $\overline{V}_\kappa : \mathcal{S}_\kappa \rightarrow \mathfrak{R}$  of the value function  $V_\kappa$  is a viscosity subsolution of the QVHJBI(\*).*

*Proof.* We adopt the method (with appropriate modifications for the infinite dimensional case) provided in [7].

Suppose  $\pi = (C, \mathcal{T}) \in \mathcal{U}_\kappa(x, \xi, \psi(0), \psi)$ . Since  $\tau(1)$  is a  $\mathbf{G}$ -stopping time, the event  $\{\tau(1) = 0\}$  is  $\mathcal{G}(0)$ -measurable. By the zero-one law (see [8, Theorem 7.17] and [9, Lemma 9.2.6]), one has

$$\text{either } \tau(1) = 0 \text{ } P\text{-a.s. or } \tau(1) > 0 \text{ } P\text{-a.s.} \tag{3.36}$$

We first assume  $\tau(1) > 0$   $P$ -a.s. Then by the Markovian property, the cost functional  $J_\kappa(x, \xi, \psi(0), \psi; \pi)$  (see [10, Lemma 5.3]) satisfies the following relation: for  $P$ -a.s.  $\omega$ ,

$$e^{-\delta\tau} J_\kappa(X(\tau), N_\tau, S(\tau), S_\tau; \pi) = E \left[ \int_\tau^\infty e^{-\delta(s+\tau)} \frac{C^\gamma(s)}{\gamma} ds \mid \mathcal{G}(\tau) \right] (\omega). \tag{3.37}$$

Hence

$$J_\kappa(x, \xi, \psi(0), \psi; \pi) = E^{x, \xi, \psi(0), \psi; \pi} \left[ \int_0^\tau e^{-\delta s} \frac{C^\gamma(s)}{\gamma} ds + e^{-\delta \tau} J_\kappa(X(\tau), N_\tau, S(\tau), S_\tau; \pi) \right] \quad (3.38)$$

for all  $\mathbf{G}$ -stopping times  $\tau \leq \tau(1)$ . It is clear that

$$V_\kappa(x, \xi, \psi(0), \psi) \geq \mathcal{M}_\kappa V_\kappa(x, \xi, \psi(0), \psi) \quad \forall (x, \xi, \psi(0), \psi) \in \mathcal{S}_\kappa. \quad (3.39)$$

We will prove that  $\overline{V}_\kappa$ , the upper semicontinuous envelop of  $V_\kappa : \mathcal{S}_\kappa \rightarrow \mathfrak{R}_+$ , is a viscosity subsolution of QVHJBI(\*). To this end, let  $\Phi : \mathcal{S}_\kappa \rightarrow \mathfrak{R}$  be any smooth function with  $\Phi \in C_{\text{lip}}^{1,0,2,2}(\mathbb{O}) \cap \mathcal{D}(\Gamma)$  on a neighborhood  $\mathbb{O}$  of  $\mathcal{S}_\kappa$  and let  $(x, \xi, \psi(0), \psi) \in \mathcal{S}_\kappa$  be such that  $\Phi \geq \overline{V}_\kappa$  on  $\mathcal{S}_\kappa$  and  $\Phi(x, \xi, \psi(0), \psi) = \overline{V}_\kappa(x, \xi, \psi(0), \psi)$ . We need to prove that

$$\overline{F}(\mathbf{A}, \Gamma, \partial_x \Phi, \Phi, (x, \xi, \psi(0), \psi)) \geq 0. \quad (3.40)$$

We consider the following two cases separately.

*Case 3.*  $\overline{V}_\kappa(x, \xi, \psi(0), \psi) \leq \mathcal{M}_\kappa \overline{V}_\kappa(x, \xi, \psi(0), \psi)$ .

Then by inequality (3.8),

$$\overline{F}(\mathbf{A}, \Gamma, \partial_x \Phi, \Phi, (x, \xi, \psi(0), \psi)) \geq 0, \quad (3.41)$$

and hence the above inequality holds at  $(x, \xi, \psi(0), \psi)$  for  $\Phi = \overline{V}_\kappa$  in this case.

*Case 4.*  $\overline{V}_\kappa(x, \xi, \psi(0), \psi) > \mathcal{M}_\kappa \overline{V}_\kappa(x, \xi, \psi(0), \psi)$ .

In this case, it suffices to prove that  $\mathcal{A}\Phi(x, \xi, \psi(0), \psi) \geq 0$ . We argue by contradiction. Suppose  $(x, \xi, \psi(0), \psi) \in \mathcal{S}_\kappa$  and  $\mathcal{A}\Phi(x, \xi, \psi(0), \psi) < 0$ . Then from the definition of  $\mathcal{A}$ , we deduce that  $\partial_x \Phi(x, \xi, \psi(0), \psi) > 0$ . Hence by continuity,  $\partial_x \Phi > 0$  on a neighborhood  $G$  of  $(x, \xi, \psi(0), \psi)$ . But then

$$\mathcal{A}\Phi = (\mathbf{A} + \Gamma - \delta)\Phi + (rx - \hat{c})\partial_x \Phi + \frac{\hat{c}^\gamma}{\gamma} \quad (3.42)$$

with  $\hat{c} = \hat{c} = (\partial_x \Phi)^{1/(\gamma-1)}$  for all  $(\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi}) \in G \cap \mathcal{S}_\kappa$ .

Hence  $\mathcal{A}\Phi$  is continuous on  $G \cap \mathcal{S}_\kappa$  and so there exists a (bounded) neighborhood  $G(\lambda)$  of  $(x, \xi, \psi(0), \psi)$  such that

$$\begin{aligned} G(\lambda) &= G(x, \xi, \psi(0), \psi; \lambda) \\ &= \left\{ (\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi}) \mid |x - \tilde{x}| < \lambda, \|\xi - \tilde{\xi}\|_N < \lambda, \|(\psi(0), \psi) - (\tilde{\psi}(0), \tilde{\psi})\| < \lambda \right\} \end{aligned} \quad (3.43)$$

for some  $\lambda > 0$  and

$$\mathcal{A}\Phi(\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi}) < \frac{1}{2}\mathcal{A}\Phi(x, \xi, \psi(0), \psi) < 0 \quad \forall (\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi}) \in G(\lambda) \cap \mathcal{S}_\kappa. \quad (3.44)$$

Now, since  $\overline{V}_\kappa(x, \xi, \psi(0), \psi) > \mathcal{M}_\kappa \overline{V}_\kappa(x, \xi, \psi(0), \psi)$ , let  $\eta$  be any number such that

$$0 < \eta < (\overline{V}_\kappa - \mathcal{M}_\kappa \overline{V}_\kappa)(x, \xi, \psi(0), \psi). \quad (3.45)$$

Since  $\overline{V}_\kappa(x, \xi, \psi(0), \psi) > \mathcal{M}_\kappa \overline{V}_\kappa(x, \xi, \psi(0), \psi) + \eta$ , we can by Lemma 3.2(v) find a sequence  $\{(x^{(n)}, \xi^{(n)}, \psi^{(n)}(0), \psi^{(n)})\}_{n=1}^\infty \subset G(\lambda) \cap \mathcal{S}_\kappa$  such that

$$\begin{aligned} (x^{(n)}, \xi^{(n)}, \psi^{(n)}(0), \psi^{(n)}) &\longrightarrow (x, \xi, \psi(0), \psi), \\ V_\kappa(x^{(n)}, \xi^{(n)}, \psi^{(n)}(0), \psi^{(n)}) &\longrightarrow \overline{V}_\kappa(x, \xi, \psi(0), \psi) \quad \text{as } n \longrightarrow \infty \end{aligned} \quad (3.46)$$

and for all  $n \geq 1$

$$\mathcal{M}_\kappa V_\kappa(x^{(n)}, \xi^{(n)}, \psi^{(n)}(0), \psi^{(n)}) < V_\kappa(x^{(n)}, \xi^{(n)}, \psi^{(n)}(0), \psi^{(n)}) - \eta. \quad (3.47)$$

Choose  $\epsilon \in (0, \eta)$ . Since  $\overline{V}_\kappa(x, \xi, \psi(0), \psi) = \Phi(x, \xi, \psi(0), \psi)$ , we can choose  $K > 0$  (a positive integer) such that for all  $n \geq K$

$$\left| V_\kappa(x^{(n)}, \xi^{(n)}, \psi^{(n)}(0), \psi^{(n)}) - \Phi(x^{(n)}, \xi^{(n)}, \psi^{(n)}(0), \psi^{(n)}) \right| < \epsilon. \quad (3.48)$$

In the following, we fix  $n \geq K$  and put

$$(\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi}) = (x^{(n)}, \xi^{(n)}, \psi^{(n)}(0), \psi^{(n)}). \quad (3.49)$$

Let  $\tilde{\pi} = (\tilde{C}, \tilde{\mathcal{F}})$  with  $\tilde{\mathcal{F}} = \{(\tilde{\tau}(i), \tilde{\zeta}(i)), i = 1, 2, \dots\}$  be an  $\epsilon$ -optimal control for  $(\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi})$ , in the sense that

$$V_\kappa(\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi}) \leq J_\kappa(\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi}; \tilde{\pi}) + \epsilon. \tag{3.50}$$

We claim that  $\tilde{\tau}(1) > 0$   $P$ -a.s. If this were false, then  $\tilde{\tau}(1) = 0$   $P$ -a.s. by the zero-one law (see (3.36)).

Then the state process  $\{Z^{\tilde{\pi}}(t) \equiv (X^{\tilde{\pi}}(t), N_t^{\tilde{\pi}}, S^{\tilde{\pi}}(t), S_t^{\tilde{\pi}}), t \geq 0\}$  makes an immediate jump from  $(\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi})$  to some point  $(\hat{x}, \hat{\xi}, \hat{\psi}(0), \hat{\psi}) \in \mathcal{S}_\kappa$  according to (1.39)–(1.41), and hence by its definition (see (1.26))

$$J_\kappa(\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi}; \tilde{\pi}) = E^{x, \xi, \psi(0), \psi; \tilde{\pi}} [J_\kappa(\hat{x}, \hat{\xi}, \hat{\psi}(0), \hat{\psi}; \tilde{\pi})]. \tag{3.51}$$

Denoting the conditional expectation given the initial state  $(\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi})$  and the strategy  $\tilde{\pi}$  by  $\tilde{E}[\dots]$ , we have

$$\begin{aligned} V_\kappa(\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi}) &\leq J_\kappa(\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi}; \tilde{\pi}) + \epsilon = \tilde{E}[J_\kappa(\hat{x}, \hat{\xi}, \hat{\psi}(0), \hat{\psi}; \tilde{\pi})] + \epsilon \\ &\leq \tilde{E}[V_\kappa(\hat{x}, \hat{\xi}, \hat{\psi}(0), \hat{\psi})] + \epsilon \leq \mathcal{M}_\kappa[V_\kappa(\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi})] + \epsilon, \end{aligned} \tag{3.52}$$

which contradicts (3.45). We therefore conclude that  $\tilde{\tau}(1) > 0$   $P$ -a.s. Fix  $R > 0$  and define the  $\mathbf{G}$ -stopping time  $\tau$  by

$$\tau = \tau(\epsilon) = \tilde{\tau}(1) \wedge R \wedge \inf \{t > 0 \mid Z^{\tilde{\pi}}(t) \notin G(\lambda)\}. \tag{3.53}$$

By the Dynkin's formula (see [1, Theorem 3.6]), we have

$$\begin{aligned} \tilde{E}\left[e^{-\delta\tau}\Phi(Z^{\tilde{\pi}}(\tau))\right] &= \Phi(\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi}) + \tilde{E}\left[\int_0^\tau e^{-\delta t}\mathcal{L}^{\tilde{c}}\Phi(Z^{\tilde{\pi}}(t))dt\right] \\ &\quad + \tilde{E}\left[e^{-\delta\tau}\left(\Phi(Z^{\tilde{\pi}}(\tau)) - \Phi(Z^{\tilde{\pi}}(\tau-))\right)\right] \end{aligned} \tag{3.54}$$

or

$$\tilde{E}\left[e^{-\delta\tau}\Phi(Z^{\tilde{\pi}}(\tau-))\right] = \Phi((\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi})) + \tilde{E}\left[\int_0^\tau e^{-\delta t}\mathcal{L}^{\tilde{c}}\Phi(Z^{\tilde{\pi}}(t))dt\right]. \tag{3.55}$$

Since  $V_\kappa \geq \mathcal{M}_\kappa V_\kappa$ ,

$$\begin{aligned}
& V_\kappa(\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi}) \\
& \leq J_\kappa(\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi}; \tilde{\pi}) + \epsilon \\
& = \tilde{E} \left[ \int_0^\tau e^{-\delta t} \frac{\tilde{C}^y(t)}{y} dt + J_\kappa(Z^{\tilde{\pi}}(\tau); \tilde{\pi}) \right] + \epsilon \\
& \leq \tilde{E} \left[ \int_0^\tau e^{-\delta t} \frac{\tilde{C}^y(t)}{y} dt + e^{-\delta \tau} V_\kappa(Z^{\tilde{\pi}}(\tau)) \right] + \epsilon \\
& = \tilde{E} \left[ \int_0^\tau e^{-\delta t} \frac{\tilde{C}^y(t)}{y} dt + e^{-\delta \tau} \{ V_\kappa(Z^{\tilde{\pi}}(\tau-)) \chi_{\{\tau < \tilde{\tau}(1)\}} + \mathcal{M}_\kappa V_\kappa(Z^{\tilde{\pi}}(\tau-)) \chi_{\{\tau = \tilde{\tau}(1)\}} \} \right] + \epsilon \\
& \leq \tilde{E} \left[ \int_0^\tau e^{-\delta t} \frac{\tilde{C}^y(t)}{y} dt + e^{-\delta \tau} V_\kappa(Z^{\tilde{\pi}}(\tau-)) \right] + \epsilon \\
& = \tilde{E} \left[ \int_0^\tau e^{-\delta t} \frac{\tilde{C}^y(t)}{y} dt + e^{-\delta \tau} \Phi(Z^{\tilde{\pi}}(\tau-)) \right] + \epsilon \\
& = \Phi(\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi}) + \tilde{E} \left[ \int_0^\tau e^{-\delta t} \left( \mathcal{L}^{\tilde{c}} \Phi(Z^{\tilde{\pi}}(t)) + \frac{\tilde{C}^y(t)}{y} \right) dt \right] \\
& \leq V_\kappa(\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi}) + \tilde{E} \left[ \int_0^\tau e^{-\delta t} \mathcal{A} \Phi(Z^{\tilde{\pi}}(t)) dt \right] + 2\epsilon.
\end{aligned} \tag{3.56}$$

We conclude from this that

$$\tilde{E} \left[ \int_0^\tau e^{-\delta t} \mathcal{A} \Phi(Z^{\tilde{\pi}}(t)) dt \right] \geq -2\epsilon. \tag{3.57}$$

Note that one can deduce from (3.44) that

$$\tilde{E} \left[ \int_0^\tau e^{-\delta t} \mathcal{A} \Phi(Z^{\tilde{\pi}}(t)) dt \right] \leq \frac{1}{2\delta} \mathcal{A} \Phi(\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi}) (1 - \tilde{E}[e^{-\delta \tau}]). \tag{3.58}$$

We claim the following. □

LEMMA 3.6.

$$0 < E^{(n)}[e^{-\delta \tau(\epsilon)}] < 1 \quad \text{when } n \rightarrow \infty, \epsilon \rightarrow 0. \tag{3.59}$$

Note that if Lemma 3.6 is true, then inequality (3.57) contradicts inequality (3.44) if  $\epsilon$  is small enough. This contradiction proves that  $\mathcal{A} \Phi(x, \xi, \psi(0), \psi) \geq 0$  and hence

$$\bar{F}(\mathbf{A}, \Gamma, (\partial_x, \partial_{\psi(0)}), \Phi, (x, \xi, \psi(0), \psi)) \geq 0. \tag{3.60}$$

Therefore, to complete the proof of Proposition 3.5 we must verify Lemma 3.6.

*Proof of Lemma 3.6.* First note that for  $t < \tau$ , we have

$$dX(t) = (rX(t) - \tilde{C}(t))dt, \quad \text{for } 0 \leq t < \tau \leq \tilde{\tau}(1). \tag{3.61}$$

Consequently, for  $t < \tau$  we have by (1.18)

$$X(t) = X(0)e^{rt} - e^{rt} \int_0^t e^{-rs} \tilde{C}(s) ds \geq X(0) - \lambda, \tag{3.62}$$

and hence, with some constant  $K < \infty$ ,

$$\begin{aligned} \int_0^\tau e^{-\delta t} \frac{\tilde{C}^\gamma(t)}{\gamma} dt &\leq \frac{1}{\gamma} \left[ \int_0^\tau e^{-\delta t} \tilde{C}^\gamma(t) dt \right]^\gamma \left[ \int_0^\tau e^{(r\gamma - \delta)/(1-\gamma)t} dt \right]^{1-\gamma} \\ &\leq K(X(0)(1 - e^{-r\tau}) + \lambda e^{-r\tau})^\gamma, \end{aligned} \tag{3.63}$$

since  $r\gamma - \delta < 0$ . Combining this with (3.38), we get

$$V_\kappa(\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi}) - \epsilon J_\kappa(\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi}; \tilde{\pi}) \tag{3.64}$$

$$\begin{aligned} &\leq E \left[ \int_0^\tau e^{-\delta t} \frac{\tilde{C}^\gamma(t)}{\gamma} dt + e^{-\delta\tau} V_\kappa(Z^{\tilde{\pi}}(\tau)) \right] \\ &\leq E \left[ K(x - (x - \lambda)e^{-r\tau})^\gamma \right] + E \left[ e^{-\delta\tau} V_\kappa(Z^{\tilde{\pi}}(\tau-)) \chi_{\{\tilde{\tau}(1) > \tau\}} \right] \end{aligned} \tag{3.65}$$

$$\begin{aligned} &\quad + E \left[ e^{-\delta\tau} (V_\kappa(Z^{\tilde{\pi}}(\tau)) - V_\kappa(Z^{\tilde{\pi}}(\tau-))) \chi_{\{\tilde{\tau}(1) \leq \tau\}} \right] \\ &\leq E \left[ K(x - (x - \lambda)e^{-r\tau})^\gamma \right] + E \left[ e^{-\delta\tau} V_\kappa(Z^{\tilde{\pi}}(\tau-)) \chi_{\{\tilde{\tau}(1) > \tau\}} \right] \\ &\quad + E \left[ e^{-\delta\tau} \mathcal{M}_\kappa V_\kappa(Z^{\tilde{\pi}}(\tau-)) \chi_{\{\tilde{\tau}(1) \leq \tau\}} \right] \end{aligned} \tag{3.66}$$

$$\begin{aligned} &\leq E \left[ K(x - (x - \lambda)e^{-r\tau})^\gamma \right] + E \left[ e^{-\delta\tau} \chi_{\{\tilde{\tau}(1) > \tau\}} \right] \times \sup_{(\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi}) \in G(\lambda)} V_\kappa(\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi}) \\ &\quad + E \left[ e^{-\delta\tau} \chi_{\{\tilde{\tau}(1) \leq \tau\}} \right] \times \sup_{(\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi}) \in G(\lambda)} \mathcal{M}_\kappa V_\kappa(\tilde{x}, \tilde{\xi}, \tilde{\psi}(0), \tilde{\psi}). \end{aligned} \tag{3.67}$$

Now if there exists a sequence  $\epsilon_k \rightarrow 0$  and a subsequence

$$\{ (x^{(n_k)}, \xi^{(n_k)}, \psi^{(n_k)}(0), \psi^{(n_k)}) \} \quad \text{of } \{ (x^{(n)}, \xi^{(n)}, \psi^{(n)}(0), \psi^{(n)}) \} \tag{3.68}$$

such that

$$E^{(n_k)} [e^{-\delta\tau(\epsilon_k)}] \rightarrow 1 \quad \text{when } k \rightarrow \infty, \tag{3.69}$$

then

$$E [e^{-\delta\tau} \chi_{\{\tilde{\tau}(1) > \tau\}}] \rightarrow 0 \quad \text{when } k \rightarrow \infty, \tag{3.70}$$



so by choosing  $(x, \xi, \psi(0), \psi) = (x^{(n_k)}, \xi^{(n)}, \psi^{(n)}(0), \psi^{(n)})$ ,  $\tau = \tau(\epsilon_k)$  and letting  $k \rightarrow \infty$ , we obtain

$$\overline{V}_\kappa(x, \xi, \psi(0), \psi) \leq K\lambda^\gamma + \sup_{(x, \xi, \psi(0), \psi) \in G(\lambda)} \mathcal{M}_\kappa V_\kappa(x, \xi, \psi(0), \psi). \quad (3.71)$$

Hence by Lemma 3.2 and inequality (3.64)

$$\begin{aligned} \overline{V}_\kappa(x, \xi, \psi(0), \psi) &\leq \lim_{\lambda \rightarrow 0} (K\lambda^\gamma + \sup_{(x, \xi, \psi(0), \psi) \in G(\lambda)} \mathcal{M}_\kappa V_\kappa(x, \xi, \psi(0), \psi)) \\ &= \overline{\mathcal{M}_\kappa V_\kappa}(x, \xi, \psi(0), \psi) \leq \mathcal{M}_\kappa \overline{V}_\kappa(x, \xi, \psi(0), \psi) < V_\kappa(x, \xi, \psi(0), \psi) - \eta. \end{aligned} \quad (3.72)$$

This contradicts inequality (3.45). This contradiction proves the proposition and completes the proof that  $\overline{V}_\kappa$  is a viscosity subsolution.  $\square$

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