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Research Article

On a Class of Forward-Backward Stochastic Differential Systems in Infinite Dimensions

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We prove that a class of fully coupled forward-backward systems in infinite dimensions has a local unique solution. After studying the regularity property of the solution, we prove that for a peculiar class of systems arising in the theory of stochastic optimal control, the solution exists in arbitrary large time interval. Finally, we investigate the connection between the solution to the systems and a stochastic optimal control problem.

1. Introduction

The object of our study will be the following system:

$$\begin{gathered}\forall \tau \in [t,T] \subset [0,T],\\ X_\tau = x + \int_t^\tau AX_\sigma \, d\sigma + \int_t^\tau F(\sigma, X_\sigma, Y_\sigma, Z_\sigma)\, d\sigma + \int_t^\tau G(\sigma, X_\sigma, Y_\sigma)\, dW(\sigma),\\ Y_\tau = \Phi(X_T) + \int_\tau^T BY_\sigma \, d\sigma - \int_\tau^T \Psi(\sigma, X_\sigma, Y_\sigma, Z_\sigma)\, d\sigma - \int_\tau^T Z_\sigma \, dW(\sigma).\end{gathered} \tag{1.1}$$

In the above equation, X takes values in a real separable Hilbert space H, Y takes values in a real separable Hilbert space K, and T is a nonnegative real number. The operators A and B are generators of strongly continuous semigroups $\{e^{tA}\}$ and $\{e^{tB}\}$, respectively, in H and K. The functions F, G, Φ, and Ψ have values, respectively, in H, $L(\Xi, H)$, K, and K and they satisfy appropriate Lipschitz conditions.

It is very well known that already in the finite-dimensional case, the solvability of fully coupled systems, that is the case of systems (1.1), is particularly delicate. Indeed, see [1] and again [2], there are examples, when G is not invertible, in which there is no hope to get a solution in an arbitrarily large time interval. By the way, this class of systems has been widely studied (in the finite-dimensional case) in [3–5]; see also [2] for a systematic review on the subject and its applications to mathematical finance and stochastic control.

Coming back to our infinite-dimensional framework, we first recall that forward equations have been widely studied; see the books [6, 7] and the bibliography therein.

More recently, also backward stochastic equation, in infinite dimension of the form

$$Y_\tau = \eta + \int_\tau^T BY_\sigma \, d\sigma - \int_\tau^T \Psi(\sigma, Y_\sigma, Z_\sigma)\, d\sigma - \int_\tau^T Z_\sigma \, dW(\sigma), \tag{1.2}$$

where B is a linear unbounded operator, have been studied by several authors; see [8–12]. In particular, we will exploit some techniques described in [10] where existence and uniqueness results for (1.2) are obtained. Equations of this kind arise in the theory of nonlinear filtering, stochastic control, (see [13]) and in mathematical finance, (see [14, 15]).

The first part of the present paper is devoted to prove existence and uniqueness—for small enough time interval $[T-\delta, T]$—of a mild solution $\{(X_\tau, Y_\tau, Z_\tau) : \tau \in [t,T]\}$ of the fully coupled system (1.1) in every $[t,T] \subset [T-\delta, T]$. In proving such a result, we have separated the case when B is dissipative from the case when B is any generator of a C_0-semigroup since different regularity results for the solution can be proved. In the case when B is dissipative, also the regular dependence on the initial state is studied. The main tool is a fixed-point technique performed in a suitable space of stochastic processes. To the author's knowledge, this is a first attempt to solve a fully coupled system in the infinite-dimensional framework, allowing the unbounded operators to appear in both equations of the system. In Section 4.4, we provide an example in which our theory applies.

Although our main motivation is the novelty of the mathematical problem, we also give an example of an application to optimal control theory where the forward equation takes value in a Hilbert space H while the backward equation is one-dimensional. Note that even this simpler case was not covered by the existing literature.

We consider an infinite-dimensional stochastic control state equation of the form

$$\begin{gathered} dX_\tau^u = AX_\tau^u \, d\tau + F(\tau, X_\tau^u)\, d\tau + G(\tau, X_\tau^u) r(\tau, X_\tau^u, u_\tau)\, d\tau + G(\tau, X_\tau^u)\, dW_\tau, \quad \tau \in [t,T], \\ X_t^u = x \in H, \end{gathered} \tag{1.3}$$

where $r : [0,T] \times H \times U \to \Xi$, with U a real separable Hilbert space. The cost functional to be minimized is

$$J(t,x,u.) = \mathbb{E}\int_t^T l(\tau, X_\tau^u, u_\tau)\, d\tau + \mathbb{E}\Phi(X_T^u), \tag{1.4}$$

where $l : [0,T] \times H \times U \to \mathbb{R}$, over all admissible controls that will be processes $\{u_\tau, \tau \in [0,T]\}$ taking values in U. In [16], authors solve the optimal control problem in its *weak formulation*, that is, when the probability space and the noise process are allowed to change.

In this paper, under suitable assumptions on the Hamiltonian function Ψ, we will solve the same optimal control problem considered in a *strong formulation*, that is, when the probability space and the noise are prescribed.

We stress on the fact that in the existing literature, optimal control in strong formulation was found only for constant and nondegenerate G (sometimes with more general Hamiltonian); see [6, 7, 16–24]. Since we assume the coefficients just Lipschitz continuous, following [21], we replace the approach based on the Hamilton-Jacobi-Bellman equation with the analysis of a fully coupled forward-backward system.

We define the Hamiltonian function

$$\Psi(t,x,z) = \inf_{u\in U}\{l(t,x,u)+zr(t,x,u)\}, \quad t\in[0,T],\ x\in H,\ z\in\Xi^*, \tag{1.5}$$

and we assume that the infimum in (1.5) is attained at $\gamma(t,x,z)$, then we introduce the coupled system

$$\begin{gathered} dX_\tau = AX_\tau\,d\tau + F(\tau,X_\tau)\,d\tau + G(\tau,X_\tau)r(\tau,X_\tau,\gamma(\tau,X_\tau,Z_\tau))\,d\tau + G(\tau,X_\tau)\,dW_\tau, \quad \tau\in[t,T], \\ X_t = x, \\ dY_\tau = Z_\tau\,dW_\tau - l(\tau,X_\tau,\gamma(\tau,X_\tau,Z_\tau))\,d\tau \quad \tau\in[t,T], \\ Y_T = \Phi(X_T). \end{gathered} \tag{1.6}$$

We show that this system has a unique, global, predictable solution $\{(X_\tau,Y_\tau,Z_\tau):\tau\in[t,T]\}$ that takes value in H, $\mathbb{R}$, and Ξ^*, respectively. Then we can conclude that $u_\tau = \gamma(\tau,X_\tau,Z_\tau)$ is an optimal control (in the strong sense), the corresponding trajectory X^u coincides with the solution X of (1.6), and the optimal cost $V(t,x)$ is given by Y_t. We stress again the fact that in system (1.6), the forward equation is infinite-dimensional so it cannot be treated with the finite-dimensional theory developed in [20], and since it is strongly coupled this case cannot be covered by the theory studied in [16].

The paper is organized as follows. In Section 2, we set notation and assumption. In Section 3, we provide some preliminary results. In Section 4, we prove the local existence theorems, some regularity properties of the solution, and we provide an example. Finally in Section 5, we apply the previous result to the above-mentioned control problem.

2. Notation and assumptions

We are given three separable real Hilbert spaces H, K, and Ξ, endowed with their inner scalar products that we will denote by $(\cdot,\cdot)_H$, $(\cdot,\cdot)_K$, and $(\cdot,\cdot)_\Xi$, respectively. $L(\Xi,H)$, $L(H)=L(H,H)$ and $L(K)=L(K,K)$, as usual are, respectively, the Banach spaces of linear and bounded operators from Ξ to H, from H to H, and from K to K endowed by the usual norms.

$L_2(\Xi,H)$ denote the Hilbert space of Hilbert-Schmidt operators from Ξ to H, endowed with the Hilbert-Schmidt norm, that is, $|T|_{L_2(\Xi,H)} = (\sum_{i=1}^\infty |Te_i|_H^2)^{1/2}$ ($\{e_i : i\in\mathbb{N}\}$ being an orthonormal basis in Ξ).

The space $L_2(\Xi,K)$ is defined in the same way.

The cylindrical Wiener process. We fix a probability basis $(\Omega, \mathcal{F}, \mathbb{P})$. A cylindrical Wiener process with value in Ξ is a family $W(t)$, $t \geq 0$, of linear mappings $\Xi \to L^2(\Omega)$ such that

(i) for every $h \in \Xi$, $\{W(t)h, \ t \geq 0\}$ is a real (continuous) Wiener process;

(ii) for every $h, k \in \Xi$ and $t \geq 0$, $\mathbb{E}(W(t)h \cdot W(t)k) = t(h,k)_{\Xi}$.

We denote by $\mathcal{F}_t$ its natural filtration augmented with the set $\mathcal{N}$ of $\mathbb{P}$-null sets of $\mathcal{F}$. As it is well known, the filtration $\mathcal{F}_t$ satisfies the usual conditions. By $\mathbb{E}^{\mathcal{F}_t}$, or by $\mathbb{E}(\cdot \mid F_t)$, we denote the conditional expectation with respect to $\mathcal{F}_t$. Finally, by $\mathcal{P}$ we denote the predictable σ-field on $\Omega \times [0,T]$.

Some classes of stochastic processes. Let S be any separable Hilbert space, with scalar product $(\cdot,\cdot)_S$ and let $\mathcal{B}(S)$ be its Borel σ-field. The following classes of processes will be used in this work.

(1) $L^p_{\mathcal{P}}(\Omega \times (t,T); S)$, $t \in [0,T]$, denotes the subset of $L^p(\Omega \times (t,T); S)$, given by all equivalence classes admitting a predictable version. This space is endowed with the natural norm

$$|Y|^p_{L^p_{\mathcal{P}}(\Omega \times (t,T);S)} = \mathbb{E}\int_t^T |Y_s|^p_S \, ds. \tag{2.1}$$

(2) $L^p_{\mathcal{P}}(\Omega; L^2((t,T);S))$, $p \in [1,+\infty]$, $t \in [0,T]$, denotes the space of equivalence classes of processes Y, admitting a predictable version such that the norm

$$|Y|^p_{L^p_{\mathcal{P}}(\Omega;L^2((t,T);S))} = \mathbb{E}\left(\int_t^T |Y_s|^2_S \, ds\right)^{p/2} \tag{2.2}$$

is finite.

(3) $C_{\mathcal{P}}([t,T]; L^p(\Omega;S))$, $p \in [1,+\infty]$, $t \in [0,T]$, denotes the space of S-valued processes Y such that $Y : [t,T] \to L^p(\Omega, S)$ is continuous and Y has a predictable modification, endowed with the norm

$$|Y|^p_{C_{\mathcal{P}}([t,T];L^p(\Omega;S))} = \sup_{s\in[t,T]} \mathbb{E}|Y_s|^p_S. \tag{2.3}$$

Elements of $C_{\mathcal{P}}([t,T]; L^p(\Omega;S))$ are identified up to modification.

(4) $L^p_{\mathcal{P}}(\Omega; C([t,T];S))$, $p \in [1,+\infty]$, $t \in [0,T]$, denotes the space of predictable processes Y with continuous paths in S, such that the norm

$$|Y|^p_{L^p_{\mathcal{P}}(\Omega;C([t,T];S))} = \mathbb{E} \sup_{s\in[t,T]} |Y_s|^p_S \tag{2.4}$$

is finite. Elements of this space are defined up to indistinguishability.

(5) $L^p(\Omega, \mathcal{F}_t, \mathbb{P}; S)$, $p \in [1,+\infty]$, for all $t \in [0,T]$, denotes the space of $\mathcal{F}_t$-measurable random variables with values in H such that their pth moment is bounded.

Statement of the problem and general assumptions on the coefficients. We set the main assumptions on the coefficients of system (1.1).

Hypothesis 2.1. (i) $A : D(A) \subset H \to H$ is the infinitesimal generator of a C_0 semigroup $\{e^{tA} : t \geq 0\}$ in H. We define with M_A a positive constant such that

$$\sup_{t\in[0,T]} |e^{tA}|_{L(H)} \leq M_A. \tag{2.5}$$

(ii) $B : D(B) \subset K \to K$ is the infinitesimal generator of a C_0 semigroup $\{e^{tB} : t \geq 0\}$ in K. We define with M_B a positive constant such that

$$\sup_{t\in[0,T]} |e^{tB}|_{L(K)} \leq M_B. \tag{2.6}$$

(iii) The mappings $F : [0,T] \times H \times K \times L_2(\Xi,K) \to H$ and $\Psi : [0,T] \times H \times K \times L_2(\Xi,K) \to K$ are measurable. Moreover, there exist positive constants L and C such that

$$\begin{gathered}
|F(\sigma,x_1,y_1,z_1) - F(\sigma,x_2,y_2,z_2)|_H \leq L\Big(|x_1 - x_2|_H + |y_1 - y_2|_K + |z_1 - z_2|_{L_2(\Xi,K)}\Big),\\
|\Psi(\sigma,x_1,y_1,z_1) - \Psi(\sigma,x_2,y_2,z_2)|_K \leq L\Big(|x_1 - x_2|_H + |y_1 - y_2|_K + |z_1 - z_2|_{L_2(\Xi,K)}\Big),\\
|F(\sigma,x,y,z)|_H \leq C(1 + |x|_H + |y|_K + |z|_{L_2(\Xi,K)}),\\
|\Psi(\sigma,x,y,z)|_K \leq C(1 + |x|_H + |y|_K + |z|_{L_2(\Xi,K)}),
\end{gathered} \tag{2.7}$$

for every $\sigma \in [0,T]$, for every $x,x_1,x_2 \in H$, $y,y_1,y_2 \in K$, and $z,z_1,z_2 \in L_2(\Xi,K)$.

(iv) $G : [0,T] \times H \times K \to L(\Xi,H)$ is a mapping such that for all $v \in \Xi$, $Gv : [0,T] \times H \times K \to H$ is measurable and $e^{sA}G(t,x,y) \in L_2(\Xi,H)$ for every $s > 0$, $t \in [0,T]$, and every $x \in H$, $y \in K$. We assume that for some constant $L > 0$ and $\gamma \in [0,1/2[$,

$$\begin{gathered}
|e^{sA}G(\sigma,x_1,y_1) - e^{sA}G(\sigma,x_2,y_2)|_{L_2(\Xi,H)} \leq s^{-\gamma}L(|x_1 - x_2|_H + |y_1 - y_2|_K),\\
|e^{sA}G(\sigma,x,y)|_{L_2(\Xi,H)} \leq s^{-\gamma}L(1 + |x|_H + |y|_K),
\end{gathered} \tag{2.8}$$

for every $s > 0$, $\sigma \in [0,T]$, $x,x_1,x_2 \in H$, $y,y_1,y_2 \in K$.

(v) Φ is a mapping $H \to K$ measurable such that for some constant $L > 0$,

$$|\Phi(x_1) - \Phi(x_2)|_K \leq L|x_1 - x_2|_H \tag{2.9}$$

for every $x_1,x_2 \in H$.

Remark 2.2. Note that from Hypothesis 2.1(v), it follows that there exists a constant $C > 0$ such that

$$|\Phi(x)|_K \leq C(1 + |x|_H) \tag{2.10}$$

for every $x \in H$.

Next we provide the definition of a *mild solution* for the system.

Definition 2.3. Given $\xi \in L^2(\Omega,\mathcal{F}_t;H)$ and $T > 0$, for all $t \in [0,T]$, a mild solution of problem (1.1), considered in $[t,T]$, is a triple $(X,Y,Z) \in C_{\mathcal{P}}([t,T];L^2(\Omega;H)) \times C_{\mathcal{P}}([t,T];L^2(\Omega;K)) \times L^2_{\mathcal{P}}(\Omega \times [t,T];L_2(\Xi,K))$ such that the following holds:

$$\forall \tau \in [t,T] \subset [0,T],$$

$$\begin{aligned} X_\tau &= e^{(\tau-t)A}\xi + \int_t^\tau e^{(\tau-\sigma)A}F(\sigma,X_\sigma,Y_\sigma,Z_\sigma)\,d\sigma + \int_t^\tau e^{(\tau-\sigma)A}G(\sigma,X_\sigma,Y_\sigma)\,dW(\sigma), \\ Y_\tau &= e^{(T-\tau)B}\Phi(X_T) - \int_\tau^T e^{(\sigma-\tau)B}\Psi(\sigma,X_\sigma,Y_\sigma,Z_\sigma)\,d\sigma - \int_\tau^T e^{(\sigma-\tau)B}Z_\sigma\,dW(\sigma). \end{aligned} \tag{2.11}$$

We will prove the following result.

Theorem 2.4. *Assume Hypothesis 2.1 holds. Then there exists a positive T_0 such that for all $T \le T_0$, for every $t \in [0,T]$, and for every $\xi \in L^2(\Omega,\mathcal{F}_t,\mathbb{P};H)$, problem (1.1) has a unique mild solution (X,Y,Z) in $[t,T]$.*

Let us assume the following.

Hypothesis 2.5. We assume $B : D(B) \subset K \to K$ to be a dissipative operator, that is,

$$(By,y)_K \le 0 \quad \forall y \in D(B). \tag{2.12}$$

Then the following regularity result holds.

Theorem 2.6. *Assume Hypotheses 2.1 and 2.5 hold, then for every $p \ge 2$, there exists a positive $T_1 \le T_0$ such that for every $[t,T] \subset [0,T_1]$ and for every $\xi \in L^p(\Omega,\mathcal{F}_t,\mathbb{P};H)$, the mild solution belongs to $L^p_{\mathcal{P}}(\Omega;C([t,T];H)) \times L^p_{\mathcal{P}}(\Omega;C([t,T];K)) \times L^p_{\mathcal{P}}(\Omega;L^2((t,T);L_2(\Xi,K)))$.*

3. Preliminary results

In this section, we provide some auxiliary results on backward stochastic equations that we will need in the proof of Theorem 2.6 and in the proof of the regular dependence with respect to the initial datum. Given $\eta \in L^2(\Omega,\mathcal{F}_T,\mathbb{P};K)$ and $f \in L^2_{\mathcal{P}}(\Omega \times (0,T);K)$, let us consider the following equation, for all $t \in [0,T]$:

$$\begin{aligned} d_\tau Y_\tau &= -BY_\tau\,d\tau + f(\tau)d\tau + Z_\tau\,dW_\tau, \quad \tau \in [t,T], \\ Y_T &= \eta. \end{aligned} \tag{3.1}$$

We introduce a weaker notion of solution, in analogy to the case of forward equations; see [6, Chapter 6].

Definition 3.1. $(Y,Z) \in L^2_{\mathcal{P}}(\Omega,C([t,T];K)) \times L^2_{\mathcal{P}}(\Omega \times (t,T);L_2(\Xi,K))$ is a weak solution to problem (3.1) if for all $\xi \in D(B^*)$ and all $\tau \in [t,T]$,

$$\langle Y_\tau,\xi\rangle = \langle \eta,\xi\rangle_K + \int_\tau^T \langle Y_s,B^*\xi\rangle_K\,ds - \int_\tau^T \langle f(s),\xi\rangle_K\,ds - \int_\tau^T \langle Z_s,\xi\rangle_K\,dW_s \quad \mathbb{P}\text{-a.s.} \tag{3.2}$$

Since we are dealing only with the backward equation, we can ask the mild solution to be more regular; see also [25, Remark 4.7].

Definition 3.2. A mild solution to (3.1) is a couple of processes (Y,Z) that belongs to $L^2_{\mathcal{P}}(\Omega, C([t,T];K)) \times L^2_{\mathcal{P}}(\Omega \times (t,T); L_2(\Xi,K))$ such that for all $\tau \in [t,T]$,

$$Y_\tau = e^{(T-\tau)B}\eta - \int_\tau^T e^{(s-\tau)B} f(s)\,ds - \int_\tau^T e^{(s-\tau)B} Z_s\,dW_s \quad \mathbb{P}\text{-a.s.} \tag{3.3}$$

The following result holds.

Lemma 3.3. *For every $t \in [0,T]$, (3.1) has a unique mild solution in $[t,T]$. Moreover, the couple (Y,Z) is a weak solution to problem (3.1).*

Proof. The existence and uniqueness of the mild solution is proven in [25, Theorem 4.4], see also [10, Lemma 2.1]. The proof of the second part of the statement follows exactly with the same arguments used in [6, Chapter 5, Section 5.2] where the case of the forward stochastic differential equation is treated. □

We are interested in proving the following regularity result.

Proposition 3.4. *Let $\eta \in L^p(\Omega, \mathcal{F}_T, \mathbb{P}; K)$ and $f \in L^p_{\mathcal{P}}(\Omega; L^2((0,T);K))$ with $p \geq 2$. Assume Hypotheses 2.1(ii) and 2.5 hold, then the mild solution of problem (3.1), for every $t \in [0,T]$, has the following regularity:*

$$(Y,Z) \in L^p_{\mathcal{P}}(\Omega; C([t,T];K)) \times L^p_{\mathcal{P}}(\Omega; L^2((t,T); L_2(\Xi,K))). \tag{3.4}$$

Moreover, the following estimates hold:

$$\mathbb{E} \sup_{s\in[t,T]} |Y_s|_K^p + \mathbb{E}\left(\int_t^T |Z_s|^2_{L_2(\Xi,K)}\,ds\right)^{p/2} \leq C\left[\mathbb{E}|\eta|_K^p + \mathbb{E}\left(\int_t^T |f(s)|_K\,ds\right)^p\right], \tag{3.5}$$

where C is a constant that depends on p, M_B, and T.

Proof. We will separate the proof into two parts and we will consider only the case when $t = 0$, the procedure being identical for all $t \in [0,T]$.
Step 1. Estimate for Y. Since $Y_\tau = \mathbb{E}[e^{(T-\tau)B}\eta - \int_\tau^T e^{(s-\tau)B} f(s)\,ds \mid \mathcal{F}_\tau]$, see [10, Lemma 2.1], one has that for every $p \geq 2$,

$$\begin{aligned}
&\mathbb{E} \sup_{\tau\in[0,T]} |Y_\tau|_K^p \\
&\quad \leq C_p\left[\mathbb{E} \sup_{\tau\in[0,T]} \left|\mathbb{E}[e^{(T-\tau)B}\eta \mid \mathcal{F}_\tau]\right|^p + \mathbb{E} \sup_{\tau\in[0,T]} \left|\mathbb{E}\left[\int_\tau^T e^{(s-\tau)B} f(s)\,ds \mid \mathcal{F}_\tau\right]\right|^p\right] \\
&\quad \leq C_p M_B^p\left[\mathbb{E}|\eta|_K^p + \mathbb{E}\left(\int_0^T |f(s)|_K\,ds\right)^p\right].
\end{aligned} \tag{3.6}$$

Here C_p depends only on p. Notice that in this step we do not need Hypothesis 2.5 that instead plays a fundamental role in estimating the process Z.

Step 2. Estimate for Z. We introduce the bounded operators $J_n \doteq n(nI - B)^{-1}$ for every $n \in \mathbb{N}^*$. It is very well known, see for instance [6, Appendix A], that

(i) for every $x \in K$, $J_n x \to x$ in K;

(ii) $|J_n|_{L(K)} \leq 1$;

(iii) for every n and every $s \in [0,T]$, $e^{sB}J_n x = J_n e^{sB}x$, for all $x \in K$.

Let us multiply each term in (3.1) by J_n and set $Y^n = J_n Y$, $f^n = J_n f$, $\eta^n = J_n \eta$, and $Z^n = J_n Z$. One has that the following equation is verified by (Y^n, Z^n) for every $\tau \in [0,T]$:

$$Y_\tau^n = e^{(T-\tau)B}\eta^n - \int_\tau^T e^{(s-\tau)B} f^n(s)ds - \int_\tau^T e^{(s-\tau)B} Z_s^n \, dW_s \quad \mathbb{P}\text{-a.s.} \tag{3.7}$$

That is, (Y^n, Z^n) is the unique mild solution to

$$\begin{gathered} d_\tau Y_\tau^n = -BY_\tau^n \, d\tau + f^n(\tau)d\tau + Z_\tau^n \, dW_\tau, \quad \tau \in [0,T], \\ Y_T^n = \eta^n. \end{gathered} \tag{3.8}$$

Moreover, by the previous lemma we know that (Y^n, Z^n) is the unique weak solution of problem (3.8) and since $Y^n \in D(B)$, we have that for every $\xi \in D(B^*)$,

$$\langle Y_\tau^n, \xi \rangle_K = \langle \eta^n, \xi \rangle_K + \int_\tau^T \langle BY_s^n, \xi \rangle_K \, ds - \int_\tau^T \langle f^n(s), \xi \rangle_K \, ds - \int_\tau^T \langle Z_s^n, \xi \rangle_K \, dW_s \quad \mathbb{P}\text{-a.s.} \tag{3.9}$$

Therefore extending by density (3.9) to all $\xi \in K$, we obtain that (Y^n, Z^n) is a strong solution.

From Step 1, we also know that $Y^n \in L^p(\Omega; C([0,T],K))$ since $|Y_s^n|_K \leq |Y_s|_K$. Now we are in the position to apply the Itô formula to $|Y^n|_K^2$. We introduce the following sequence of stopping times:

$$\tau_n^m = \inf\left\{t : \int_0^t |Z_s^n|^2_{L_2(\Xi,K)} \geq m\right\} \wedge T. \tag{3.10}$$

Note that since $Z^n \in L^2_{\mathcal{P}}(\Omega \times (0,T); L_2(\Xi,K))$, then at every fixed $n \geq 1$,

$$\lim_{m \to +\infty} \tau_n^m = T \quad \mathbb{P}\text{-a.s.} \tag{3.11}$$

For fixed m and n, we have

$$\begin{aligned} &|Y_0^n|_K^2 + \int_0^{\tau_n^m} |Z_s^n|^2_{L_2(\Xi,K)} \, ds \\ &\quad = |Y_{\tau_n^m}^n|_K^2 + 2\int_0^{\tau_n^m} \langle BY_s^n, Y_s^n \rangle_K \, ds - 2\int_0^{\tau_n^m} \langle f^n(s), Y_s^n \rangle_K \, ds - 2\int_0^{\tau_n^m} \langle Z_s^n \, dW(s), Y_s^n \rangle_K. \end{aligned} \tag{3.12}$$

Therefore, since B is dissipative, one has for some constant c_p depending only on p that

$$
\begin{aligned}
&|Y_0^n|_K^2 + \int_0^{\tau_n^m} |Z_s^n|^2_{L_2(\Xi,K)}\, ds \\
&\quad = |Y_{\tau_n^m}^n|_K^2 + 2\int_0^{\tau_n^m} \langle BY_s^n, Y_s^n\rangle_K\, ds - 2\int_0^{\tau_n^m} \langle f^n(s), Y_s^n\rangle_K\, ds - 2\int_0^{\tau_n^m} \langle Z_s^n\, dW(s), Y_s^n\rangle_K \\
&\quad \leq c_p\left(\sup_{s\in[0,T]} |Y_s^n|_K^2 + \left(\int_0^{\tau_n^m} |Y_s^n|_K\, |f^n(s)|_K\, ds\right) + \left|\int_0^{\tau_n^m} \langle Z_s^n\, dW(s), Y_s^n\rangle_K\right|\right).
\end{aligned}
\tag{3.13}
$$

As a consequence of BDG inequality, for some constant κ_p depending on p, one has

$$
\begin{aligned}
\mathbb{E}\left|\int_0^{\tau_n^m} \langle Z_s^n\, dW(s), Y_s^n\rangle_K\right|^{p/2} &\leq \kappa_p \mathbb{E}\left(\int_0^{\tau_n^m} |Z_s^n|^2_{L_2(\Xi,K)} |Y_s^n|^2\, ds\right)^{p/4} \\
&\leq \kappa_p \mathbb{E} \sup_{s\in[0,T]} |Y_s^n|_K^{p/2}\left(\int_0^{\tau_n^m} |Z_s^n|^2_{L_2(\Xi,K)}\, ds\right)^{p/4}.
\end{aligned}
\tag{3.14}
$$

Therefore, one gets that there exists a constant C_p depending on p, such that

$$
\begin{aligned}
&\mathbb{E}\left(\int_0^{\tau_n^m} |Z_s^n|^2_{L_2(\Xi,K)}\, ds\right)^{p/2} \\
&\quad \leq C_p \mathbb{E} \sup_{s\in[0,T]} |Y_s^n|_K^p + C_p\left(\int_0^{\tau_n^m} |f^n(s)|_K\, ds\right)^p + \frac{1}{2}\mathbb{E}\left(\int_0^{\tau_n^m} |Z_s^n|^2_{L_2(\Xi,K)}\, ds\right)^{p/2}.
\end{aligned}
\tag{3.15}
$$

Thus Fatou's lemma and property (ii) of J_n imply that letting m tend to infinity,

$$
\begin{aligned}
\mathbb{E}\left(\int_0^T |Z_s^n|^2_{L_2(\Xi,K)}\, ds\right)^{p/2} &\leq 2C_p\left[\mathbb{E} \sup_{s\in[0,T]} |Y_s^n|_K^p + \mathbb{E}\left(\int_0^T |f^n(s)|_K\, ds\right)^p\right] \\
&\leq 2C_p\left[\mathbb{E} \sup_{s\in[0,T]} |Y_s|_K^p + \mathbb{E}\left(\int_0^T |f(s)|_K\, ds\right)^p\right].
\end{aligned}
\tag{3.16}
$$

Since $\lim_{n\to+\infty} Z_s^n(\omega) = Z_s(\omega)$ $\mathbb{P}$-a.s. for a.e. $s \in [0,T]$, again by Fatou's lemma we have that letting n tend to infinity,

$$
\mathbb{E}\left(\int_0^T |Z_s|^2_{L_2(\Xi,K)}\, ds\right)^{p/2} \leq 2C_p \mathbb{E}\left[\sup_{s\in[0,T]} |Y_s|_K^p + \mathbb{E}\left(\int_0^T |f(s)|_K\, ds\right)^p\right].
\tag{3.17}
$$

Combining this last inequality with (3.6), we conclude the proof of the proposition. Similar estimates in the finite-dimensional case, holding also for $p \in (1,2)$ and more general f, can be found in [26, Lemma 3.1]. □

Now let us consider the following generalization to (3.3):

$$d_\tau Y_\tau = -BY_\tau\, d\tau + f(\tau, Y_\tau, Z_\tau)\, d\tau, + Z_\tau\, dW_\tau \quad \tau \in [t,T], \qquad Y_T = \eta. \tag{3.18}$$

The definition of mild solution of (3.18) is the obvious extension of Definition 3.2. We can prove the following.

Proposition 3.5. *Besides Hypotheses 2.1(ii) and 2.5, assume that for some constant $L > 0$ such that*

$$|f(s,y,z) - f(s,y_1,z_1)| \le L(|y - y_1|_K + |z - z_1|_{L_2(\Xi,K)}) \tag{3.19}$$

for every $s \in [0,T]$, $y, y_1 \in K$, and $z, z_1 \in L_2(\Xi,K)$, and that

$$\mathbb{E}\Big(\int_t^T |f(s,0,0)|_K\, ds\Big)^p < +\infty, \tag{3.20}$$

then (3.18) has a unique mild solution in $[t,T]$ such that

$$\mathbb{E}\sup_{s\in[t,T]} |Y_s|_K^p + \mathbb{E}\Big(\int_t^T |Z_s|^2_{L_2(\Xi,K)}\, ds\Big)^{p/2} \le C\Big[\mathbb{E}|\eta|_K^p + \mathbb{E}\Big(\int_t^T |f(s,0,0)|_K\, ds\Big)^p\Big], \tag{3.21}$$

where C is a constant the depends on p, M_B, and T.

Proof. Take $t = 0$ being the procedure identical for any t. We set

$$\mathcal{Y} = L^p_{\mathcal{P}}(\Omega; C([0,\delta];K)) \times L^p_{\mathcal{P}}(\Omega; L^2((0,\delta); L_2(\Xi,K))), \tag{3.22}$$

where $\delta > 0$ will be chosen later in the proof. Let $\Gamma : \mathcal{Y} \to \mathcal{Y}$ be the map such that for any $(Y,Z) \in \mathcal{Y}$, $\Gamma(Y,Z) = (Y^1, Z^1)$ is the mild solution to

$$Y^1_\tau = e^{(T-\tau)B}\eta - \int_\tau^T e^{(s-\tau)B} f(s, Y_s, Z_s)\, ds - \int_\tau^T e^{(s-\tau)B} Z^1_s\, dW_s \quad \mathbb{P}\text{-a.s.} \tag{3.23}$$

that exists by Proposition 3.4. We will prove that Γ is a contraction in $\mathcal{Y}$ for sufficiently small δ. Let us denote $(V^1, W^1) = \Gamma(V, W)$. By Proposition 3.4 and from the hypothesis on f, we have that

$$\begin{aligned} &\mathbb{E}\sup_{s\in[0,\delta]} |Y^1_s - V^1_s|_K^p + \mathbb{E}\Big(\int_0^\delta |Z_s - Z^1_s|^2_{L_2(\Xi,K)}\, ds\Big)^{p/2} \\ &\quad\le C\Big[\mathbb{E}\Big(\int_0^\delta |f(s,Y_s,Z_s) - f(s,V_s,W_s)|_K\, ds\Big)^p\Big] \\ &\quad\le CL^p\Big[\delta^{p-1}\mathbb{E}\sup_{s\in[0,\delta]} \nu |Y_s - V_s|_K^p + \delta^{p/2}\mathbb{E}\Big(\int_0^\delta |Z_s - W_s|^2_{L_2(\Xi,K)}\Big)^{p/2}\Big], \end{aligned} \tag{3.24}$$

where the constant C depends on M_B and p. Thus we can choose δ small enough such that Γ is a contraction and we find a unique fixed point $(\overline{Y},\overline{Z})$ solution to our equation in $[0,\delta]$. Since δ depends only on M_B, L, and p, we can repeat the same procedure in $[\delta,2\delta]$ and so on in order to cover the whole interval $[0,T]$. Thus we have obtained a solution $(\overline{Y},\overline{Z})$ in $[0,T]$, patching together the solutions obtained in every interval. The uniqueness follows from the local uniqueness in each interval of length δ. It remains to show the estimate. Since Γ is a contraction in $[0,\delta]$, we have

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,\delta]}|\overline{Y}_s|_K^p+\mathbb{E}\left(\int_0^\delta|\overline{Z}_s|^2_{L_2(\Xi,K)}ds\right)^{p/2}\\ \le 2\|\Gamma(0,0)\|^p_{\mathcal{Y}}\le 2C\left[\mathbb{E}|\eta|_K^p+\mathbb{E}\left(\int_0^\delta|f(s,0,0)|_K\,ds\right)^p\right].\end{aligned}\tag{3.25}$$

In the second interval $[\delta,2\delta]$, the solution $(\overline{Y},\overline{Z})$ is again the fixed point of a contraction map, thus

$$\begin{aligned}\mathbb{E}\sup_{s\in[\delta,2\delta]}|\overline{Y}_s|_K^p+\mathbb{E}\left(\int_\delta^{2\delta}|\overline{Z}_s|^2_{L_2(\Xi,K)}ds\right)^{p/2}&\le 2C\left[\mathbb{E}|Y_\delta|_K^p+\mathbb{E}\left(\int_\delta^{2\delta}|f(s,0,0)|_K\,ds\right)^p\right]\\ &\le \widetilde{C}\left[\mathbb{E}|\eta|_K^p+\mathbb{E}\left(\int_0^{2\delta}|f(s,0,0)|_K\,ds\right)^p\right],\end{aligned}\tag{3.26}$$

where $\widetilde{C}$ is a constant depending on known parameters. Iterating this procedure in a finite number of intervals until we cover the whole interval $[0,T]$, we get estimate (3.21) and we conclude the proof. □

4. Proofs of theorems

We will prove first Theorem 2.6 and then Theorem 2.4.

4.1. Proof of Theorem 2.6.

We set

$$\mathcal{H}_t^T \doteq L^p_{\mathcal{P}}(\Omega,C([t,T],K))\times L^p_{\mathcal{P}}(\Omega;L^2((t,T);L_2(\Xi,K)))\tag{4.1}$$

for an arbitrary $p>2/(1-2\gamma)$, where γ was introduced in Hypothesis 2.1(iv). For simplicity, we take $t=0$, the procedure being identical for all $t\in[0,T]$. We define the map $\Gamma:\mathcal{H}_0^T\to\mathcal{H}_0^T:(Y,Z)\to(\overline{Y},\overline{Z})$ in two steps.

(1) For $\xi\in L^p(\Omega,\mathcal{F}_t,\mathbb{P};H)$ fixed, we define the process $\{\overline{X}_\tau:\tau\in[0,T]\}$ that depends on Y and Z as the solution to

$$\overline{X}_\tau=e^{\tau A}\xi+\int_0^\tau e^{(\tau-\sigma)A}F(\sigma,\overline{X}_\sigma,Y_\sigma,Z_\sigma)\,d\sigma+\int_0^\tau e^{(\tau-\sigma)A}G(\sigma,\overline{X}_\sigma,Y_\sigma)\,dW(\sigma),\quad \tau\in[0,T],\tag{4.2}$$

in [16, Proposition 3.2] it is shown that this equation has a unique solution $\overline{X}\in L^p_{\mathcal{P}}(\Omega;C([0,T];H))$ at every fixed $(Y,Z)\in\mathcal{H}_0^T$.

(2) Given $\overline{X}$, that depends on (Y,Z), we define $(\overline{Y},\overline{Z})$ as the solution to

$$\overline{Y}_\tau = e^{(T-\tau)B}\Phi(\overline{X}_T) - \int_\tau^T e^{(\sigma-\tau)B}\Psi(\sigma,\overline{X}_\sigma,\overline{Y}_\sigma,\overline{Z}_\sigma)\,d\sigma - \int_\tau^T e^{(\sigma-\tau)B}\overline{Z}_\sigma\,dW(\sigma), \quad \tau\in[0,T]. \tag{4.3}$$

The existence and uniqueness of a solution $(\overline{Y},\overline{Z})$ in $\mathcal{H}_0^T$, at every fixed $\overline{X} \in L^p_{\mathcal{P}}(\Omega;C([0,T];H))$, are proven in [16, Proposition 4.3].

We will prove that there exists a T_1 that depends only on L, M_A, and M_B such that for any $T \le T_1$, the map $\Gamma : \mathcal{H}_0^T \to \mathcal{H}_0^T$ is a contraction.

We start from the forward equation, we have

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[0,T]}|\overline{X}_s-\overline{U}_s|_H^p \\
&\quad\le c_p\Bigg[\mathbb{E}\sup_{\tau\in[0,T]}\left|\int_0^\tau e^{(\tau-\sigma)A}\big(F(\sigma,\overline{X}_\sigma,Y_\sigma,Z_\sigma)-F(\sigma,\overline{U}_\sigma,V_\sigma,W_\sigma)\big)\,d\sigma\right|^p \\
&\qquad+\mathbb{E}\sup_{\tau\in[0,T]}\left|\int_0^\tau e^{(\tau-\sigma)A}\big(G(\sigma,\overline{X}_\sigma,Y_\sigma)-G(\sigma,\overline{U}_\sigma,V_\sigma)\big)\,dW(\sigma)\right|^p\Bigg].
\end{aligned} \tag{4.4}$$

By standard estimates, we obtain that

$$\begin{aligned}
&\mathbb{E}\sup_{\tau\in[0,T]}\left|\int_0^\tau e^{(\tau-\sigma)A}\big(F(\sigma,\overline{X}_\sigma,Y_\sigma,Z_\sigma)-F(\sigma,\overline{U}_\sigma,V_\sigma,W_\sigma)\big)\,d\sigma\right|^p \\
&\quad\le \tilde{c}_p T^{p/2}L^pM_A^p\Bigg[T^{(p-2)/2}\left(\mathbb{E}\sup_{s\in[0,T]}|\overline{X}_s-\overline{U}_s|_H^p+\mathbb{E}\sup_{s\in[0,T]}|Y_s-V_s|_K^p\right) \\
&\qquad+\mathbb{E}\left(\int_0^T|Z_\sigma-W_\sigma|^2\right)^{p/2}\Bigg].
\end{aligned} \tag{4.5}$$

Using now the factorization method, see [6, Chapter 5] or [16, Proposition 3.2], we have that

$$\begin{aligned}
&\mathbb{E}\sup_{\tau\in[0,T]}\left|\int_0^\tau e^{(\tau-\sigma)A}\big(G(\sigma,\overline{X}_\sigma,Y_\sigma)-G(\sigma,\overline{U}_\sigma,V_\sigma)\big)\,dW(\sigma)\right|^p \\
&\quad\le \bar{c}_pM_A^{2p}L^pT^{p/2-1-\gamma p}\left[\mathbb{E}\sup_{s\in[0,T]}|\overline{X}_s-\overline{U}_s|_H^p+\mathbb{E}\sup_{s\in[0,T]}|Y_s-V_s|_K^p\right].
\end{aligned} \tag{4.6}$$

We can assume that $T \leq 1$, therefore by combining these estimates we have that there exists a constant $\gamma = \gamma(p, M_A, L)$ such that

$$\begin{aligned}(1-\gamma T^{p/2-1-\gamma p})\mathbb{E}\sup_{s\in[0,T]}|\overline{X}_s - \overline{U}_s|_H^p \\ \leq \gamma T^{p/2-1-\gamma p}\mathbb{E}\sup_{s\in[0,T]}|Y_s - V_s|_K^p + \gamma T^{p/2}\mathbb{E}\left(\int_0^T |Z_\sigma - W_\sigma|^2\right)^{p/2}.\end{aligned}\tag{4.7}$$

Now we consider the backward equation.

We recall that, see [10] for instance, the following relation holds:

$$\begin{aligned}\overline{Y}_\tau - \overline{V}_\tau = \mathbb{E}[e^{B(T-\tau)}(\Phi(\overline{X}_T) - \Phi(\overline{U}_T)) \mid \mathcal{F}_\tau] \\ + \mathbb{E}\left[\int_\tau^T e^{B(s-\tau)}(\Psi(s,\overline{X}_s,\overline{Y}_s,\overline{Z}_s) - \Psi(s,\overline{U}_s,\overline{V}_s,\overline{W}_s))\,ds \mid \mathcal{F}_\tau\right].\end{aligned}\tag{4.8}$$

Thus we deduce that there exists a constant $\gamma_1 = \gamma_1(p, M_B, L)$ such that for all $T \leq T_1$,

$$\begin{aligned}(1-\gamma_1 T^{p-1})\mathbb{E}\sup_{s\in[0,T]}|\overline{Y}_s - \overline{V}_s|_H^p \\ \leq \gamma_1\mathbb{E}\sup_{s\in[0,T]}|\overline{X}_s - \overline{U}_s|_K^p + \gamma_1 T^{p/2}\mathbb{E}\left(\int_0^T |\overline{Z}_\sigma - \overline{W}_\sigma|^2\right)^{p/2}.\end{aligned}\tag{4.9}$$

It remains to estimate $\overline{Z} - \overline{W}$. First of all we notice that for every fixed $\overline{X}, \overline{U} \in L^p_{\mathcal{P}}(\Omega, C([0,T],H))$ by Proposition 3.5 there exists a unique mild solution of the following equations:

$$\begin{aligned}\overline{Y}_\tau &= e^{(T-\tau)B}\Phi(\overline{X}_T) - \int_\tau^T e^{(\sigma-\tau)B}\Psi(\sigma,\overline{X}_\sigma,\overline{Y}_\sigma,\overline{Z}_\sigma)\,d\sigma - \int_\tau^T e^{(\sigma-\tau)B}\overline{Z}_\sigma\,dW(\sigma), \quad s\in[0,T], \\ \overline{V}_\tau &= e^{(T-\tau)B}\Phi(\overline{U}_T) - \int_\tau^T e^{(\sigma-\tau)B}\Psi(\sigma,\overline{U}_\sigma,\overline{V}_\sigma,\overline{W}_\sigma)\,d\sigma - \int_\tau^T e^{(\sigma-\tau)B}\overline{W}_\sigma\,dW(\sigma), \quad s\in[0,T].\end{aligned}\tag{4.10}$$

Therefore estimate (3.5) with $\eta = \Phi(\overline{X}_T) - \Phi(\overline{U}_T)$ and $f(\sigma) = \Psi(\sigma,\overline{X}_\sigma,\overline{Y}_\sigma,\overline{Z}_\sigma) - \Psi(\sigma,\overline{U}_\sigma,\overline{V}_\sigma,\overline{W}_\sigma)$ imply that there exists a constant $\gamma_2 = \gamma_2(p, L)$, such that

$$\begin{aligned}\mathbb{E}\left(\int_0^T |\overline{Z}_\sigma - \overline{W}_\sigma|^2\,d\sigma\right)^{p/2} \leq \gamma_2\Bigg[T^{p-1}\mathbb{E}\sup_{s\in[0,T]}|\overline{Y}_s - \overline{V}_s|_H^p + (1+T^{p-1})\mathbb{E}\sup_{s\in[0,T]}|\overline{X}_s - \overline{U}_s|_H^p \\ + T^{p/2}\mathbb{E}\left(\int_0^T |\overline{Z}_\sigma - \overline{W}_\sigma|^2\right)^{p/2}\Bigg].\end{aligned}\tag{4.11}$$

Estimates (4.7), (4.9), and (4.11) imply that there exists a $T_1 > 0$, depending on p, M_A, M_B, and L, such that the map Γ is a contraction in $\mathcal{H}_0^T$ for all $T \leq T_1$. Thus for every $p > 2/(1-2\gamma)$, there exists a unique solution to (1.1). This solution clearly belongs to $L^p_{\mathcal{P}}(\Omega, C([0,T],K)) \times L^p_{\mathcal{P}}(\Omega; L^2((0,T); L_2(\Xi,K)))$ for all $p \geq 2$. The uniqueness in this bigger class of processes is a consequence of the next theorem, therefore we will chose $T_1 \leq T_0$, where T_0 is given in Theorem 2.4. This concludes the proof.

4.2. Proof of Theorem 2.4. The proof of this theorem follows the same procedure of the previous one. We set

$$\mathcal{K}_t^T = C_{\mathcal{P}}\left([t,T]; L^2(\Omega;K)\right) \times L^2_{\mathcal{P}}\left((t,T)\times\Omega; L_2(\Xi,K)\right). \tag{4.12}$$

One has to prove that the map Γ, considered now as a map from $\mathcal{K}_t^T$ into itself, is a contraction for a sufficiently small T, that again depends only on L, M_A, and M_B. For simplicity, we start from $t = 0$.

We obtain by standard estimates that

$$\begin{aligned}
&\left(1 - 6TM_A^2L^2 - 4M_A^2L^2T^{1-2\gamma}\right) \sup_{\tau\in[0,T]} \mathbb{E}\left|\overline{X}_\tau - \overline{U}_\tau\right|_H^2 \\
&\quad\leq \left(6TM_A^2L^2 + 4M_A^2L^2T^{1-2\gamma}\right) \sup_{s\in[0,T]} \mathbb{E}\left|Y_s - V_s\right|_K^2 + 6TM_A^2L^2\mathbb{E}\int_0^T \left|Z_s - W_s\right|^2_{L_2(\Xi,K)} ds.
\end{aligned} \tag{4.13}$$

Now thanks again to (4.8), we have that

$$\begin{aligned}
&\left(1 - 6TM_B^2L^2\right) \sup_{\tau\in[0,T]} \mathbb{E}\left|\overline{Y}_\tau - \overline{V}_\tau\right|_K^2 \\
&\quad\leq \left(2M_B^2L^2 + 6TM_B^2L^2\right) \sup_{\tau\in[0,T]} \mathbb{E}\left|\overline{X}_\tau - \overline{U}_\tau\right|_H^2 + 6TM_B^2L^2\mathbb{E}\int_0^T \left|\overline{Z}_s - \overline{W}_s\right|^2_{L_2(\Xi,K)} ds.
\end{aligned} \tag{4.14}$$

It remains to treat the term $\mathbb{E}\int_0^T |\overline{Z}_s - \overline{W}_s|^2_{L_2(\Xi,K)} ds$. Since we have to deal with the convolution term, we follow the technique introduced by Hu and Peng in [10] that is based on the martingale representation theorem. We have the following representation for the processes Z and W:

$$\begin{aligned}
Z_s &= e^{(T-s)B}V(s) - \int_s^T e^{(\sigma-s)B}K(s,\sigma)d\sigma \quad \mathbb{P}\text{-a.s. and for a.e. } s\in[0,T], \\
W_s &= e^{(T-s)B}\widetilde{V}(s) - \int_s^T e^{(\sigma-s)B}\widetilde{K}(s,\sigma)d\sigma \qquad \mathbb{P}\text{-a.s. and for a.e. } s\in[0,T],
\end{aligned} \tag{4.15}$$

where $V,\widetilde{V} \in L^2_{\mathcal{P}}(\Omega \times (0,T);K)$ and $K,\widetilde{K} \in L^2_{\mathcal{P}}(\Omega \times (0,T) \times (0,T);K)$ are related to Φ and Ψ, respectively, as follows:

$$\begin{aligned}
&\left.\begin{aligned}
\int_\tau^T V(s)dW_s &= \Phi(X_T) - \mathbb{E}(\Phi(X_T) \mid \mathcal{F}_\tau),\\
\int_\tau^T \widetilde{V}(s)dW_s &= \Phi(U_T) - \mathbb{E}(\Phi(U_T) \mid \mathcal{F}_\tau),
\end{aligned}\right\} \quad \tau \in [0,T],\\
&\int_\tau^s K(r,s)dW_r = \Psi(s,\overline{X}_s,\overline{Y}_s,\overline{Z}_s) - \mathbb{E}(\Psi(s,\overline{X}_s,\overline{Y}_s,\overline{Z}_s) \mid \mathcal{F}_\tau), \quad 0 \le \tau \le s \le T,\\
&\int_\tau^s \widetilde{K}(r,s)dW_r = \Psi(s,\overline{U}_s,\overline{V}_s,\overline{W}_s) - \mathbb{E}(\Psi(s,\overline{U}_s,\overline{V}_s,\overline{W}_s) \mid \mathcal{F}_\tau), \quad 0 \le \tau \le s \le T.
\end{aligned} \tag{4.16}$$

One can deduce the following estimates:

$$\begin{aligned}
\mathbb{E}\int_0^T |\overline{Z}_s - \overline{W}_s|^2_{L_2(\Xi,K)}\,ds &\le 2\bigg[\mathbb{E}\int_0^T |e^{(T-s)B}[V(s)-\widetilde{V}(s)]|^2_{L_2(\Xi,K)}\,ds\\
&\qquad + \mathbb{E}\int_0^T \bigg|\int_s^T e^{(\sigma-s)B}[K(s,\sigma)-\widetilde{K}(s,\sigma)]d\sigma\bigg|^2_{L_2(\Xi,K)}\,ds\bigg]\\
&\le 8M_B^2\bigg[L^2\mathbb{E}|\overline{X}_T - \overline{U}_T|^2_H + T\int_0^T \mathbb{E}|\Psi(\sigma,\overline{X}_\sigma,\overline{Y}_\sigma,\overline{Z}_\sigma)\\
&\qquad - \Psi(\sigma,\overline{U}_\sigma,\overline{V}_\sigma,\overline{W}_\sigma)|^2_K\,d\sigma\bigg]\\
&\le 8M_B^2L^2\bigg[(1+3T^2)\sup_{\tau\in[0,T]}\mathbb{E}|\overline{X}_\tau - \overline{U}_\tau|^2_H + 3T^2\sup_{\tau\in[0,T]}\mathbb{E}|\overline{Y}_\tau - \overline{V}_\tau|^2_H\\
&\qquad + 3T\mathbb{E}\int_0^T |\overline{Z}_s - \overline{W}_s|^2_{L_2(\Xi,K)}\,ds\bigg].
\end{aligned} \tag{4.17}$$

We have finally obtained

$$\begin{aligned}
&(1-24M_B^2TL^2)\mathbb{E}\int_0^T |\overline{Z}_s - \overline{W}_s|^2_{L_2(\Xi,K)}\,ds\\
&\quad \le 8M_B^2L^2\bigg[(1+3T^2)\sup_{\tau\in[0,T]}\mathbb{E}|\overline{X}_\tau - \overline{U}_\tau|^2_H + 3T^2\sup_{\tau\in[0,T]}\mathbb{E}|\overline{Y}_\tau - \overline{V}_\tau|^2_K\bigg].
\end{aligned} \tag{4.18}$$

By the three inequalities (4.13), (4.14), and (4.18) there exists a positive number T_0 that depends on L, M_A, and M_B, such that the map Γ is a contraction in $\mathcal{K}_0^T$ for every $T \le T_0$, and this concludes the proof of the theorem.

4.3. Regular dependence on the parameters. In this paragraph, we study the differentiability of the solution to the forward-backward system with respect to the initial state. More precisely, we will prove that under appropriate assumptions, the solution is *Gâteaux differentiable* with respect to x. We introduce the following class of functions.

Definition 4.1. We say that a mapping $F : X \to V$ belongs to the class $\mathcal{G}^1(X;V)$ if it is continuous, Gâteaux differentiable on X, and $\nabla F : X \to L(X,V)$ is strongly continuous.

We need to generalize this definition to functions depending on several variables.

Definition 4.2. We say that a mapping $F : X \times Y \to V$ belongs to the class $\mathcal{G}^{1,0}(X \times Y;V)$ if it is continuous, Gâteaux differentiable with respect to x on $X \times Y$, and $\nabla_x F : X \times Y \to L(X,V)$ is strongly continuous.

We make further assumptions on the coefficients.

Hypothesis 4.3. We assume that for every $t \in [0,T]$,

(i) $F(t,\cdot,\cdot,\cdot) \in \mathcal{G}^{1,1,1}(H \times K \times L_2(\Xi,K);H)$;

(ii) for every $s > 0$, $e^{sA}G(t,\cdot,y) \in \mathcal{G}^1(H;L_2(\Xi,H))$ for every $y \in K$ and $e^{sA}G(t,x,\cdot) \in \mathcal{G}^1(K;L_2(\Xi,K))$ for every $x \in H$;

(iii) $\Psi(t,\cdot,\cdot,\cdot) \in \mathcal{G}^{1,1,1}(H \times K \times L_2(\Xi,K);K)$;

(iv) $\Phi \in \mathcal{G}^1(H;K)$.

As in [16, paragraph 3.2], we set

$$S(\tau) = e^{\tau A} \quad \text{for } \tau \geq 0, \qquad S(\tau) = I \quad \text{for } \tau < 0, \tag{4.19}$$

and we consider the system

$$\begin{aligned} X_\tau &= S(\tau - t)x + \int_0^\tau 1_{[t,T]}(\sigma)S(\tau-\sigma)F(\sigma,X_\sigma,Y_\sigma,Z_\sigma)\,d\sigma \\ &\quad + \int_0^\tau 1_{[t,T]}(\sigma)S(\tau-\sigma)G(\sigma,X_\sigma,Y_\sigma)\,dW_\sigma, \\ Y_\tau &= e^{(T-\tau)B}\Phi(X_T) - \int_\tau^T e^{(\sigma-\tau)B}Z_\sigma\,dW_\sigma - \int_\tau^T e^{(\sigma-\tau)B}\Psi(\sigma,X_\sigma,Y_\sigma,Z_\sigma)\,d\sigma, \quad \tau \in [0,T], \end{aligned} \tag{4.20}$$

under assumptions 2.1 and 2.5, system (4.20) for every $p \geq 2$ has a unique solution $(X,Y,Z) \in L^p_{\mathcal{P}}(\Omega;C([0,T];H)) \times L^p_{\mathcal{P}}(\Omega;C([0,T];K)) \times L^p_{\mathcal{P}}(\Omega;L^2((0,T);L_2(\Xi,K)))$ for a sufficiently small T. Moreover, the restriction to the time interval $[t,T]$ is the unique solution to (1.1).

From now on, we will denote by $(X(\tau,t,x),Y(\tau,t,x),Z(\tau,t,x))$, $\tau \in [0,T]$ the solution to (4.20).

Proposition 4.4. *Assume Hypotheses 2.1, 2.5, and 4.3 hold. Then, for every $p \geq 2$, one has the following.*

(1) *The map* $(t,x) \to (X(\cdot,t,x),Y(\cdot,t,x),Z(\cdot,t,x))$

$$\begin{aligned} &\text{belongs to } \mathcal{G}^{0,1}\big([0,T] \times H; L^p_{\mathcal{P}}(\Omega;C([0,T];H)) \times L^p_{\mathcal{P}}(\Omega;C([0,T];K)) \\ &\qquad \times L^p_{\mathcal{P}}(\Omega;L^2((0,T);L_2(\Xi,K)))\big). \end{aligned} \tag{4.21}$$

(2) *Let $\nabla_x X$, $\nabla_x Y$, and $\nabla_x Z$ be the partial Gâteaux derivatives, for every direction $h \in H$, the directional derivative processes $(\nabla_x X(\tau,t,x)h, \nabla_x Y(\tau,t,x)h, \nabla_x Z(\tau,t,x)h)$, $\tau \in [0,T]$, solve, $\mathbb{P}$-a.s., the system*

$$\begin{aligned}
&\nabla_x X(\tau,t,x)h \\
&\quad = e^{(\tau-t)A}h \\
&\qquad + \int_t^\tau e^{(\tau-\sigma)A}\nabla_x F\big(\sigma,X(\sigma,t,x),Y(\sigma,t,x),Z(\sigma,t,x)\big)\nabla_x X(\sigma,t,x)h\,d\sigma \\
&\qquad + \int_t^\tau e^{(\tau-\sigma)A}\nabla_y F\big(\sigma,X(\sigma,t,x),Y(\sigma,t,x),Z(\sigma,t,x)\big)\nabla_x Y(\sigma,t,x)h\,d\sigma \\
&\qquad + \int_t^\tau e^{(\tau-\sigma)A}\nabla_z F\big(\sigma,X(\sigma,t,x),Y(\sigma,t,x),Z(\sigma,t,x)\big)\nabla_x Z(\sigma,t,x)h\,d\sigma \\
&\qquad + \int_t^\tau e^{(\tau-\sigma)A}\nabla_x G\big(\sigma,X(\sigma,t,x),Y(\sigma,t,x)\big)\nabla_x X(\sigma,t,x)h\,dW_\sigma \\
&\qquad + \int_t^\tau e^{(\tau-\sigma)A}\nabla_y G\big(\sigma,X(\sigma,t,x),Y(\sigma,t,x)\big)\nabla_x Y(\sigma,t,x)h\,dW_\sigma, \quad \tau \in [t,T], \\
&\qquad\qquad \nabla_x X(\tau,t,x)h = h, \quad \tau \in [0,t),
\end{aligned} \tag{4.22}$$

$$\begin{aligned}
&\nabla_x Y(\tau,t,x)h \\
&\quad = e^{(T-\tau)B}\nabla_x \Phi\big(X(T,t,x)\big)\nabla_x X(T,t,x)h - \int_\tau^T e^{(\sigma-\tau)B}\nabla_x Z(\sigma,t,x)h\,dW_\sigma \\
&\qquad - \int_\tau^T e^{(\sigma-\tau)B}\nabla_x \Psi\big(\sigma,X(\sigma,t,x),Y(\sigma,t,x),Z(\sigma,t,x)\big)\nabla_x X(\sigma,t,x)h\,d\sigma \\
&\qquad - \int_\tau^T e^{(\sigma-\tau)B}\nabla_y \Psi\big(\sigma,X(\sigma,t,x),Y(\sigma,t,x),Z(\sigma,t,x)\big)\nabla_x Y(\sigma,t,x)h\,d\sigma \\
&\qquad - \int_\tau^T e^{(\sigma-\tau)B}\nabla_z \Psi\big(\sigma,X(\sigma,t,x),Y(\sigma,t,x),Z(\sigma,t,x)\big)\nabla_x Z(\sigma,t,x)h\,d\sigma, \quad \tau \in [0,T].
\end{aligned} \tag{4.23}$$

(3) *The following estimate holds for every $h \in H$:*

$$\begin{aligned}
&\mathbb{E}\sup_{\tau\in[0,T]}\big|\nabla_x X(\tau,t,x)h\big|_H^p \\
&\qquad + \mathbb{E}\sup_{\tau\in[0,T]}\big|\nabla_x Y(\tau,t,x)h\big|_K^p + \mathbb{E}\bigg(\int_0^T \big|\nabla_x Z(\tau,t,x)h\big|_{L_2(\Xi,K)}^2\bigg)^{p/2} \le c|h|_H^p,
\end{aligned} \tag{4.24}$$

where $c > 0$ depends on p, M_A, M_B, L, and T.

Proof. We start by recalling how the solution to (4.20) is built. We set again $\mathcal{H}_0^T = L^p_{\mathcal{P}}(\Omega; C([0,T];K)) \times L^p_{\mathcal{P}}(\Omega; L^2((0,T);L_2(\Xi,K)))$ for some $p > 2$. For every $x \in H$, $t \in [0,T]$, and $(Y^1,Z^1) \in \mathcal{H}_0^T$, we define a map $X \to \Gamma_1(X;Y^1,Z^1,t,x) : L^p_{\mathcal{P}}(\Omega;C([0,T];H)) \to L^p_{\mathcal{P}}(\Omega; C([0,T];H))$ as follows:

$$\begin{aligned}\Gamma_1(X;Y^1,Z^1,t,x)(\tau) &= S(\tau-t)x + \int_0^\tau 1_{[t,T]}(\sigma)S(\tau-\sigma)F(\sigma,X_\sigma,Y^1_\sigma,Z^1_\sigma)\,d\sigma \\ &\quad + \int_0^\tau 1_{[t,T]}(\sigma)S(\tau-\sigma)G(\sigma,X_\sigma,Y^1_\sigma)\,dW_\sigma.\end{aligned} \tag{4.25}$$

This map is a contraction and we denote by $\Lambda^1(Y^1,Z^1,t,x) = \{\Lambda^1(s;Y^1,Z^1,t,x) : s \in [0,T]\}$ its (unique) fixed point. We introduce a second map $(X,\eta) \to \Gamma_2(X,\eta) : L^p_{\mathcal{P}}(\Omega;C([0,T];H)) \times L^p(\Omega,\mathcal{F}_T,\mathbb{P};K) \to \mathcal{H}_0^T$ defined as follows: $\Gamma_2(X,\eta) = (\overline{Y},\overline{Z})$ is the unique solution to

$$\begin{aligned}\overline{Y}(\tau,X,\eta) &= e^{(T-\tau)B}\eta - \int_\tau^T e^{(\sigma-\tau)B}\Psi(\sigma,X_\sigma,\overline{Y}(\sigma,X,\eta),\overline{Z}(\sigma,X,\eta))\,d\sigma \\ &\quad - \int_\tau^T e^{(\sigma-\tau)B}\overline{Z}(\sigma,X,\eta)\,dW_\sigma.\end{aligned} \tag{4.26}$$

Finally the map Γ is expressed in terms of Γ_1 and Γ_2 as follows:

$$\Gamma(Y^1,Z^1,t,x) = \Gamma_2(\Lambda^1(Y^1,Z^1,t,x),\Phi(\Lambda^1(T;Y^1,Z^1,t,x))). \tag{4.27}$$

Then the solution to (4.20) is the unique solution to $(Y^1,Z^1) = \Gamma(Y^1,Z^1,t,x)$. Therefore, since we want to prove that the fixed point $Y(t,x)$, $Z(t,x)$ of $\Gamma(\cdot,\cdot,t,x)$ is differentiable with respect to x, we apply the parameter depending contraction principle; see [27] or [16, Proposition 2.4]. Thus we have to prove that $\Gamma \in \mathcal{G}^{1,0,1}(\mathcal{H}_0^T \times [0,T] \times H; \mathcal{H}_0^T)$. In order to carry on with this program we will proceed as follows:

(1) we prove that Γ_2 is $\mathcal{G}^{1,1}(L^p(\Omega;C([0,T];H)) \times L^p(\Omega,\mathcal{F}_T,\mathbb{P};K);\mathcal{H}_0^T)$;

(2) we show that $(Y^1,Z^1,t,x) \to \Lambda^1(Y^1,Z^1,t,x)$ is $\mathcal{G}^{1,0,1}(\mathcal{H}_0^T \times [0,T] \times H; L^p_{\mathcal{P}}(\Omega; C([0,T];H)))$;

(3) finally we apply the chain rule, see for instance [16, Lemma 2.1], and we conclude the proof.

We will just sketch the proof of each step being very similar to [16, Propositions 3.3 and 4.8].

(1) First of all we note that Proposition 3.5 applies to (4.26) and from the hypothesis on Ψ, we have that, for fixed η and X,

$$\begin{aligned}&\mathbb{E}\sup_{s\in[0,T]} |\bar{Y}(s,X,\eta)|_K^p + \mathbb{E}\left(\int_0^T |\overline{Z}(s,X,\eta)|^2\,ds\right)^{p/2} \\ &\quad \le C_1\left(\mathbb{E}|\eta|^p + \mathbb{E}\left(\int_0^T |\Psi(s,X,0,0)|_K\right)^p\right) \\ &\quad \le C_2\left(\mathbb{E}|\eta|^p + 1 + \|X\|^p_{L^p(\Omega;C([0,T];H))}\right),\end{aligned} \tag{4.28}$$

where the positive constants C_1 and C_2 depend on M_B, T, C, and p. We introduce the following equation, for any $N \in L^p_{\mathcal{P}}(\Omega; C([0,T];H))$ and $\zeta \in L^p(\Omega, \mathcal{F}_T, \mathbb{P}; K)$ and $(\overline{Y}, \overline{Z}) \in \mathcal{H}_0^T$ fixed:

$$\begin{aligned}
&\widehat{Y}(\tau, X, \eta) + \int_\tau^T e^{(\sigma-\tau)B} \widehat{Z}(\sigma, X, \eta) dW_\sigma \\
&\quad = -\int_\tau^T e^{(\sigma-\tau)B} \nabla_x \Psi(\sigma, X, \overline{Y}, \overline{Z}) N_\sigma \, d\sigma \\
&\qquad - \int_\tau^T e^{(\sigma-\tau)B} \nabla_y \Psi(\sigma, X, \overline{Y}, \overline{Z}) \widehat{Y}(\sigma, X, \eta) d\sigma \\
&\qquad - \int_\tau^T e^{(\sigma-\tau)B} \nabla_z \Psi(\sigma, X, \overline{Y}, \overline{Z}) \widehat{Z}(\sigma, X, \eta) d\sigma + \zeta.
\end{aligned} \tag{4.29}$$

From Proposition 3.5, it follows that this equation has a unique solution $(\widehat{Y}, \widehat{Z})$ such that

$$\mathbb{E} \sup_{s\in[0,T]} |\widehat{Y}_s|_K^p + \mathbb{E}\left(\int_0^T |\widehat{Z}_s|^2_{L_2(\Xi,K)} ds\right)^{p/2} \le \kappa\left[\mathbb{E}|\zeta|_K^p + \|N\|^p_{L^p_{\mathcal{P}}(\Omega;C([0,T];H))}\right], \tag{4.30}$$

where κ is a constant the depends on p, M_B, L, and T. From the hypothesis on the coefficients one can deduce—using the same arguments exploited in [16, Proposition 4.8]—that $(X, N, \overline{Y}, \overline{Z}, \zeta) \to (\widehat{Y}(X, N, \overline{Y}, \overline{Z}, \zeta), \widehat{Z}(X, N, \overline{Y}, \overline{Z}, \zeta))$ is continuous form $L^p_{\mathcal{P}}(\Omega; C([0,T];H)) \times L^p_{\mathcal{P}}(\Omega; C([0,T];H)) \times \mathcal{H}_0^T \times L^p(\Omega, \mathcal{F}_T; K) \to \mathcal{H}_0^T$. It remains to show that the directional derivatives of $(\overline{Y}(X,\eta), \overline{Z}(X,\eta))$ in direction (N,ζ) coincide with the processes $(\widehat{Y}(X, N, \overline{Y}(X,\eta), \overline{Z}(X,\eta), \zeta), \widehat{Z}(X, N, \overline{Y}(X,\eta), \overline{Z}(X,\eta), \zeta))$. For every $\epsilon > 0$, the processes

$$\begin{aligned}
Y^\epsilon &= \frac{1}{\epsilon}[\overline{Y}(X+\epsilon N, \eta+\epsilon\zeta) - \overline{Y}(X,\eta)] - \widehat{Y}(X, N, \overline{Y}(X,\eta), \overline{Z}(X,\eta), \zeta), \\
Z^\epsilon &= \frac{1}{\epsilon}[\overline{Z}(X+\epsilon N, \eta+\epsilon\zeta) - \overline{Z}(X,\eta)] - \widehat{Z}(X, N, \overline{Y}(X,\eta), \overline{Z}(X,\eta), \zeta)
\end{aligned} \tag{4.31}$$

solve the following equation:

$$Y^\epsilon(\tau) = -\int_\tau^T e^{(\sigma-\tau)B} \nu^\epsilon(\sigma) d\sigma - \int_\tau^T e^{(\sigma-\tau)B} Z^\epsilon(\sigma) dW_\sigma \tag{4.32}$$

with

$$\begin{aligned}
\nu^\epsilon(\sigma) = &\frac{1}{\epsilon}[\Psi(\sigma, X(\sigma)+\epsilon N(\sigma), \overline{Y}^\epsilon(\sigma), \overline{Z}^\epsilon(\sigma))] - \nabla_x \Psi(\sigma, X(\sigma), \overline{Y}(\sigma), \overline{Z}(\sigma)) N_\sigma \\
&+ \frac{1}{\epsilon}[\Psi(\sigma, X(\sigma), \overline{Y}^\epsilon(\sigma), \overline{Z}^\epsilon(\sigma)) - \Psi(\sigma, X(\sigma), \overline{Y}(\sigma), \overline{Z}(\sigma))] \\
&- \nabla_y \Psi(\sigma, X, \overline{Y}(\sigma), \overline{Z}(\sigma)) \widehat{Y}(\sigma) - \nabla_z \Psi(\sigma, X, \overline{Y}(\sigma), \overline{Z}(\sigma)) \widehat{Z}(\sigma).
\end{aligned} \tag{4.33}$$

Thus by Proposition 3.4, there exists a positive constant κ_1 depending on M_B, T, p such that

$$\mathbb{E}\sup_{s\in[0,T]}\left|Y_s^{\epsilon}\right|_K^p+\mathbb{E}\left(\int_0^T\left|Z_s^{\epsilon}\right|_{L_2(\Xi,K)}^2 ds\right)^{p/2}\leq \kappa_1\|\nu^{\epsilon}\|_{L_{\mathcal{P}}^p(\Omega;C([0,T];H))}^p. \tag{4.34}$$

Then following [16, Proposition 4.8], one can prove that

$$\lim_{\epsilon\to 0}\|\nu^{\epsilon}\|_{L_{\mathcal{P}}^p(\Omega;C([0,T];H))}=0. \tag{4.35}$$

Note that from estimates (4.28), (4.30), and the continuity of $(\widehat{Y}(X,N,\overline{Y},\overline{Z},\zeta),\widehat{Z}(X,N,\overline{Y},\overline{Z},\zeta))$ with respect to its parameters, we are in the position to apply [16, Lemma 2.2] and we conclude the first step.

(2) Since $\Lambda^1(Y^1,Z^1,x,t)$ is the fixed point of $\Gamma_1(\cdot,Y^1,Z^1,x,t)$, we can proceed applying the parameter depending contraction principle. The proof of a very similar result can be found in [16, Proposition 3.3], so will be omitted here. Thus we have that $\Lambda^1(Y^1,Z^1,x,t)$—for every $(Y^1,Z^1)\in\mathcal{H}_0^T$—is Gâteaux differentiable and $\nabla_x\Lambda^1$ solves the following equation—we omit the dependence on the variables Y^1 and Z^1 for the sake of clearness:

$$\begin{aligned}
&\nabla_x\Lambda^1(\tau;t,x)h\\
&\quad=e^{(\tau-t)A}h\\
&\qquad+\int_t^T e^{(\tau-\sigma)A}\nabla_xF(\sigma,\Lambda^1(\sigma;t,x),Y^1(\sigma,t,x),Z^1(\sigma,t,x))\nabla_x\Lambda^1(\sigma;t,x)h\,d\sigma\\
&\qquad+\int_t^T e^{(\tau-\sigma)A}\nabla_yF(\sigma,\Lambda^1(\sigma;t,x),Y^1(\sigma,t,x),Z^1(\sigma,t,x))\nabla_xY^1(\sigma,t,x)h\,d\sigma\\
&\qquad+\int_t^T e^{(\tau-\sigma)A}\nabla_zF(\sigma,\Lambda^1(\sigma;t,x),Y^1(\sigma,t,x),Z^1(\sigma,t,x))\nabla_xZ^1(\sigma,t,x)h\,d\sigma\\
&\qquad+\int_t^T e^{(\tau-\sigma)A}\nabla_xG(\sigma,\Lambda^1(\sigma;t,x),Y^1(\sigma,t,x))\nabla_x\Lambda^1(\sigma;t,x)h\,dW_\sigma\\
&\qquad+\int_t^T e^{(\tau-\sigma)A}\nabla_yG(\sigma,\Lambda^1(\sigma;t,x),Y^1(\sigma,t,x))\nabla_xY^1(\sigma,t,x)h\,dW_\sigma,\quad \tau\in[t,T],\\
&\qquad\qquad\nabla_x\Lambda^1(\tau;t,x)h=h,\quad \tau\in[0,t).
\end{aligned} \tag{4.36}$$

Moreover, one has that for every $h\in H$,

$$\mathbb{E}\sup_{\tau\in[0,T]}\left|\nabla_x\Lambda^1(\tau;t,x)h\right|_H^p\leq C|h|_H^p. \tag{4.37}$$

(3) Let us consider the fixed point $X(s,t,x) = \Lambda^1(s;Y,Z,t,x)$ corresponding to the couple Y, Z that is the fixed point of the map Γ. We can choose $N_\sigma = \nabla_x X(\sigma,t,x)h$ and $\zeta = \nabla_x \Phi(X(T,t,x))\nabla_x X(T,t,x)h$ for every $h \in H$, then $(\nabla_x X(\sigma,t,x)h, \nabla_x Y(\sigma,t,x)h, \nabla_x Z(\sigma,t,x)h)$ is the unique solution to system (4.22)-(4.23) and combining estimates (4.30) and (4.37), we get estimate (4.24).

This concludes the proof. □

4.4. Example. Let $\mathbb{Z}$ be the one-dimensional lattice of integers. We introduce an infinite collection of forward-backward systems,

$$
\begin{aligned}
dX^n(t) &= a_n X^n(t) - \sum_{j:|j-n|\le 1} [X^n(t) - X^j(t) - Y^j(t)]dt + dW^n(t), \quad n \in \mathbb{Z},\ t \in [0,T], \\
dY^n(t) &= -b_n Y^n(t) + \sum_{j:|j-n|\le 1} [Y^n(t) - Y^j(t) - X^j(t)]dt + Z^n(t)dW^n(t), \\
X^n(0) &= x^n, \quad Y^n(T) = \phi(X^n(T)), \quad n \in \mathbb{N}.
\end{aligned}
\tag{4.38}
$$

Let $l^2(\mathbb{Z})$ be the set of square summable sequences of real numbers, to fit our assumption 2.1 we assume that the following hold.

Hypothesis 4.5. (1) W^n are independent real Wiener processes.

(2) $\{a_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers such that

$$
\left\{x \in l^2(\mathbb{Z}) : \sum_{n\in\mathbb{Z}} a_n^2 x_n^2 < +\infty\right\}
\tag{4.39}
$$

is dense in $l^2(\mathbb{Z})$.

(3) $\{b_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers such that

$$
\left\{x \in l^2(\mathbb{Z}) : \sum_{n\in\mathbb{Z}} b_n^2 x_n^2 < +\infty\right\}
\tag{4.40}
$$

is dense in $l^2(\mathbb{Z})$.

We can thus formulate problem (4.38) as a forward-backward system:

$$
\begin{aligned}
dX_t &= AX_t + F(X_t, Y_t)dt + dW_t, \quad t \in [0,T], \\
dY_t &= BY_t + \Psi(X_t, Y_t)dt + Z\,dW_t, \quad t \in [0,T], \\
X_0 &= x, \qquad Y_T = \Phi(X_T).
\end{aligned}
\tag{4.41}
$$

We set $H = K = l^2(\mathbb{Z})$, $W(t) = \{W^n(t)\}_{n\in\mathbb{N}}$, then $W(t)$ is a cylindrical Wiener process with values in $l^2(\mathbb{Z})$. We define A and B by

$$
A(x) = (a_n x_n)_{n\in\mathbb{N}}, \qquad B(y) = (-b_n y_n)_{n\in\mathbb{N}}
\tag{4.42}
$$

with domains, respectively,

$$D(A) = \left\{ x \in l^2(\mathbb{Z}) : \sum_{n\in\mathbb{Z}} a_n^2 x_n^2 < +\infty \right\}, \qquad D(B) = \left\{ y \in l^2(\mathbb{Z}) : \sum_{n\in\mathbb{Z}} b_n^2 y_n^2 < +\infty \right\}, \tag{4.43}$$

these two operators are sectorial with dense domain, and B is negative, thus they obviously fulfill Hypothesis 2.1. The function $F : H \times K \to H$ is given by $F(t,x,y) = -(\sum_{j:|j-n|\le 1}[x^n - x^j - y^j])_{n\in\mathbb{Z}}$ and $\Psi : H \times K \to K$ is defined as $\Psi(t,x,y) = (\sum_{j:|j-n|\le 1}[y^n - x^j - y^j])_{n\in\mathbb{Z}}$, and it is immediate to check that these functions are Lipschitz continuous functions.

In this case, Theorem 2.6 applies and there exists a T_1 such that for any $T \le T_1$ and any $p \ge 2$, for any $x \in l^2(\mathbb{Z})$, there exists a unique solution in $L^p_{\mathcal{P}}(\Omega; C([t,T];H)) \times L^p_{\mathcal{P}}(\Omega; C([t,T];K)) \times L^p_{\mathcal{P}}(\Omega; L^2((t,T); L_2(\Xi,K)))$.

5. Application of forward-backward systems to stochastic optimal control

5.1. Setting of the problem, assumptions, and auxiliary results. Let U be a real and separable Hilbert space and $\mathcal{U}$ a subset of U.

Let Hypothesis 2.1 be in force and consider a controlled process X^u in H on a time interval $[t,T] \subset [0,T]$, governed by the following Itô stochastic differential equation of the form

$$\begin{gathered} dX_\tau^u = AX_\tau^u\,d\tau + F(\tau,X_\tau^u)\,d\tau + G(\tau,X_\tau^u)r(\tau,X_\tau^u,u_\tau)\,d\tau + G(\tau,X_\tau^u)\,dW_\tau, \quad \tau \in [t,T], \\ X_t^u = x \in H. \end{gathered} \tag{5.1}$$

An *admissible control* is defined as a predictable process that takes value in $\mathcal{U}$ and that is square integrable with respect to $d\mathbb{P} \times dt$. We recall that the cost functional to be minimized is

$$J(t,x,u.) = \mathbb{E}\int_t^T l(\tau,X_\tau^u,u_\tau)\,d\tau + \mathbb{E}\Phi(X_T^u), \tag{5.2}$$

where $l : [0,T] \times H \times U \to \mathbb{R}$ and Φ is defined in Hypothesis 2.1(v). Then the value function is the following:

$$V(t,x) = \inf_{u.} J(t,x,u.), \quad t \in [0,T],\ x \in H, \tag{5.3}$$

where as usual the infimum is taken over all admissible controls. We introduce the Hamiltonian function

$$\Psi(t,x,z) = \inf_{u\in U}\{l(t,x,u) + zr(t,x,u)\} \quad t \in [0,T],\ x \in H,\ z \in \Xi^*. \tag{5.4}$$

We stress the fact that the probability basis $(\Omega, \mathcal{F}, \mathbb{P})$, where the Wiener process W_t appearing in (5.1) is defined, is prescribed so we are considering the control problem in its strong formulation.

Besides Hypothesis 2.1, we assume also that the following hold.

Hypothesis 5.1. (1) The maps $r:[0,T]\times H\times U\to\Xi$ and $l:[0,T]\times H\times U\to\mathbb{R}$ are Borel measurable and there exists a constant $C>0$ such that

$$\begin{gathered} ||r(t,x,u)-r(t,x',u')||_{\Xi}+|l(t,x,u)-l(t,x',u')|\le C(\|x-x'\|_H+\|u-u'\|_U),\\ |e^{\sigma A}G(t,x)|_{L(\Xi,H)}+|r(t,x,u)|_{\Xi}+|l(t,0,0)|\le C,\\ l(t,0,u)\ge -C \end{gathered}\tag{5.5}$$

for every $\sigma>0$, $t\in[0,T]$, $x,x'\in H$, $u,u'\in U$.

(2) There exists a Borel measurable function $\gamma:[0,T]\times H\times\Xi^*\to U$ such that

$$\Psi(t,x,z)=l(t,x,\gamma(t,x,z))+zr(t,x,\gamma(t,x,z)),\quad t\in[0,T],\ x\in H,\ z\in\Xi^*\tag{5.6}$$

and such that for some constant $C>0$,

$$\begin{gathered} ||\gamma(t,x,z)-\gamma(t,x',z')||_U\le C(\|x-x'\|_H+\|z-z'\|_{\Xi^*}),\\ ||\gamma(t,0,0)||_U\le C \end{gathered}\tag{5.7}$$

for every $t\in[0,T]$, $x,x'\in H$, $z,z'\in\Xi^*$.

Note that Hypothesis 5.1(1) implies that for every admissible control $u.$ the functional $J(t,x,u.)$ takes values in $(-\infty,+\infty]$ and is not identically $+\infty$. See also [20, Section 2.1] for more details. In the rest of the section, Hypotheses 2.1 and 5.1 will always be in force.

Lemma 5.2. *There exists a constant $c>0$, depending only on known parameters, such that*

$$|\Psi(t,0,0)|\le c,\qquad |\Psi(t,x,z)-\Psi(t,x',z')|\le c|z-z'|+c|x-x'|(1+|z|+|z'|),\tag{5.8}$$

for every $t\in[0,T]$, $x,x'\in H$, and $z,z'\in\Xi^*$.

For the proof see [20, Lemma 2.3].

Now let us consider a standard Wiener space $\widetilde{W}$, defined on a complete probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{P})$. For $0\le t\le\tau\le T$, we denote by $\widetilde{\mathcal{F}}_t$ (resp., $\widetilde{\mathcal{F}}_{[t,\tau]}$) the σ-algebra generated by $\widetilde{W}_s$, $s\in[0,t]$ (resp., $s\in[t,\tau]$), completed by the null sets of $\widetilde{\mathcal{F}}$. For fixed $t\in[0,T]$ and $x\in H$, we consider the equation

$$\widetilde{X}_\tau=e^{(\tau-t)A}x+\int_t^\tau e^{(\tau-\sigma)A}F(\sigma,\widetilde{X}_\sigma)d\sigma+\int_t^\tau e^{(\tau-\sigma)A}G(\sigma,\widetilde{X}_\sigma)d\widetilde{W}_\sigma,\quad \tau\in[t,T],\tag{5.9}$$

by [16, Proposition 3.2] there exists a unique solution $\widetilde{X}_\tau$ such that

$$\mathbb{E}\sup_{\tau\in[t,T]}||\widetilde{X}_\tau||_H^2\le C(1+\|x\|_H^2).\tag{5.10}$$

This property and Lemma 5.2 imply that

$$\mathbb{E}\int_t^T ||\Psi(\sigma,\widetilde{X}_\sigma,0)||_H^2 < +\infty. \tag{5.11}$$

Thus, following again [16, Proposition 4.3], we have that the system that comprehends (5.9) and

$$\widetilde{Y}_\tau = \Phi(\widetilde{X}_T) + \int_\tau^T \Psi(\sigma,\widetilde{X}_\sigma,\widetilde{Z}_\sigma)\,d\sigma - \int_t^\tau \widetilde{Z}_\sigma\, d\widetilde{W}_\sigma, \quad \tau \in [t,T], \tag{5.12}$$

has a unique mild solution $\{(\widetilde{X}_\tau(t,x),\widetilde{Y}_\tau(t,x),\widetilde{Z}_\tau(t,x)),\tau \in [t,T]\}$, we will use this notation when we want to stress the dependence on the data. We set $J^*(t,x) = \widetilde{Y}_t(t,x)$, note that $J^*(t,x)$ is a deterministic value and depends only on the law of $\widetilde{Y}$, that is, only on t, x, F, G, Ψ, Φ. Using an infinite-dimensional version of Girsanov theorem, see [6, Theorem 10.14], one has the following.

Proposition 5.3. *For every $t \in [0,T]$ and $x \in H$, and for every admissible control $u \in L^2_{\mathcal{P}}(\Omega \times [0,T];\mathcal{U})$, we have that $J^*(t,x) \le J(t,x,u)$. Moreover, the following relation holds, for every control u and every $x \in H$:*

$$\begin{aligned} J^*(t,x) = J(t,x,u) + \mathbb{E}\int_t^T &\big[\Psi(\sigma,X^u_\sigma,Z^u_\sigma) - Z^u_\sigma r(\sigma,X^u_\sigma,u_\sigma) \\ &- l(\sigma,X^u_\sigma,u_\sigma)\big]\,d\sigma \quad \mathbb{P}\text{-a.s. for a.e. } t \in [0,T], \end{aligned} \tag{5.13}$$

where $\{(X^u_\tau(t,x),Y^u_\tau(t,x),Z^u_\tau(t,x)),\ \tau \in [t,T]\}$ is the solution to

$$\begin{aligned} X^u_\tau &= e^{(\tau-t)A}x + \int_t^\tau e^{(\tau-\sigma)A}F(\sigma,X^u_\sigma)\,d\sigma + \int_t^\tau e^{(\tau-\sigma)A}G(\sigma,X^u_\sigma)r(\sigma,X^u_\sigma,u_\sigma)\,d\sigma \\ &\quad + \int_t^\tau e^{(\tau-\sigma)A}G(\sigma,X^u_\sigma)\,dW_\sigma, \quad \tau \in [t,T], \\ Y^u_\tau &= \Phi(X^u_T) + \int_\tau^T \big[\Psi(\sigma,X^u_\sigma,Z^u_\sigma) - Z^u_\sigma r(\sigma,X^u_\sigma,u_\sigma)\big]\,d\sigma - \int_\tau^T Z^u_\sigma\, dW_\sigma \quad \tau \in [t,T]. \end{aligned} \tag{5.14}$$

For the proofs see [20, Proposition 2.5].

Note that the procedure described above allow to solve the problem considered in a weak formulation; see [16, Theorem 7.2]. Next results will allow us to solve the problem in the strong formulation.

Proposition 5.4. *Suppose that for some $(t,x) \in [0,T] \times H$, the system*

$$\begin{aligned} X_\tau &= e^{(\tau-t)A}x + \int_t^\tau e^{(\tau-\sigma)A}F(\sigma,X_\sigma)\,d\sigma + \int_t^\tau e^{(\tau-\sigma)A}G(\sigma,X_\sigma)r(\sigma,X_\sigma,\gamma(\sigma,X_\sigma,Z_\sigma))\,d\sigma \\ &\quad + \int_t^\tau e^{(\tau-\sigma)A}G(\sigma,X_\sigma)\,dW_\sigma, \quad \tau \in [t,T], \\ Y_\tau &= \Phi(X_T) + \int_\tau^T l(\sigma,X_\sigma,\gamma(\sigma,X_\sigma,Z_\sigma))\,d\sigma - \int_\tau^T Z_\sigma\,dW_\sigma, \quad \tau \in [t,T], \end{aligned} \tag{5.15}$$

has a solution $\{(X_\tau(t,x),Y_\tau(t,x),Z_\tau(t,x)) : \tau \in [t,T]\}$. Then setting $\overline{u}_\tau = \gamma(\tau,X_\tau(t,x), Z_\tau(t,x))$, $\tau \in [t,T]$, the process $\overline{u}$ is optimal for the control problem starting from x at time t with optimal cost $J(t,x,\overline{u}) = J^*(t,x) = V(t,x) = Y_t(t,x)$.

Proof. It is clear that $\overline{u}_\tau = \gamma(\tau,X_\tau(t,x),Z_\tau(t,x))$ is an admissible control. Finally, Hypotheses 5.1(2) and (5.13), evaluated at $\overline{u}$, imply that $\overline{u}$ is optimal, that is, $J(t,x,\overline{u}) = J^*(t,x) = V(t,x)$. Note that system (5.15) rewritten with respect to $\widetilde{W}_\tau = W_\tau + \int_{\tau\wedge t}^\tau r(\sigma,X_\sigma(t,x), \gamma(X_\sigma(t,x),Z_\sigma(t,x)))d\sigma$ coincides with (5.9)–(5.12), thus $Y_t(t,x) = \widetilde{Y}_t(t,x) = J^*(t,x) = V(t,x)$. □

Remark 5.5. Note that the uniqueness of the solution of the system is not required in the last statement since, morally speaking, only the law of Y_τ—that depends only on the coefficients T, F, G, Ψ, Φ—plays a role.

5.2. Global unique solvability for a class of forward-backward systems. Let us fix $T > 0$ and for every $t \in [0,T]$ and $x \in H$, consider

$$\begin{aligned} X_\tau &= e^{(\tau-t)A}x + \int_t^\tau e^{(\tau-\sigma)A}F(\sigma,X_\sigma)\,d\sigma + \int_t^\tau e^{(\tau-\sigma)A}G(\sigma,X_\sigma)r(\sigma,X_\sigma,\gamma(\sigma,X_\sigma,Z_\sigma))\,d\sigma \\ &\quad + \int_t^\tau e^{(\tau-\sigma)A}G(\sigma,X_\sigma)\,dW_\sigma, \quad \tau \in [t,T], \\ Y_\tau &= \Phi(X_T) + \int_\tau^T l(\sigma,X_\sigma,\gamma(\sigma,X_\sigma,Z_\sigma))\,d\sigma - \int_\tau^T Z_\sigma\,dW_\sigma, \quad \tau \in [t,T]. \end{aligned} \tag{5.16}$$

Exploiting the result proved in Theorem 2.6 and the interpretation of $Y_t(t,x)$ given in Proposition 5.4, one has the following.

Theorem 5.6. *For every $t \in [0,T]$ and $x \in H$ and every $p \geq 2$, there exists a unique mild solution of the forward-backward system (5.16) in $L^p_{\mathcal{P}}(\Omega;C([t,T];H)) \times L^p_{\mathcal{P}}(\Omega;C([t,T]; K)) \times L^p_{\mathcal{P}}(\Omega;L^2((t,T);L^2(\Xi,K)))$.*

Proof. First of all, we note that changing if necessary the constant, one can find a positive $L>0$ such that

$$\begin{aligned}|\Phi(x)-\Phi(\widetilde{x})| &\leq L\|x-\widetilde{x}\|_H,\\ |V(t,x)-V(t,\widetilde{x})| &\leq L\|x-\widetilde{x}\|_H\end{aligned} \tag{5.17}$$

for every $x,\widetilde{x}\in H$ and all $t\in[0,T]$.

In order to prove the second inequality, we recall that the function $V(t,x)$ coincides with $\widetilde{Y}_t(t,x)$, where $\{(\widetilde{X}_\tau(t,x),\widetilde{Y}_\tau(t,x),\widetilde{Z}_\tau(t,x));\ \tau\in[t,T]\}$ is the solution of system (5.9)–(5.12) starting in t at x. First we have that for every $\tau\in[0,T]$ and $x,\widetilde{x}\in H$, one has

$$\widetilde{\mathbb{E}}||\widetilde{X}_\tau(t,x)-\widetilde{X}_\tau(t,\widetilde{x})||_H^2\leq c\|x-\widetilde{x}\|_H^2. \tag{5.18}$$

Indeed, for any admissible u,

$$\begin{aligned}\widetilde{X}_\tau(t,x)-\widetilde{X}_\tau(t,\widetilde{x}) &= e^{(\tau-t)A}x-e^{(\tau-t)A}\widetilde{x}+\int_t^\tau e^{(\tau-\sigma)A}[F(\sigma,\widetilde{X}_\sigma(t,x))-F(\sigma,\widetilde{X}_\sigma(t,\widetilde{x}))]d\sigma\\ &\quad+\int_t^\tau e^{(\tau-\sigma)A}[G(\sigma,\widetilde{X}_\sigma(t,x))-G(\sigma,\widetilde{X}_\sigma(t,\widetilde{x}))]d\widetilde{W}_\sigma.\end{aligned} \tag{5.19}$$

Thus,

$$\begin{aligned}&\widetilde{\mathbb{E}}||\widetilde{X}_\tau(t,x)-\widetilde{X}_\tau(t,\widetilde{x})||_H^2\\ &\quad\leq c_2\Big\{M_A^2\|x-\widetilde{x}\|_H^2+TM_A^2L^2\int_t^\tau\widetilde{\mathbb{E}}||\widetilde{X}_\sigma(t,x)-\widetilde{X}_\sigma(t,\widetilde{x})||_H^2\,d\sigma\\ &\qquad+\widetilde{\mathbb{E}}\int_t^\tau||e^{(\tau-\sigma)A}[G(\sigma,\widetilde{X}_\sigma(t,x))-G(\sigma,\widetilde{X}_\sigma(t,\widetilde{x}))]||_{L_2(\Xi,H)}^2\,d\sigma\Big\}\\ &\quad\leq c_2\Big\{M_A^2\|x-\widetilde{x}\|_H+TM_A^2L^2\int_t^\tau\widetilde{\mathbb{E}}||\widetilde{X}_\sigma(t,x)-\widetilde{X}_\sigma(t,\widetilde{x})||_H^2\,d\sigma\\ &\qquad+L^2\int_t^\tau\widetilde{\mathbb{E}}||\widetilde{X}_\sigma(t,x)-\widetilde{X}_\sigma(t,\widetilde{x})||_H^2(\tau-\sigma)^{-2\gamma}\,d\sigma\Big\}.\end{aligned} \tag{5.20}$$

Thus, by Gromwall's lemma, there is a constant c depending on T, L, M_A such that for every $\tau\in[0,T]$,

$$\widetilde{\mathbb{E}}||\widetilde{X}_\tau(t,x)-\widetilde{X}_\tau(t,\widetilde{x})||_H^2\leq c\|x-\widetilde{x}\|_H^2. \tag{5.21}$$

Now, applying the Itô formula to $|\widetilde{Y}_\tau(t,x)-\widetilde{Y}_\tau(t,\widetilde{x})|^2$, one gets that

$$\begin{aligned}\widetilde{\mathbb{E}}\left|\widetilde{Y}_\tau(t,x)-\widetilde{Y}_\tau(t,\widetilde{x})\right|^2+\frac{1}{2}\widetilde{\mathbb{E}}\int_\tau^T\left|\widetilde{Z}_\sigma(t,x)-\widetilde{Z}_\sigma(t,\widetilde{x})\right|^2d\sigma\\ \le L^2\widetilde{\mathbb{E}}\|\widetilde{X}_T(t,x)-\widetilde{X}_T(t,\widetilde{x})\|_H^2+L^2\int_\tau^T\widetilde{\mathbb{E}}\|\widetilde{X}_\sigma(t,x)-\widetilde{X}_\sigma(t,\widetilde{x})\|_H^2d\sigma\\ +3L^2\int_\tau^T\widetilde{\mathbb{E}}\left|\widetilde{Y}_\sigma(t,x)-\widetilde{Y}_\sigma(t,\widetilde{x})\right|^2d\sigma.\end{aligned}\tag{5.22}$$

Thus again by the Gromwall lemma and (5.18), there exists a positive constant L, such that

$$\widetilde{\mathbb{E}}\left|\widetilde{Y}_\tau(t,x)-\widetilde{Y}_\tau(t,\widetilde{x})\right|^2\le L^2|x-\widetilde{x}|^2,\quad \tau\in[t,T].\tag{5.23}$$

For $\tau=t$ in particular, one has

$$\left|\widetilde{Y}_t(t,x)-\widetilde{Y}_t(t,\widetilde{x})\right|=\left|V(t,x)-V(t,\widetilde{x})\right|\le L|x-\widetilde{x}|\tag{5.24}$$

for every $x,\widetilde{x}$ in H.

Now we can conclude the proof in three steps.

Local existence. From Theorem 2.6, we know that for every $p\ge 2$, there exists a $\delta>0$ such that for every $t\ge T-\delta$, system (5.16) has a unique solution $\{(X_\tau(t,x),Y_\tau(t,x),Z_\tau(t,x)),\ \tau\in[t,T]\}$ with values in $L^p_{\mathcal{P}}(\Omega;C([t,T];H))\times L^p_{\mathcal{P}}(\Omega;C([t,T];K))\times L^p_{\mathcal{P}}(\Omega;L^2((t,T);L_2(\Xi,K)))$.

Global existence. Assume that $t\in[T-2\delta,T-\delta[$: indeed if $t\ge T-\delta$, there is nothing to prove, while if $t<T-2\delta$, then we can proceed repeating the construction below and after a finite number of steps we obtain the required solution in $[t,T]$ for arbitrary $t\in[0,T]$. We proceed in some steps.

(1) For every $\tau\in[T-\delta,T]$, consider the system

$$\begin{aligned}X^1_\tau&=e^{(\tau-(T-\delta))A}x+\int_{T-\delta}^\tau e^{(\tau-\sigma)A}F(\sigma,X^1_\sigma)d\sigma+\int_{T-\delta}^\tau e^{(\tau-\sigma)A}G(\sigma,X^1_\sigma)r(\sigma,X^1_\sigma,\gamma(\sigma,X^1_\sigma,Z^1_\sigma))d\sigma\\ &\quad+\int_{T-\delta}^\tau e^{(\tau-\sigma)A}G(\sigma,X^1_\sigma)dW_\sigma,\quad \tau\in[T-\delta,T],\\ Y^1_\tau&=\Phi(X^1_T)+\int_\tau^T l(\sigma,X^1_\sigma,\gamma(\sigma,X^1_\sigma,Z^1_\sigma))d\sigma-\int_\tau^T Z^1_\sigma dW_\sigma\quad \tau\in[T-\delta,T].\end{aligned}\tag{5.25}$$

Thus, by Theorem 2.6, there exists a unique solution $\{(X^1_\tau(t,x),Y^1_\tau(t,x),Z^1_\tau(t,x)),\ \tau\in[T-\delta,T]\}$ and applying Proposition 5.4, one has that $Y^1_{T-\delta}(T-\delta,x)=V(T-\delta,x)$ for all $x\in H$.

(2) Consider for $\tau \in [t, T-\delta]$ the following forward-backward system with initial condition x and final condition V:

$$X_\tau = e^{(\tau-t)A}x + \int_t^\tau e^{(\tau-\sigma)A}F(\sigma, X_\sigma)\,d\sigma + \int_t^\tau e^{(\tau-\sigma)A}G(\sigma, X_\sigma)r(\sigma, X_\sigma, \gamma(\sigma, X_\sigma, Z_\sigma))\,d\sigma$$
$$+ \int_t^\tau e^{(\tau-\sigma)A}G(\sigma, X_\sigma)\,dW_\sigma, \quad \tau \in [t, T-\delta],$$
$$Y_\tau = V(T-\delta, X_{T-\delta}) + \int_\tau^{T-\delta} l(\sigma, X_\sigma, \gamma(\sigma, X_\sigma, Z_\sigma))\,d\sigma - \int_\tau^{T-\delta} Z_\sigma\,dW_\sigma, \quad \tau \in [t, T-\delta]. \tag{5.26}$$

By Theorem 2.6, this system has a unique solution $\{(X_\tau(t,x), Y_\tau(t,x), Z_\tau(t,x)),\ \tau \in [t, T-\delta]\}$ since V is Lipschitz continuous with the same Lipschitz constant of ϕ by previous considerations.

(3) Finally we conclude solving again the system of step (1) with boundary condition $X_{T-\delta}(t,x)$ and Φ that

$$\begin{aligned} X^1{}_\tau &= e^{(\tau-(T-\delta))A}X_{T-\delta}(t,x) + \int_{T-\delta}^\tau e^{(\tau-\sigma)A}F(\sigma, X^1_\sigma)\,d\sigma \\ &\quad + \int_{T-\delta}^\tau e^{(\tau-\sigma)A}G(\sigma, X^1_\sigma)r(\sigma, X^1_\sigma, \gamma(\sigma, X^1_\sigma, Z^1_\sigma))\,d\sigma \\ &\quad + \int_{T-\delta}^\tau e^{(\tau-\sigma)A}G(\sigma, X^1_\sigma)\,dW_\sigma, \quad \tau \in [T-\delta, T], \end{aligned} \tag{5.27}$$
$$Y^1_\tau = \Phi(X^1_T) + \int_\tau^T l(\sigma, X^1_\sigma, \gamma(\sigma, X^1_\sigma, Z^1_\sigma))\,d\sigma - \int_\tau^T Z^1_\sigma\,dW_\sigma, \quad \tau \in [T-\delta, T].$$

Note that $X_{T-\delta}(t,x) \in L^p(\Omega, \mathscr{F}_{T-\delta}, \mathbb{P}, H)$ therefore satisfies the hypothesis of Theorem 2.6 and we find a solution $\{(X^1_\tau(T-\delta, X_{T-\delta}(t,x)), Y^1_\tau(T-\delta, X_{T-\delta}(t,x)), Z^1_\tau(T-\delta, X_{T-\delta}(t,x))),\ \tau \in [T-\delta, T]\}$ to the system. Thus the triplet of processes

$$X_\tau = \begin{cases} X_\tau(t,x), & \tau \in [t, T-\delta], \\ X^1_\tau(T-\delta, X_{T-\delta}), & \tau \in [T-\delta, T], \end{cases}$$
$$Y_\tau = \begin{cases} Y_\tau(t,x), & \tau \in [t, T-\delta], \\ Y^1_\tau(T-\delta, X_{T-\delta}), & \tau \in [T-\delta, T], \end{cases} \tag{5.28}$$
$$Z_\tau = \begin{cases} Z_\tau(t,x), & \tau \in [t, T-\delta], \\ Z^1_\tau(T-\delta, X_{T-\delta}), & \tau \in [T-\delta, T], \end{cases}$$

is a solution to system (5.16) in $[t, T]$ with boundary conditions x and Φ for any $p \geq 2$.

Global uniqueness. It is a consequence of the local uniqueness.

Let us assume that $t \in [T-2\delta, T-\delta)$ and that there are two different solutions in $[t,T]$ with initial datum ξ and final datum ϕ. We denote the two solutions by $(X^1(t,x), Y^1(t,x), Z^1(t,x))$ and $(X^2(t,x), Y^2(t,x), Z^2(t,x))$. Clearly, $(X^1(t,x), Y^1(t,x), Z^1(t,x))$ and $(X^2(t,x), Y^2(t,x), Z^2(t,x))$ are solutions of the system in $[T-\delta, T]$ with respect to the initial datum $X^1_{T-\delta}(t,x)$ $[X^2_{T-\delta}(t,x)]$ and final datum ϕ. The uniqueness proved in Theorem 2.6 implies that $Y^1_{T-\delta} = V(T-\delta, X^1_{T-\delta})$ and $Y^2_{T-\delta} = V(T-\delta, X^2_{T-\delta})$. Thus $(X^1(t,x), Y^1(t,x), Z^1(t,x))$ and $(X^2(t,x), Y^2(t,x), Z^2(t,x))$ are both solutions in $[t, T-\delta]$ with respect to x and V, therefore again by Theorem 2.6 and the Lipschitz continuity of V, the two solutions coincide in $[t, T-\delta]$. This implies in particular that $X^1_{T-\delta}(t,x) = X^2_{T-\delta}(t,x)$, thus the two solutions have to coincide also in $[T-\delta, T]$ having the same initial condition and the same final datum, again by Theorem 2.6. □

We are now in the position to solve the control problem.

PROPOSITION 5.7. *For every $t \in [0,T]$ and $x \in H$, let $\{(X_\tau(t,x), Y_\tau(t,x), Z_\tau(t,x)), \tau \in [t,T]\}$ be the solution to system (5.16) in $[t,T]$ with boundary conditions x and Φ. Set*

$$\overline{u}_\tau = \gamma(\tau, X_\tau(t,x), Z_\tau(t,x)), \quad \tau \in [t,T], \tag{5.29}$$

then $\overline{u}$ is an optimal control for the control problem starting from x at time t. If $X^{\overline{u}}$ is the corresponding solution, then $\mathbb{P}$-a.s., $X^{\overline{u}}_\tau = X_\tau(t,x)$ for $\tau \in [t,T]$. Finally, the optimal cost $V(t,x) = J(t,x,\overline{u})$ is equal to $Y_t(t,x)$.

Proof. Let us consider the forward equation corresponding to $\overline{u}$:

$$\begin{aligned} X_\tau = e^{(\tau-t)A}x &+ \int_t^\tau e^{(\tau-\sigma)A} F(\sigma, X_\sigma)\,d\sigma + \int_t^\tau e^{(\tau-\sigma)A} G(\sigma, X_\sigma) r(\sigma, X_\sigma, \overline{u}_\tau)\,d\sigma \\ &+ \int_t^\tau e^{(\tau-\sigma)A} G(\sigma, X_\sigma)\,dW_\sigma, \quad \tau \in [t,T]. \end{aligned} \tag{5.30}$$

This equation, thanks to the regularity of the coefficients, has a unique solution, therefore $X^{\overline{u}}$ must coincide with the first component $\{X_\tau(t,x),\ \tau \in [t,T]\}$ of the solution of system (5.16). By Theorem 5.6, system (5.16) admits a unique solution $\{(X_\tau(t,x), Y_\tau(t,x), Z_\tau(t,x)),\ \tau \in [t,T]\}$, thus we can apply Proposition 5.4 to conclude the proof. □

Remark 5.8. Following [19, Proposition 3.2], one can find a Borel measurable function $\zeta : [0,T] \times H \to \Xi^*$, such that

$$\zeta(\tau, X_\tau(t,x)) = Z_\tau(t,x) \qquad \mathbb{P}\text{-a.s. for a.e. } \tau \in [t,T]. \tag{5.31}$$

Thus setting $\underline{u}(t,x) = \gamma(t,x,\zeta(t,x))$ for every $t \in [0,T]$ and $x \in H$, one has

$$\overline{u}_\tau = \gamma(\tau, X_\tau(t,x), Z_\tau(t,x)) = \underline{u}(\tau, X_\tau(t,x)) \tag{5.32}$$

and $X_\tau(t,x)$ solves the following *closed-loop equation*:

$$X_\tau = e^{(\tau-t)A}x + \int_t^\tau e^{(\tau-\sigma)A}F(\sigma,X_\sigma)\,d\sigma + \int_t^\tau e^{(\tau-\sigma)A}G(\sigma,X_\sigma)r(\sigma,X_\sigma,\underline{u}(\sigma,X_\sigma))\,d\sigma \\ + \int_t^\tau e^{(\tau-\sigma)A}G(\sigma,X_\sigma)\,dW_\sigma, \quad \tau \in [t,T]. \tag{5.33}$$

Moreover, if F, G, Ψ, and Φ satisfy Hypothesis 4.3 with $K = \mathbb{R}$, then the feedback law is expressed in terms of the value function,

$$Z_\tau(t,x) = \nabla_x V(\tau,X_\tau(t,x))G(\tau,X_\tau(t,x)) \quad \mathbb{P}\text{-a.s. for a.e. } \tau \in [t,T]. \tag{5.34}$$

This relation is true for system (5.9)–(5.12) corresponding to $\overline{u}$; see [16]. Then noting that $Z_\tau(t,x) = Z_\tau^{\overline{u}}(t,x)$ $\mathbb{P}$-a.s. for a.e. $\tau \in [t,T]$, we obtain relation (5.34).

Example 5.9. We consider the controlled stochastic differential equation with delay in $\mathbb{R}^n$:

$$dx(\tau) = \left[\int_{-1}^0 x(\tau+\theta)a(d\theta) + f(\tau,x(\tau)) + \sigma(\tau,x(\tau))r(\tau,x(\tau),u(\tau))\right]d\tau \\ + \sigma(\tau,x(\tau))\,dW_\tau, \quad \tau \in [0,T], \tag{5.35}$$
$$x(0) = \mu_0, \qquad x(\theta) = \nu_0(\theta), \quad \text{for a.e. } \theta \in (-1,0),$$

and a cost functional of the form

$$J(0,\mu_0,\nu_0,u) = \mathbb{E}\int_0^T \ell(\tau,x(\tau),u(\tau))\,d\tau + \mathbb{E}\phi(x(T)), \tag{5.36}$$

that we minimize over all predictable controls u with values in $U \subset \mathbb{R}^N$.

We assume the following:

(1) $\mu_0 \in \mathbb{R}^n$, $\nu_0 \in L^2((-1,0);\mathbb{R}^n)$;

(2) $\{W_t : t \geq 0\}$ is a cylindrical Wiener process in $\mathbb{R}^d$ defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$, $(\mathcal{F}_t)_{t\geq 0}$ is its natural filtration completed with the null sets of $\mathcal{F}$;

(3) U is a Borel and bounded subset of $\mathbb{R}^N$, say $U = [-\delta,\delta]^N$, for some $\delta > 0$ and u is a $(\mathcal{F}_t)_{t\geq 0}$-predictable process with values in U;

(4) a is an $L(\mathbb{R}^n,\mathbb{R}^n)$-valued finite measure on $[-1,0]$;

(5) $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ is measurable and there exists a constant $L > 0$ such that

$$|f(t,0)| \leq L, \quad |f\nu(t,x_1) - f(t,x_2)| \leq L|x_1 - x_2|, \quad t \in [0,T],\ x_1,x_2 \in \mathbb{R}^n; \tag{5.37}$$

(6) $\sigma : [0,T] \times \mathbb{R}^n \to L(\mathbb{R}^d,\mathbb{R}^n)$ is measurable and there exists a positive constant L such that

$$|\sigma(t,x)| \leq L, \quad |\sigma(t,x_1) - \sigma(t,x_2)| \leq L|x_1 - x_2|, \quad t \in [0,T],\ x,x_1,x_2 \in \mathbb{R}^n; \tag{5.38}$$

(7) $r : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^d$ is measurable and for some constant $C > 0$,

$$\begin{aligned} &|r(t,x,u)| \leq C, \quad t \in [0,T],\ u \in U,\ x \in \mathbb{R}^n, \\ &|r(t,x_1,u_1) - r(t,x_2,u_2)| \\ &\qquad \leq C(|x_1 - x_2| + |u_1 - u_2|), \quad t \in [0,T],\ u,u_1,u_2 \in U,\ x_1,x_2 \in \mathbb{R}^n; \end{aligned} \tag{5.39}$$

(8) let $\ell(t,x,u) = \ell^0(t,x) + |u|^2$, $r(t,x,u) = Bu$ with $\ell^0 : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ measurable, such that $\ell^0(\cdot,0)$ is bounded, $\ell^0(t,\cdot)$ is Lipschitz continuous uniformly with respect to t, and $B \in L(\mathbb{R}^N, \mathbb{R}^d)$. Then the infimum over all $u \in U$ of $\ell(t,x,u) + zr(t,x,\gamma(t,x,u))$ is attained at a unique point:

$$\gamma(z) = \begin{cases} -\left(\dfrac{1}{2}\right) B^* z^* & \text{if } |B^* z^*| \leq 2\delta, \\ -\delta |B^* z^*|^{-1} B^* z^* & \text{if } |B^* z^*| > 2\delta, \end{cases} \tag{5.40}$$

and clearly γ is a Lipschitz continuous function;

(9) $\phi : \mathbb{R}^n \to \mathbb{R}$ verifies, for some constant $L > 0$,

$$|\phi(x_1) - \phi(x_2)| \leq L|x_1 - x_2|, \quad x_1,x_2 \in \mathbb{R}^n. \tag{5.41}$$

Following [7, Chapter 10], see also [28], we set $H = \mathbb{R}^n \times L^2((-1,0);\mathbb{R}^n)$,

$$\begin{aligned} \mathcal{D}(A) &= \left\{ \begin{pmatrix} \mu \\ \nu \end{pmatrix} \in H : \nu \in W^{1,2}((-1,0);\mathbb{R}^n),\ \nu(0) = \mu \right\}, \\ A \begin{pmatrix} \mu \\ \nu \end{pmatrix} &= \begin{pmatrix} \displaystyle\int_{-1}^{0} \nu(\theta) a(d\theta) \\ \dfrac{d\nu}{d\theta} \end{pmatrix}. \end{aligned} \tag{5.42}$$

It is known that A generates a strongly continuous semigroup in H; see again [7]. Moreover, if we set, for $t \in [0,T]$, $\mu \in \mathbb{R}^n$, $\nu \in L^2((-1,0);\mathbb{R}^n)$, $u \in U$,

$$\begin{aligned} &x_0 = \begin{pmatrix} \mu_0 \\ \nu_0 \end{pmatrix}, \qquad F\left(t, \begin{pmatrix} \mu \\ \nu \end{pmatrix}\right) = \begin{pmatrix} f(t,\mu) \\ 0 \end{pmatrix}, \qquad G\left(t, \begin{pmatrix} \mu \\ \nu \end{pmatrix}\right) = \begin{pmatrix} \sigma(t,\mu) \\ 0 \end{pmatrix}, \\ &R\left(t, \begin{pmatrix} \mu \\ \nu \end{pmatrix}, u\right) = r(t,\mu,u), \qquad l\left(t, \begin{pmatrix} \mu \\ \nu \end{pmatrix}, u\right) = \ell(t,\mu,u), \qquad \Phi \begin{pmatrix} \mu \\ \nu \end{pmatrix} = \phi(\mu), \end{aligned} \tag{5.43}$$

then (5.35) is equivalent (see [7, 28]) to

$$\begin{gathered} dX_\tau = (AX_\tau + F(\tau,X_\tau) + G(\tau,X_\tau)R(\tau,X_\tau,u_\tau))d\tau + G(\tau,X_\tau)dW_\tau, \quad \tau \in [0,T], \\ X_0 = x_0, \end{gathered} \tag{5.44}$$

where $X_\tau = \binom{x(\tau)}{x_\tau(\cdot)}$, with $x_\tau(\theta) = x(\tau+\theta)$, for every $\theta \in [-1,0]$. The cost functional becomes

$$J(0,x_0,u) = \mathbb{E}\int_0^T l(\tau,X_\tau,u_\tau)\,d\tau + \mathbb{E}\,\Phi(X_T). \tag{5.45}$$

Moreover, it is easy to verify that Hypotheses 2.1 and 5.1 hold. Thus Proposition 5.7 can be applied to obtain the existence of the optimal control in strong formulation and the feedback.

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