Hindawi Publishing Corporation
Journal of Applied Mathematics and Stochastic Analysis
Volume 2007, Article ID 82753, 33 pages
doi:10.1155/2007/82753

Research Article

Hereditary Portfolio Optimization with Taxes and Fixed Plus Proportional Transaction Costs—Part I

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Received 23 June 2006; Revised 26 October 2006; Accepted 27 October 2006

This is the first of the two companion papers which treat an infinite time horizon hereditary portfolio optimization problem in a market that consists of one *savings* account and one *stock* account. Within the *solvency* region, the investor is allowed to consume from the *savings* account and can make transactions between the two assets subject to paying capital gain taxes as well as a fixed plus proportional transaction cost. The *investor* is to seek an optimal consumption-trading strategy in order to maximize the expected utility from the total discounted consumption. The portfolio optimization problem is formulated as an infinite dimensional stochastic classical-impulse control problem. The quasi-variational HJB inequality (QVHJBI) for the value function is derived in this paper. The second paper contains the verification theorem for the optimal strategy. It is also shown there that the value function is a viscosity solution of the QVHJBI.

1. Introduction

This is the first of the two companion papers (see [1] for the second paper) which treat an infinite time horizon hereditary portfolio optimization problem in a financial market that consists of one *savings* account and one *stock* account. It is assumed that the *savings* account compounds continuously with a constant interest rate $r > 0$ and the unit price process, $\{S(t),\ t \geq 0\}$, of the underlying *stock* follows a nonlinear stochastic hereditary differential equation (see (2.5)) with an infinite but fading memory. The main purpose of the *stock* account is to keep track of the inventories, (i.e., the time instants and the base prices at which shares were purchased or short-sold) of the underlying stock for purpose of calculating the capital gain taxes, and so forth. In the stock price dynamics, we assume that both $f(S_t)$ (the mean rate of return) and $g(S_t)$ (the volatility coefficient) depend on the entire history of stock prices S_t over the time interval $(-\infty, t]$ instead of just the current stock price $S(t)$ at time $t \geq 0$ alone. Within the solvency region $\mathcal{S}_\kappa$ (to be defined in (2.29)) and under the requirements of paying fixed plus proportional transaction costs and capital gain taxes, the *investor* is allowed to consume from his *savings* account in accordance with a consumption rate process $C = \{C(t),\ t \geq 0\}$ and can make transactions between his *savings* and *stock* accounts according to a trading strategy $\mathcal{T} = \{(\tau(i), \zeta(i)),\ i = 1, 2, \ldots\}$, where $\tau(i),\ i = 0, 1, 2, \ldots$ denotes the sequence of transaction times and $\zeta(i)$ stands for quantities of the transaction at time $\tau(i)$ (see Definitions 2.4 and 2.5).

The *investor* will follow the following set of consumption, transaction, and taxation rules (Rules 1–6). Note that an action of the *investor* in the market is called a transaction if it involves trading of shares of the *stock* such as buying and selling.

Rule 1. At the time of each transaction, the *investor* has to pay a transaction cost that consists of a fixed cost $\kappa > 0$ and a proportional transaction cost with the cost rate of $\mu \geq 0$ for both selling and buying shares of the *stock*. All the purchases and sales of any number of stock shares will be considered one transaction if they are executed at the same time instant and therefore incur only one fixed fee $\kappa > 0$ (in addition to a proportional transaction cost).

Rule 2. Within the solvency region $\mathcal{S}_\kappa$, the *investor* is allowed to consume and to borrow money from his *savings* account for *stock* purchases. He can also sell and/or buy back at the current price shares of the *stock* he bought and/or short-sold at a previous time.

Rule 3. The proceeds for the sales of the *stock* minus the transaction costs and capital gain taxes will be deposited in his *savings* account and the purchases of stock shares together with the associated transaction costs and capital gain taxes (if short shares of the *stock* are bought back at a profit) will be financed from his *savings* account.

Rule 4. Without loss of generality, it is assumed that the interest income in the *savings* account is tax-free by using the effective interest rate $r > 0$, where the effective interest rate equals the interest rate paid by the bank minus the tax rate for the interest income.

Rule 5. At the time of a transaction (say $t \geq 0$), the *investor* is required to pay a capital gain tax (resp., be paid as a capital-loss credit) in the amount that is proportional to the amount of profit (resp., loss). A sale of stock shares is said to result in a profit if the current stock price $S(t)$ is higher than the base price $B(t)$ of the stock and it is a loss otherwise. The base price $B(t)$ is defined to be the price at which the stock shares were previously bought or short-sold, that is, $B(t) = S(t - \tau(t))$ where $\tau(t) > 0$ is the time duration for which those shares (long or short) have been held at time t. The *investor* will also pay capital gain taxes (resp., be paid as capital-loss credits) for the amount of profit (resp., loss) by short-selling shares of the *stock* and then buying back the shares at a lower (resp., higher) price at a later time. The tax will be paid (or the credit will be given) at the buying back time. Throughout the end, a negative amount of tax will be interpreted as a capital loss credit. The capital gain tax and capital loss credit rates are assumed to be the same as $\beta > 0$ for simplicity. Therefore, if $|m|$ ($m > 0$ stands for buying and $m < 0$ stands for selling) shares of the stock are traded at the current price $S(t)$ at the base $B(t) = S(t - \tau(t))$, then the amount of tax due at the transaction time is given by

$$|m|\beta(S(t) - S(t - \tau(t))). \tag{1.1}$$

Rule 6. The tax and/or credit will not exceed all other gross proceeds and/or total costs of the stock shares, that is,

$$\begin{aligned} m(1-\mu)S(t) &\geq \beta m \,|\, S(t) - S(t-\tau(t)) \,|, \quad \text{if } m \geq 0, \\ m(1+\mu)S(t) &\leq \beta m \,|\, S(t) - S(t-\tau(t)) \,|, \quad \text{if } m < 0, \end{aligned} \tag{1.2}$$

where $m \in \Re$ denotes the number of shares of the stock traded with $m \geq 0$ being the number of shares purchased and $m < 0$ being the number of shares sold.

Convention 1. Throughout the end, we assume that $\mu + \beta < 1$.

Under the above assumptions and Rules 1–6, the *investor's* objective is to seek an optimal consumption-trading strategy $(C^*, \mathcal{T}^*)$ in order to maximize

$$E\left[\int_0^\infty e^{-\delta t} \frac{C^\gamma(t)}{\gamma} dt\right], \tag{1.3}$$

the expected utility from the total discounted consumption over the infinite time horizon, where $\delta > 0$ represents the discount rate and $0 < \gamma < 1$ represents the *investor's* risk aversion factor.

Due to the fixed plus proportional transaction costs and the hereditary nature of the stock dynamics and inventories, the problem will be formulated as a combination of a classical control (for consumptions) problem and an impulse control (for the transactions) problem in infinite dimensions. A classical-impulse control problem in finite dimensions is treated in [2]. In this paper a quasi-variational Hamilton-Jocobi-Bellman inequality (QVHJBI) for the value function together with its boundary conditions is derived. The second paper (see [1]) establishes the verification theorem for the optimal investment trading strategy. In there, it is also shown that the value function is a viscosity solution of the QVHJBI (see (QVHJBI $(*)$) in Section 4.3.4). Due to the complexity of the analysis involved, the uniqueness result and finite-dimensional approximations for the viscosity solution of (QVHJBI $(*)$) will be treated separately in a future paper.

In recent years, there has been extensive amount of research on the optimal consumption-trading problems with proportional transaction costs (see, e.g., [3–6], and references contained therein) and fixed plus proportional transaction costs (see, e.g., [7]) within the geometric Brownian motion financial market. In all these papers, the objective has been to maximize the expected utility from the total discounted or averaged consumption over the infinite time horizon without considering the issues of capital gain taxes (resp., capital loss credits) when stock shares are sold at a profit (resp., loss). In different contents, the issues of capital gain taxes have been studied in [8–15], and references contained therein. In particular, [9, 10] considered the effect of capital gain taxes and capital loss credits on capital market equilibrium without consumption and transaction costs. These two papers illustrated that under some conditions, it may be more profitable to cut one's losses short and never to realize a gain because of capital loss credits and capital gain taxes as some conventional wisdom will suggest. In [8] the optimal transaction time problem with proportional transaction costs and capital-gain taxes was considered in order to maximize the long-run growth rate of the investment (or the so-called Kelley criterion), that is,

$$\lim_{t\to\infty}\frac{1}{t}E[\log V(t)], \tag{1.4}$$

where $V(t)$ is the value of the investment measured at time $t>0$. This paper is quite different from ours in that the unit price of the *stock* is described by a geometric Brownian motion, and all shares of the stock owned by the investor are to be sold at a chosen transaction time and all of its proceeds from the sale are to be used to purchase new shares of the *stock* immediately after the sale without consumption. Fortunately due to the nature of the geometric Brownian motion market, the authors of that paper were able to obtain some explicit results.

In recent years, the interest in stock price dynamics described by stochastic delay equations has increased tremendously (see, e.g., [16, 17]). To the best of the author's knowledge, this is the first paper that treats the optimal consumption-trading problem in which the hereditary nature of the stock price dynamics and the issue of capital gain taxes are taken into consideration. Due to drastically different nature of the problem and the techniques involved, the hereditary portfolio optimization problem with taxes and proportional transaction costs (i.e., $\kappa=0$ and $\mu,\nu>0$) remains to be solved.

This paper is organized as follows. The description of the stock price dynamics, the admissible consumption-trading strategies, and the formulation of the hereditary portfolio optimization problem are given in Section 2. In Section 3, the properties of the controlled state process are further explored and corresponding infinite-dimensional Markovian solution of the price dynamics is investigated. Section 4 contains the derivations of the QVHJBI together with its boundary conditions (QVHJBI $(*)$) using a Bellman-type dynamic programming principle.

The verification theorem for the optimal consumption-trading strategy and the proof that the value function is a viscosity solution of the (QVHJBI $(*)$) are contained in the second paper [1].

2. The hereditary portfolio optimization problem

Throughout the end, we use the following convention.

Convention 2. If $t\geq 0$ and $\phi:\Re\to\Re$ is a measurable function, define

$$\phi_t:(-\infty,0]\longrightarrow\Re\quad\text{by } \phi_t(\theta)=\phi(t+\theta),\quad \theta\in(-\infty,0]. \tag{2.1}$$

2.1. Hereditary price structure with infinite memory. Throughout the end of this paper, let $\rho:(-\infty,0]\to[0,\infty)$ be the *influence function with relaxation property* that satisfies the following conditions.

Condition 1. ρ is summable on $(-\infty,0]$, that is, $0 < \int_{-\infty}^{0} \rho(\theta)d\theta < \infty$.

Condition 2. For every $\lambda \le 0$, one has

$$\overline{K}(\lambda) = \text{ess} \sup_{\theta\in(-\infty,0]} \frac{\rho(\theta+\lambda)}{\rho(\theta)} \le \overline{K} < \infty, \qquad \underline{K}(\lambda) = \text{ess} \sup_{\theta\in(-\infty,0]} \frac{\rho(\theta)}{\rho(\theta+\lambda)} < \infty. \tag{2.2}$$

Under Conditions 1-2, it can be shown that ρ is essentially bounded and strictly positive on $(-\infty,0]$. Furthermore,

$$\lim_{\theta\to-\infty} \theta\rho(\theta) = 0. \tag{2.3}$$

The following are two examples of $\rho : (-\infty,0] \to [0,\infty)$ that satisfy Conditions 1 and 2:

(i) $\rho(\theta) = e^{\theta}$,
(ii) $\rho(\theta) = 1/(1+\theta^2)$, $-\infty < \theta \le 0$.

Let $\mathfrak{R} \times L^2_\rho(-\infty,0)$ (or simply $\mathfrak{R} \times L^2_\rho$ for short) be the history space of the stock price dynamics, where L^2_ρ is the class of ρ-weighted Hilbert space of measurable functions $\phi : (-\infty,0) \to \mathfrak{R}$ such that

$$\int_{-\infty}^{0} |\phi(\theta)|^2 \rho(\theta)d\theta < \infty. \tag{2.4}$$

For $t \in (-\infty,\infty)$, let $S(t)$ denote the unit price of the *stock* at time t. It is assumed that the unit *stock* price process $\{S(t),\ t \in (-\infty,\infty)\}$ satisfies the following stochastic hereditary differential equation with an infinite but fading memory:

$$dS(t) = S(t)[f(S_t)dt + g(S_t)dW(t)], \quad t \ge 0. \tag{2.5}$$

In the above equation, the process $\{W(t),\ t \ge 0\}$ is one-dimensional standard Brownian motion defined on a complete filtered probability space $(\Omega,\mathcal{F},P;\mathbf{F})$, where $\mathbf{F} = \{\mathcal{F}(t),\ t \ge 0\}$ is the P-augmented natural filtration generated by the Brownian motion $\{W(t),\ t \ge 0\}$. Note that $f(S_t)$ and $g(S_t)$ in (2.5) represent, respectively, the *mean growth rate* and the *volatility rate* of the *stock* price at time $t \ge 0$. Note that the *stock* is said to have a hereditary price structure with infinite but fading memory because both the drift term $S(t)f(S_t)$ and the diffusion term $S(t)g(S_t)$ in the right-hand side of (2.5) explicitly depend on the entire past history prices $(S(t),S_t) \in \mathfrak{R} \times L^2_\rho$ in a weighted fashion by the function ρ satisfying Conditions 1-2.

Note that we have used the following notation in the above:

$$\mathfrak{R}_+ = [0,\infty), \qquad L^2_{\rho,+} = \{\phi \in L^2_\rho \mid \phi(\theta) \ge 0\ \forall\theta \in (-\infty,0)\}. \tag{2.6}$$

It is assumed for simplicity and to guarantee the existence and uniqueness of a strong solution $S(t)$, $t \ge 0$, that the initial price function $(S(0),S_0) = (\psi(0),\psi) \in \mathfrak{R}_+ \times L^2_{\rho,+}$ is given and the functions $f,g : L^2_\rho \to [0,\infty)$ are continuous, and satisfy the following Lipschitz and linear growth conditions (see, e.g., [18–22] for the theory of stochastic functional differential equations with an infinite or a bounded memory).

Assumption 2.1 (linear growth condition). There exists a constant $c_1 > 0$ such that

$$0 \leq |\phi(0)f(\phi) + \phi(0)g(\phi)| \leq c_1(1 + \|(\phi(0),\phi)\|) \quad \forall (\phi(0),\phi) \in \Re_+ \times L^2_{\rho,+}. \tag{2.7}$$

Assumption 2.2 (Lipschitz condition). There exists a constant $c_2 > 0$ such that

$$\begin{aligned} &|\phi(0)f(\phi) - \varphi(0)f(\varphi)| + |\phi(0)g(\phi) - \varphi(0)g(\varphi)| \\ &\quad \leq c_2\|(\phi(0),\phi) - (\varphi(0),\varphi)\| \quad \forall (\phi(0),\phi),(\varphi(0),\varphi) \in \Re \times L^2_{\rho}, \end{aligned} \tag{2.8}$$

where

$$\|(\phi(0),\phi)\| = \sqrt{|\phi(0)|^2 + \int_{-\infty}^0 |\phi(\theta)|^2 \rho(\theta)d\theta}. \tag{2.9}$$

Assumption 2.3. There exist positive constants α and σ such that

$$0 < r < f(\phi) \leq \alpha, \quad 0 < \sigma \leq g(\phi), \quad \forall \phi \in L^2_{\rho,+}. \tag{2.10}$$

Note that the lower bound of the *mean rate of return* f in Assumption 2.3 is imposed to make sure that the *stock* account has a higher mean growth rate than the interest rate $r > 0$ for the *savings* account. Otherwise, it will be more profitable and less risky for the *investor* to put all his money in the *savings* account for the purpose of optimizing the expected utility from the total consumption.

Although the modeling of *stock* prices is still under intensive investigations, it is not the intention of this paper to address the validity of the model stock price dynamics treated in this paper but to illustrate the hereditary optimization problem that is explicitly dependent upon the entire past history of the stock prices for computing capital gain taxes or capital loss credits. The term "hereditary portfolio optimization" is therefore coined in this paper for the first time. We, however, mention here that stochastic hereditary equation similar to (2.5) was first used to model the behavior of elastic material with infinite memory and that stochastic functional differential equations with bounded memory have been used to model stock price dynamics in option pricing problems (see [16, 17]).

It can be shown that, for each initial historical price function $(\psi(0),\psi) \in \Re_+ \times L^2_{\rho,+}$, the price process $\{S(t),\ t \geq 0\}$ is a positive, continuous, and $\mathbf{F}$-adapted process defined on $(\Omega,\mathcal{F},P;\mathbf{F})$ but it is not Markovian with respect to any filtration that makes sense. For this reason, we frequently consider the corresponding $\Re_+ \times L^2_{\rho,+}$-valued process $\{(S(t),S_t),\ t \geq 0\}$ instead of the real-valued process $\{S(t),\ t \geq 0\}$. However, following approaches similar to that of [20, Section 3], it can be shown under Conditions 1-2 and Assumptions 2.1–2.3 that the $\Re_+ \times L^2_{\rho,+}$-valued process $\{(S(t),S_t),\ t \geq 0\}$ is strong Markovian with respect to the filtration $\mathbf{G}$, where $\mathbf{G} = \{\mathcal{G}(t),\ t \geq 0\}$ is the filtration generated by $\{S(t),\ t \geq 0\}$, that is,

$$\mathcal{G}(t) = \sigma(S(s),\ 0 \leq s \leq t)\,(= \sigma((S(s),S_s),\ 0 \leq s \leq t)), \quad \forall t \geq 0. \tag{2.11}$$

We also note here that, since security exchanges have only existed in a finite past, it is realistic but not technically required to assume that the initial historical price function

$(\psi(0),\psi)$ has the property that

$$\psi(\theta) = 0 \quad \forall \theta \le \overline{\theta} < 0 \text{ for some } \overline{\theta} < 0. \tag{2.12}$$

2.2. The stock inventory space. The space of stock inventories, **N**, will be the space of bounded functions $\xi : (-\infty, 0] \to \Re$ of the following form:

$$\xi(\theta) = \sum_{k=0}^{\infty} n(-k) \mathbf{1}_{\{\tau(-k)\}}(\theta), \quad \theta \in (-\infty, 0], \tag{2.13}$$

where $\{n(-k),\ k = 0,1,2,\ldots\}$ is a sequence in $\Re$ with $n(-k) = 0$ for all but finitely many k,

$$-\infty < \cdots < \tau(-k) < \cdots < \tau(-1) < \tau(0) = 0, \tag{2.14}$$

and $\mathbf{1}_{\{\tau(-k)\}}$ is the indicator function at $\tau(-k)$.

Note that the function $\xi : (-\infty, 0] \to \Re$ defined above denotes the inventory of the investor's stock account. In particular, when $\theta = \tau(-k)$, $\xi(\theta) = n(-k)$ is the number of shares of the stock purchased (resp., short-sold) if $n(-k) > 0$ (resp., $n(-k) < 0$) at time $\tau(-k)$, of course $\xi(\theta) = 0$ if $\theta \ne \tau(-k)$ for all $k = 0,1,2,\ldots$.

Let $\|\cdot\|_N$ (the norm of the space **N**) be defined by

$$\|\xi\|_N = \sup_{\theta \in (-\infty, 0]} |\xi(\theta)|, \quad \forall \xi \in \mathbf{N}. \tag{2.15}$$

As illustrated in Sections 2.3 and 2.5, **N** is the space in which the *investor's* stock inventory lives. The assumption that $n(-k) = 0$ for all but finitely many k implies that the *investor* can only have finitely many open positions in his stock account. However, the number of open positions may increase from time to time. Note that the *investor* is said to have an open long (resp., short) position at time τ if he still owns (resp., owes) all or part of the stock shares that were originally purchased (resp., short-sold) at a previous time τ. The only way to close a position is to sell what he owns and buy back what he owes.

If $\eta : \Re \to \Re$ is a bounded function of the form

$$\eta(t) = \sum_{k=-\infty}^{\infty} n(k) \mathbf{1}_{\{\tau(k)\}}(t), \quad -\infty < t < \infty, \tag{2.16}$$

where

$$-\infty < \cdots < \tau(-k) < \cdots < 0 = \tau(0) < \tau(1) < \cdots < \tau(k) < \cdots < \infty, \tag{2.17}$$

then for each $t \ge 0$, we define, using Convention 2, the function $\eta_t : (-\infty, 0] \to \Re$ by

$$\eta_t(\theta) = \eta(t + \theta), \quad \theta \in (-\infty, 0]. \tag{2.18}$$

In this case,

$$\eta_t(\theta) = \sum_{k=-\infty}^{\infty} n(k)\mathbf{1}_{\{\tau(k)\}}(t+\theta) = \sum_{k=-\infty}^{Q(t)} n(k)\mathbf{1}_{\{\tau(k)\}}(\theta), \quad \theta \in (-\infty,0], \tag{2.19}$$

where $Q(t) = \sup\{k \geq 0 \mid \tau(k) \leq t\}$.

2.3. Consumption-trading strategies. Let $(X(0-),N_{0-},S(0),S_0) = (x,\xi,\psi(0),\psi) \in \Re \times \mathbf{N} \times \Re_+ \times L^2_{\rho,+}$ be the *investor's* initial portfolio immediately prior to $t = 0$. That is, the investor starts with $x \in \Re$ dollars in his *savings* account, the initial stock inventory,

$$\xi(\theta) = \sum_{k=0}^{\infty} n(-k)\mathbf{1}_{\{\tau(-k)\}}(\theta), \quad \theta \in (-\infty,0), \tag{2.20}$$

and the initial profile of historical stock prices $(\psi(0),\psi) \in \Re_+ \times L^2_{\rho,+}$, where $n(-k) > 0$ (resp., $n(-k) < 0$) represents an open long (resp., short) position at $\tau(-k)$. Within the *solvency region* $\mathcal{S}_\kappa$ (see (2.29)), the *investor* is allowed to consume from his *savings* account and can make transactions between his *savings* and *stock* accounts under Rules 1–6 and according to a consumption-trading strategy $\pi = (C,\mathcal{T})$ defined below.

Definition 2.4. The pair $\pi = (C,\mathcal{T})$ is said to be a consumption-trading strategy if

(i) the consumption rate process $C = \{C(t),\ t \geq 0\}$ is a nonnegative $\mathbf{G}$-progressively measurable process such that

$$\int_0^T C(t)dt < \infty \quad P\text{-a.s. } \forall T > 0; \tag{2.21}$$

(ii) $\mathcal{T} = \{(\tau(i),\zeta(i)),\ i = 1,2,\ldots\}$ is a trading strategy with $\tau(i)$, $i = 1,2,\ldots$, being a sequence of trading times that are $\mathbf{G}$-stopping times such that

$$0 = \tau(0) \leq \tau(1) < \cdots < \tau(i) < \cdots, \qquad \lim_{i\to\infty} \tau(i) = \infty \quad P\text{-a.s.}, \tag{2.22}$$

and for each $i = 0,1,\ldots$,

$$\zeta(i) = (\ldots,m(i-k),\ldots,m(i-2),m(i-1),m(i)) \tag{2.23}$$

is an $\mathbf{N}$-valued $\mathcal{G}(\tau(i))$-measurable random vector (instead of a random variable in $\Re$) that represents the trading quantities at the trading time $\tau(i)$. In the above, $m(i) > 0$ (resp., $m(i) < 0$) is the number of stock shares newly purchased (resp., short-sold) at the current time $\tau(i)$ and at the current price of $S(\tau(i))$ and, for $k = 1,2,\ldots$, $m(i-k) > 0$ (resp., $m(i-k) < 0$) is the number of stock shares bought back (resp., sold) at the current time $\tau(i)$ and at the current price of $S(\tau(i))$ in his open short (resp., long) position at the previous time $\tau(i-k)$ and at the base price of $S(\tau(i-k))$.

For each stock inventory ξ of the form expressed (2.13), Rules 1–6 also dictate that the investor can purchase or short sell new shares and/or buy back (resp., sell) all or part of what he owes (resp., owns). Therefore, the trading quantity $\{m(-k),\ k = 0,1,\dots\}$ must satisfy the constraint set $\mathcal{R}(\xi) \subset \mathbf{N}$ defined by

$$\mathcal{R}(\xi) = \Bigg\{\zeta \in \mathbf{N} \mid \zeta = \sum_{k=0}^{\infty} m(-k)\mathbf{1}_{\{\tau(-k)\}},\ -\infty < m(0) < \infty,$$
$$\text{either } n(-k) > 0,\ m(-k) \le 0,\ n(-k) + m(-k) \ge 0 \tag{2.24}$$
$$\text{or } n(-k) < 0,\ m(-k) \ge 0,\ n(-k) + m(-k) \le 0 \text{ for } k \ge 1\Bigg\}.$$

2.4. Solvency region. Throughout the end of this paper, the investor's state space $\mathbf{S}$ is taken to be $\mathbf{S} = \Re \times \mathbf{N} \times \Re_+ \times L^2_{\rho,+}$. An element $(x,\xi,\psi(0),\psi) \in \mathbf{S}$ is called a portfolio, where $x \in \Re$ is investor's holding in his *savings* account, ξ is the investor's stock inventory, and $(\psi(0),\psi) \in \Re_+ \times L^2_{\rho,+}$ is the profile of historical stock prices. Define the function $H_\kappa : \mathbf{S} \to \Re$ as follows:

$$H_\kappa(x,\xi,\psi(0),\psi) = \max\{G_\kappa(x,\xi,\psi(0),\psi),\ \min\{x, n(-k),\ k = 0,1,2,\dots\}\}, \tag{2.25}$$

where $G_\kappa : \mathbf{S} \to \Re$ is the liquidating function defined by

$$G_\kappa(x,\xi,\psi(0),\psi) = x - \kappa + \sum_{k=0}^{\infty}\big[\min\{(1-\mu)n(-k),(1+\mu)n(-k)\}\psi(0)$$
$$- n(-k)\beta(\psi(0) - \psi(\tau(-k)))\big]. \tag{2.26}$$

In the right-hand side of the above expression,

$$x - \kappa = \text{the amount in his } savings \text{ account after deducting the fixed transaction cost } \kappa; \tag{2.27}$$

and for each $k = 0,1,\dots$,

$\min\{(1-\mu)n(-k),(1+\mu)n(-k)\}\psi(0)$
= the proceed for selling $n(-k) > 0$ or buying back $n(-k) < 0$ shares of the stock net of proportional transactional cost;

$-n(-k)\beta(\psi(0) - \psi(\tau(-k)))$
= the capital gain tax to be paid for selling the $n(-k)$ shares of the stock with the current price of $\psi(0)$ and base price of $\psi(\tau(-k))$. (2.28)

Therefore, $G_\kappa(x,\xi,\psi(0),\psi)$ defined in (2.26) represents the cash value (if the assets can be liquidated at all) after closing all open positions and paying all transaction costs (fixed plus proportional transactional costs) and taxes.

The *solvency region* $\mathcal{S}_\kappa$ of the portfolio optimization problem is defined as

$$\begin{aligned}\mathcal{S}_\kappa &= \{(x,\xi,\psi(0),\psi) \in \mathbf{S} \mid H_\kappa(x,\xi,\psi(0),\psi) \ge 0\} \\ &= \{(x,\xi,\psi(0),\psi) \in \mathbf{S} \mid G_\kappa(x,\xi,\psi(0),\psi) \ge 0\} \cup \mathbf{S}_+,\end{aligned} \tag{2.29}$$

where $\mathbf{S}_+ = \Re_+ \times \mathbf{N}_+ \times \Re_+ \times M^2_{\rho,+}$ and $\mathbf{N}_+ = \{\xi \in \mathbf{N} \mid \xi(\theta) \ge 0,\ \forall \theta \in (-\infty,0]\}$.

Note that within the *solvency region* $\mathcal{S}_\kappa$, there are positions that cannot be closed at all, namely, those $(x,\xi,\psi(0),\psi) \in \mathcal{S}_\kappa$ such that

$$(x,\xi,\psi(0),\psi) \in \mathbf{S}_+, \qquad G_\kappa(x,\xi,\psi(0),\psi) < 0. \tag{2.30}$$

This is due to the insufficiency of funds to pay for the transaction costs and/or taxes, and so forth. Observe that the *solvency* region $\mathcal{S}_\kappa$ is an unbounded and nonconvex subset of the state space $\mathbf{S}$. The boundary $\partial\mathcal{S}_\kappa$ will be described in detail in Section 4.3.

2.5. Portfolio dynamics and admissible strategies. At time $t \ge 0$, the investor's portfolio in the financial market will be denoted by the quadruplet $(X(t),N_t,S(t),S_t)$, where $X(t)$ denotes the *investor's* holdings in his *savings* account, $N_t \in \mathbf{N}$ is the *inventory* of his *stock* account, and $(S(t),S_t)$ describes the profile of the unit prices of the *stock* over the past history $(-\infty,t]$ as described in Section 2.1.

Given the initial portfolio

$$(X(0-),N_{0-},S(0),S_0) = (x,\xi,\psi(0),\psi) \in \mathbf{S} \tag{2.31}$$

and applying a consumption-trading strategy $\pi = (C,\mathcal{T})$ (see Definition 2.4), the portfolio dynamics of $\{Z(t) = (X(t),N_t,S(t),S_t),\ t \ge 0\}$ can then be described as follows.

Firstly, the *savings* account holding $\{X(t),\ t \ge 0\}$ satisfies the following differential equation between the trading times:

$$dX(t) = [rX(t) - C(t)]dt, \quad \tau(i) \le t < \tau(i+1), \quad i = 0,1,2,\dots, \tag{2.32}$$

and the following jumped quantity at the trading time $\tau(i)$:

$$\begin{aligned}X(\tau(i)) = X(\tau(i)-) - \kappa &\\ - \sum_{k=0}^{\infty} m(i-k)\big[(1-\mu)S(\tau(i)) &- \beta\big(S(\tau(i)) \\ &- S(\tau(i-k))\big)\big]\mathbf{1}_{\{n(i-k)>0,\,-n(i-k)\le m(i-k)\le 0\}} \\ - \sum_{k=0}^{\infty} m(i-k)\big[(1+\mu)S(\tau(i)) &- \beta\big(S(\tau(i)) \\ &- S(\tau(i-k))\big)\big]\mathbf{1}_{\{n(i-k)<0,\,0\le m(i-k)\le -n(i-k)\}}.\end{aligned} \tag{2.33}$$

As a reminder, $m(i) > 0$ (resp., $m(i) < 0$) means buying (resp., selling) new stock shares at $\tau(i)$ and $m(i-k) > 0$ (resp., $m(i-k) < 0$) means buying back (resp., selling) some or all of what he owed (resp., owned).

Secondly, the inventory of the *investor's* stock account at time $t \geq 0$, $N_t \in \mathbf{N}$, does not change between the trading times and can be expressed as the following equation:

$$N_t = N_{\tau(i)} = \sum_{k=-\infty}^{Q(t)} n(k)\mathbf{1}_{\tau(k)}, \quad \text{if } \tau(i) \leq t < \tau(i+1),\ i = 0,1,2\ldots, \tag{2.34}$$

where $Q(t) = \sup\{k \geq 0 \mid \tau(k) \leq t\}$.

It has the following jumped quantity at the trading time $\tau(i)$:

$$N_{\tau(i)} = N_{\tau(i)-} \oplus \zeta(i), \tag{2.35}$$

where $N_{\tau(i)-} \oplus \zeta(i) : (-\infty,0] \to \mathbf{N}$ is defined by

$$\begin{aligned}
&\big(N_{\tau(i)-} \oplus \zeta(i)\big)(\theta) \\
&\quad = \sum_{k=0}^{\infty} \hat{n}(i-k)\mathbf{1}_{\{\tau(i-k)\}}\big(\tau(i)+\theta\big) = m(i)\mathbf{1}_{\{\tau(i)\}}\big(\tau(i)+\theta\big) \\
&\qquad + \sum_{k=1}^{\infty} \big[n(i-k) + m(i-k)\big(\mathbf{1}_{\{n(i-k)<0,0\leq m(i-k)\leq -n(i-k)\}} \\
&\qquad\qquad + \mathbf{1}_{\{n(i-k)>0,-n(i-k)\leq m(i-k)\leq 0\}}\big)\big] \\
&\qquad \cdot \mathbf{1}_{\{\tau(i-k)\}}\big(\tau(i)+\theta\big), \quad \theta \in (-\infty,0].
\end{aligned} \tag{2.36}$$

Thirdly, since the *investor* is small, the unit stock price process $\{S(t),\ t \geq 0\}$ will not be in anyway affected by the *investor's* action in the market and is again described as in (2.5).

Definition 2.5. If the *investor* starts with an initial portfolio,

$$\big(X(0-), N_{0-}, S(0), S_0\big) = \big(x, \xi, \psi(0), \psi\big) \in \mathcal{S}_\kappa. \tag{2.37}$$

The consumption-trading strategy $\pi = (C, \mathcal{T})$ defined in Definition 2.4 is said to be *admissible* at $(x,\xi,\psi(0),\psi)$ if

$$\begin{aligned}
&\zeta(i) \in \mathcal{R}\big(N_{\tau(i)-}\big), \quad \forall i = 1,2,\ldots, \\
&\big(X(t), N_t, S(t), S_t\big) \in \mathcal{S}_\kappa, \quad \forall t \geq 0.
\end{aligned} \tag{2.38}$$

The class of consumption-investment strategies admissible at $(x,\xi,\psi(0),\psi) \in \mathcal{S}_\kappa$ will be denoted by $\mathcal{U}_\kappa(x,\xi,\psi(0),\psi)$.

2.6. The problem statement. Given the initial state $(X(0-), N_{0-}, S(0), S_0) = (x,\xi,\psi(0),\psi) \in \mathcal{S}_\kappa$, the *investor's* objective is to find an admissible consumption-trading strategy

$\pi^* \in \mathcal{U}_\kappa(x,\xi,\psi(0),\psi)$ that maximizes the following expected utility from the total discounted consumption:

$$J_\kappa(x,\xi,\psi(0),\psi;\pi) = E^{x,\xi,\psi(0),\psi;\pi}\left[\int_0^\infty e^{-\delta t}\frac{C^\gamma(t)}{\gamma}\,dt\right] \tag{2.39}$$

among the class of admissible consumption-trading strategies $\mathcal{U}_\kappa(x,\xi,\psi(0),\psi)$, where $E^{x,\xi,\psi(0),\psi;\pi}[\cdots]$ is the expectation with respect to $P^{x,\xi,\psi(0),\psi;\pi}\{\cdots\}$, the probability measure induced by the controlled (by π) state process $\{(X(t),N_t,S(t),S_t),\ t \geq 0\}$ and conditioned on the initial state

$$(X(0-),N_{0-},S(0),S_0) = (x,\xi,\psi(0),\psi). \tag{2.40}$$

In the above, $\delta > 0$ denotes the discount factor, and $0 < \gamma < 1$ indicates that the utility function $U(c) = c^\gamma/\gamma$, for $c > 0$, is a function of HARA (hyperbolic absolute risk aversion) type that was considered in most of optimal consumption-trading literature (see, e.g., [3–5, 7, 6]) with or without a fixed transaction cost. The admissible (consumption-trading) strategy $\pi^* \in \mathcal{U}_\kappa(x,\xi,\psi(0),\psi)$ that maximizes $J_\kappa(x,\xi,\psi(0),\psi;\pi)$ is called an optimal (consumption-trading) strategy and the function $V_\kappa : \mathcal{S}_\kappa \to \Re_+$ defined by

$$V_\kappa(x,\xi,\psi(0),\psi) = \sup_{\pi\in\mathcal{U}_\kappa(x,\xi,\psi(0),\psi)} J_\kappa(x,\xi,\psi(0),\psi;\pi) = J_\kappa(x,\xi,\psi(0),\psi;\pi^*) \tag{2.41}$$

is called the value function of the hereditary portfolio optimization problem.

The hereditary portfolio optimization problem considered in this paper is then formalized as follows.

Problem 1. For each given initial state $(x,\xi,\psi(0),\psi) \in \mathcal{S}_\kappa$, identify the optimal strategy π^* and its corresponding value function $V_\kappa : \mathcal{S}_\kappa \to \Re_+$.

3. The controlled state process

Given an initial state $(x,\xi,\psi(0),\psi) \in \mathcal{S}_\kappa$ and an admissible consumption-investment strategy $\pi = (C,\mathcal{T}) \in \mathcal{U}_\kappa(x,\xi,\psi(0),\psi)$, the $\mathcal{S}_\kappa$-valued controlled state process will be denoted by $\{Z(t) = (X(t),N_t,S(t),S_t),\ t \geq 0\}$. Note that the dependence of the controlled state process on the initial state $(x,\xi,\psi(0),\psi)$ and the admissible consumption-trading strategy π will be suppressed for notational simplicity.

The main purpose of this section is to establish the Markovian and the Dynkin formula for the controlled state process $\{Z(t),\ t \geq 0\}$. Note that the $\Re_+ \times L^2_{\rho,+}$-valued process $\{(S(t),S_t),\ t \geq 0\}$ described by (2.5) is uncontrollable by the *investor* and is therefore independent of the consumption-trading strategy $\pi \in \mathcal{U}_\kappa(x,\xi,\psi(0),\psi)$ but is dependent on the initial historical price function $(S(0),S_0) = (\psi(0),\psi) \in \Re_+ \times L^2_{\rho,+}$.

3.1. The properties of the stock prices. To study the Markovian properties of the $\Re_+ \times L^2_{\rho,+}$-valued solution process $\{(S(t),S_t),\ t \geq 0\}$ where $S_t(\theta) = S(t+\theta)$, $\theta \in (-\infty,0]$, and $(S(0),S_0) = (\psi(0),\psi)$, we need the following notation and ancillary results.

Let $(\Re \times L^2_\rho)^*$ be the space of bounded linear functionals (or the topological dual of the space $\Re \times L^2_\rho$) equipped with the operator norm $\|\cdot\|^*$ defined by

$$\|\Phi\|^* = \sup_{(\phi(0),\phi)\neq(0,\mathbf{0})} \frac{|\Phi(\phi(0),\phi)|}{\|(\phi(0),\phi)\|}, \quad \Phi \in (\Re \times L^2_\rho)^*. \tag{3.1}$$

Note that $(\Re \times L^2_\rho)^*$ can be identified with $\Re \times L^2_\rho$ by the well-known Riesz representation theorem.

Let $(\Re \times L^2_\rho)^\dagger$ be the space of bounded bilinear functionals $\Phi : (\Re \times L^2_\rho) \times (\Re \times L^2_\rho) \to \Re$ (i.e., $\Phi((\phi(0),\phi),(\cdot,\cdot))$, $\Phi((\cdot,\cdot),(\phi(0),\phi)) \in (\Re \times L^2_\rho)^*$ for each $(\phi(0),\phi) \in \Re \times L^2_\rho$), equipped with the operator norm $\|\cdot\|^\dagger$ defined by

$$\|\Phi\|^\dagger = \sup_{(\phi(0),\phi)\neq(0,\mathbf{0})} \frac{\|\Phi((\cdot,\cdot),(\phi(0),\phi))\|^*}{\|(\phi(0),\phi)\|} = \sup_{(\phi(0),\phi)\neq(0,\mathbf{0})} \frac{\|\Phi((\phi(0),\phi),(\cdot,\cdot))\|}{\|(\phi(0),\phi)\|}. \tag{3.2}$$

Let $\Phi : \Re \times L^2_\rho \to \Re$. The function Φ is said to be Fréchet differentiable at $(\phi(0),\phi) \in \Re \times L^2_\rho$ if for each $(\varphi(0),\varphi) \in \Re \times L^2_\rho$,

$$\Phi((\phi(0),\phi)+(\varphi(0),\varphi)) - \Phi(\phi(0),\phi) = D\Phi(\phi(0),\phi)(\varphi(0),\varphi) + o(\|(\varphi(0),\varphi)\|), \tag{3.3}$$

where $D\Phi : \Re \times L^2_\rho \to (\Re \times L^2_\rho)^*$ and $o : \Re \to \Re$ is a function such that

$$\frac{o(\|(\varphi(0),\varphi)\|)}{\|(\varphi(0),\varphi)\|} \longrightarrow 0 \quad \text{as } \|(\varphi(0),\varphi)\| \longrightarrow 0. \tag{3.4}$$

In this case, $D\Phi(\phi(0),\phi) \in (\Re \times L^2_\rho)^*$ is called the (first-order) Fréchet derivative of Φ at $(\phi(0),\phi) \in \Re \times L^2_\rho$. The function Φ is said to be continuously Fréchet differentiable if its Fréchet derivative $D\Phi : \Re \times L^2_\rho \to (\Re \times L^2_\rho)^*$ is continuous under the operator norm $\|\cdot\|^*$. The function Φ is said to be twice Fréchet differentiable at $(\phi(0),\phi) \in \Re \times L^2_\rho$ if its Fréchet derivative $D\Phi(\phi(0),\phi) : \Re \times L^2_\rho \to \Re$ exists and there exists a bounded bilinear functional $D^2\Phi(\phi(0),\phi) : (\Re \times L^2_\rho) \times (\Re \times L^2_\rho) \to \Re$ where for each $(\varphi(0),\varphi),(\varsigma(0),\varsigma) \in \Re \times L^2_\rho$,

$$D^2\Phi(\phi(0),\phi)((\cdot,\cdot),(\varphi(0),\varphi)), D^2\Phi(\phi(0),\phi)((\varsigma(0),\varsigma),(\cdot,\cdot)) \in (\Re \times L^2_\rho)^*, \tag{3.5}$$

and where

$$\begin{aligned}(D\Phi((\phi(0),\phi)+(\varphi(0),\varphi)) &- D\Phi(\phi(0),\phi))(\varsigma(0),\varsigma) \\ &= D^2\Phi(\phi(0),\phi)((\varsigma(0),\varsigma),(\varphi(0),\varphi)) + o(\|(\varsigma(0),\varsigma)\|,\|(\varphi(0),\varphi)\|).\end{aligned} \tag{3.6}$$

Here, $o:\Re\times\Re\to\Re$ is such that

$$\frac{o(\cdot,\|(\varphi(0),\varphi)\|)}{\|(\varphi(0),\varphi)\|}\longrightarrow 0,\quad \text{as } \|(\varphi(0),\varphi)\|\longrightarrow 0,$$
$$\frac{o(\|(\varphi(0),\varphi)\|,\cdot)}{\|(\varphi(0),\varphi)\|}\longrightarrow 0,\quad \text{as } \|(\varphi(0),\varphi)\|\longrightarrow 0. \tag{3.7}$$

In this case, the bounded bilinear functional $D^2\Phi(\phi(0),\phi):(\Re\times L^2_\rho)\times(\Re\times L^2_\rho)\to\Re$ is the second order Fréchet derivative of Φ at $(\phi(0),\phi)\in\Re\times L^2_\rho$.

The second-order Fréchet derivative $D^2\Phi$ is said to be globally Lipschitz on $\Re\times L^2_\rho$ if there exists a constant $K>0$ such that

$$\begin{aligned}&\|D^2\Phi(\phi(0),\phi)-D^2\Phi(\varphi(0),\varphi)\|^\dagger\\ &\quad\le K\|(\phi(0),\phi)-(\varphi(0),\varphi)\|,\quad \forall\,(\phi(0),\phi),(\varphi(0),\varphi)\in\Re\times L^2_\rho.\end{aligned}\tag{3.8}$$

Assuming all the partial and/or Frechet derivatives of the following exist, the actions of the first-order Fréchet derivative $D\Phi(\phi(0),\phi)$ and the second-order Fréchet $D^2\Phi(\phi(0),\phi)$ can be expressed as

$$\begin{aligned}D\Phi(\phi(0),\phi)(\varphi(0),\varphi)&=\varphi(0)\partial_{\phi(0)}\Phi(\phi(0),\phi)+D_\phi\Phi(\phi(0),\phi)\varphi,\\ D^2\Phi(\phi(0),\phi)&((\varphi(0),\varphi),(\varsigma(0),\varsigma))\\ &=\varphi(0)\partial^2_{\phi(0)}\Phi(\phi(0),\phi)\varsigma(0)+\varsigma(0)\partial_{\phi(0)}D_\phi\Phi(\phi(0),\phi)\varphi\\ &\quad+\varphi(0)D_\phi\partial_{\phi(0)}\Phi(\phi(0),\phi)(\varphi,\varsigma)+D^2_\phi\Phi(\phi(0),\phi)\varsigma,\end{aligned}\tag{3.9}$$

where $\partial_{\phi(0)}\Phi$ and $\partial^2_{\phi(0)}\Phi$ are the first- and second-order partial derivatives of Φ with respect to its first variable $\phi(0)\in\Re$, $D_\phi\Phi$ and $D^2_\phi\Phi$ are the first- and second-order Fréchet derivatives with respect to its second variable $\phi\in L^2_\rho$, $\partial_{\phi(0)}D_\phi\Phi$ is the second-order derivative first with respect to ϕ in the Fréchet sense and then with respect to $\phi(0)$, and so forth.

Let $C^{2,2}(\Re\times L^2_\rho)$ be the space of functions $\Phi:\Re\times L^2_\rho\to\Re$ that are twice continuously differentiable with respect to both its first and second variables. The space of $\Phi\in C^{2,2}(\Re\times L^2_\rho)$ with $D^2\Phi$ being globally Lipschitz will be denoted by $C^{2,2}_{\text{lip}}(\Re\times L^2_\rho)$.

3.1.1. The weak infinitesimal generator Γ. For each $(\phi(0),\phi)\in\Re\times L^2_\rho$, define $\tilde{\phi}:(-\infty,\infty)\to\Re$ by

$$\tilde{\phi}(t)=\begin{cases}\phi(0), & \text{for } t\in[0,\infty),\\ \phi(t), & \text{for } t\in(-\infty,0).\end{cases}\tag{3.10}$$

Then for each $\theta \in (-\infty, 0]$ and $t \in [0, \infty)$,

$$\tilde{\phi}_t(\theta) = \tilde{\phi}(t+\theta) \begin{cases} \phi(0), & \text{for } t+\theta \geq 0, \\ \phi(t+\theta), & \text{for } t+\theta < 0. \end{cases} \tag{3.11}$$

A bounded measurable function $\Phi : \Re \times L^2_\rho \to \Re$, that is, $\Phi \in C_b(\Re \times L^2_\rho)$, is said to belong to $\mathcal{D}(\Gamma)$, the domain of the weak infinitesimal operator Γ, if the following limit exists for each fixed $(\phi(0), \phi) \in \Re \times L^2_\rho$:

$$\Gamma(\Phi)(\phi(0), \phi) \equiv \lim_{t \downarrow 0} \frac{\Phi(\phi(0), \tilde{\phi}_t) - \Phi(\phi(0), \phi)}{t}. \tag{3.12}$$

Remark 3.1. Note that $\Phi \in C^{2,2}_{\text{lip}}(\Re \times L^2_\rho)$ does not guarantee that $\Phi \in \mathcal{D}(\Gamma)$. For example, let $\overline{\theta} > 0$ and define a simple tame function $\Phi : \Re \times L^2_\rho \to \Re$ by

$$\Phi(\phi(0), \phi) = \phi(-\overline{\theta}) \quad \forall (\phi(0), \phi) \in \Re \times L^2_\rho. \tag{3.13}$$

Then it can be shown that $\Phi \in C^{2,2}_{\text{lip}}(\Re \times L^2_\rho)$ and yet $\Phi \notin \mathcal{D}(\Gamma)$.

It will be shown in the proof of Theorem 3.5, however, that any tame function of the above form can be approximated by a sequence of quasi-tame functions that are in $\mathcal{D}(\Gamma)$.

Again, consider the associated Markovian $\Re \times L^2_\rho$-valued process $\{(S(t), S_t),\ t \geq 0\}$ described by (2.5) with the initial historical price function $(S(0), S_0) = (\psi(0), \psi) \in \Re \times L^2_\rho$. We have the following result for its weak infinitesimal generator $\mathbf{A} + \Gamma$ (see, e.g., [18, 21, 22]).

Theorem 3.2. *If* $\Phi \in C^{2,2}_{\text{lip}}(\Re \times L^2_\rho) \cap \mathcal{D}(\Gamma)$, *then*

$$\lim_{t \downarrow 0} \frac{E[\Phi(S(t), S_t) - \Phi(\psi(0), \psi)]}{t} = (\mathbf{A} + \Gamma)\Phi(\psi(0), \psi), \tag{3.14}$$

where

$$\mathbf{A}\Phi(\psi(0), \psi) = \frac{1}{2}\partial^2_{\psi(0)}\Phi(\psi(0), \psi)\psi^2(0)g^2(\psi) + \partial_{\psi(0)}\Phi(\psi(0), \psi)\psi(0)f(\psi), \tag{3.15}$$

and $\Gamma(\Phi)(\psi(0), \psi)$ *is as defined in (3.12).*

It seems from a glance at (3.15) that $\mathbf{A}\Phi(\psi(0), \psi)$ requires only the existence of the first- and second-order partial derivatives $\partial_{\psi(0)}\Phi$ and $\partial^2_{\psi(0)}\Phi$ of $\Phi(\psi(0), \psi)$ with respect to its first variable $\psi(0) \in \Re$. However, detailed derivations of the formula reveal that a stronger condition that $\Phi \in C^{2,2}_{\text{lip}}(\Re \times L^2_\rho)$ is required.

We have the following Dynkin formula (see [20–22]).

Theorem 3.3. *Let* $\Phi \in C^{2,2}_{\text{lip}}(\Re \times L^2_\rho) \cap \mathcal{D}(\Gamma)$. *Then*

$$E[e^{-\delta\tau}\Phi(S(\tau), S_\tau)] = \Phi(\psi(0), \psi) + E\left[\int_0^\tau e^{-\delta t}(\mathbf{A} + \Gamma - \delta I)\Phi(S(t), S_t)\,dt\right], \tag{3.16}$$

for every P-a.s. finite $\mathbf{G}$*-stopping time* τ.

The function $\Phi \in C^{2,2}_{\text{lip}}(\Re \times L^2_\rho) \cap \mathfrak{D}(\Gamma)$ that has the following special form is referred to as a quasi-tame function:

$$\Phi(\phi(0),\phi) = \Psi(m(\phi(0),\phi)), \tag{3.17}$$

where

$$\begin{aligned} m(\phi(0),\phi) = \Big(\phi(0), \int_{-\infty}^{0} \eta_1(\phi(\theta))\lambda_1(\theta)d\theta, \ldots, \\ \int_{-\infty}^{0} \eta_n(\phi(\theta))\lambda_n(\theta)d\theta\Big) \quad \forall (\phi(0),\phi) \in \Re \times L^2_\rho, \end{aligned} \tag{3.18}$$

for some positive integer n and some functions $m \in C(\Re \times L^2_\rho; \Re^{n+1})$, $\eta_i \in C^\infty(\Re)$, $\lambda_i \in C^1((-\infty,0])$ with

$$\lim_{\theta \to -\infty} \lambda_i(\theta) = \lambda_i(-\infty) = 0 \tag{3.19}$$

for $i = 1,2,\ldots,n$, and $\Psi \in C^\infty(\Re^{n+1})$ of the form $\Psi(x, y_1, y_2, \ldots, y_n)$.

We have the following Ito formula in case $\Phi \in \Re \times L^2_\rho$ is a quasi-tame function in the sense defined above.

Theorem 3.4. *Let $\{(S(t),S_t),\ t \geq 0\}$ be the $\Re \times L^2_\rho$-valued solution process corresponding to (2.5) with an initial historical price function $(\psi(0),\psi) \in \Re \times L^2_\rho$. If $\Phi \in C(\Re \times L^2_\rho)$ is a quasi-time function, then $\Phi \in \mathfrak{D}(\mathbf{A}) \cap \mathfrak{D}(\Gamma)$ and*

$$\begin{aligned} e^{-\delta\tau}\Phi(S(\tau),S_\tau) = \Phi(\psi(0),\psi) + \int_0^\tau e^{-\delta t}(\mathbf{A}+\Gamma-\delta I)\Phi(S(t),S_t)dt \\ + \int_0^\tau e^{-\delta t}\Phi_x(S(t),S_t)S(t)f(S_t)dW(t) \end{aligned} \tag{3.20}$$

for every finite $\mathbf{G}$-stopping time τ, where I is the identity operator.

Moreover, if $\Phi \in C(\Re \times L^2_\rho)$ is of the form described in (3.17)-(3.18), then

$$\begin{aligned} (\mathbf{A}+\Gamma)\Phi(\psi(0),\psi) &= \sum_{i=1}^{n} \Psi_{y_i}(m(\psi(0),\psi)) \\ &\quad \times \Big(\eta_i(\psi(0))\lambda_i(0) - \int_{-\infty}^{0} \eta_i(\psi(\theta))\dot{\lambda}_i(\theta)d\theta\Big) \\ &\quad + \Psi_x(m(\psi(0),\psi))\psi(0)f(\psi) + \frac{1}{2}\Psi_{xx}(m(\psi(0),\psi))\psi^2(0)g^2(\psi), \end{aligned} \tag{3.21}$$

where Ψ_x, Ψ_{y_i}, and Ψ_{xx} denote the partial derivatives of $\Psi(x,y_1,\ldots,y_n)$ with respect to its appropriate variables.

Proof. The Ito formula for a quasi-tame function $\Phi : \Re \times L^2([-h,0]) \to \Re$ for the $\Re \times L^2([-h,0])$ solution process $\{(x(t),x_t),\ t \geq 0\}$ of a stochastic function differential equation with a bounded delay $h > 0$ is obtained in an unpublished dissertation by Arriojas [18] (the same result can also be obtained from [21, 22] with some modifications). The same arguments can be easily extended to the infinite memory stochastic hereditary differential equation (2.5) considered in this paper. To avoid further lengthening the paper, we omit the proof here. □

In the following, we will prove that above Ito's formula also holds for any tame function $\Phi : \Re \times \mathbf{C} \to \Re$ of the following form:

$$\Phi(\phi(0),\phi) = \Psi(m(\phi(0),\phi)) = \Psi(\phi(0),\phi(-\theta_1),\dots,\phi(-\theta_n)), \tag{3.22}$$

where $\mathbf{C}$ is the space continuous function $\phi : (-\infty,0] \to \Re$ equipped with uniform topology, $0 < \theta_1 < \theta_2 < \cdots < \theta_n < \infty$, and $\Psi(x,y_1,\dots,y_n)$ is such that $\Psi \in C^\infty(\Re^{n+1})$.

Theorem 3.5. *Let $\{(S(t),S_t),\ t \ge 0\}$ be the $\Re \times L^2_\rho$-valued process corresponding to (2.5) with an initial historical price function $(\psi(0),\psi) \in \Re \times L^2_\rho$. If $\Phi : \Re \times \mathbf{C} \to \Re$ is a tame function defined by (3.22), then $\Phi \in \mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\Gamma)$ and*

$$\begin{aligned} e^{-\delta\tau}&\Psi(S(\tau),S(\tau-\theta_1),\dots,S(\tau-\theta_n)) \\ &= \Psi(\psi(0),\psi(-\theta_1),\dots,\psi(-\theta_n)) \\ &\quad + \int_0^\tau e^{-\delta t}(\mathbf{A}-\delta I)\Psi(S(t),S(t-\theta_1),\dots,S(t-\theta_n))\,dt \\ &\quad + \int_0^\tau e^{-\delta t}\Psi_x(S(t),S(t-\theta_1),\dots,S(t-\theta_n))S(t)f(S_t)\,dW(t) \end{aligned} \tag{3.23}$$

for every finite $\mathbf{G}$-stopping time τ, where

$$\begin{aligned} (\mathbf{A}+\Gamma)&\Psi(\psi(0),\psi(-\theta_1),\dots,\psi(-\theta_n)) \\ &= \Psi_x(\psi(0),\psi(-\theta_1),\dots,\psi(-\theta_n))\psi(0)f(\psi) \\ &\quad + \frac{1}{2}\Psi_{xx}(\psi(0),\psi(-\theta_1),\dots,\psi(-\theta_n))\psi^2(0)g^2(\psi), \end{aligned} \tag{3.24}$$

with Ψ_x and Ψ_{xx} being the first- and second-order derivatives with respect to x of $\Psi(x,y_1,\dots,y_n)$.

Proof. Without loss of generality, we will assume in order to simplify the notation that $\Psi : \Re \times \Re \to \Re$ with $\Psi(x,y)$ and there is only one delay in the function $\Psi(\phi(0),\phi(-\overline{\theta}))$ for some fixed $\overline{\theta} \in (0,\infty)$.

We will approximate the function $\Psi(\phi(0),\phi(-\overline{\theta}))$ by a sequence of quasi-tame functions as follows.

Throughout the end of this proof, we define for each $\phi \in L^2_\rho$ and each $k = 1,2,\dots$ the function $\phi^{(k)}(-\overline{\theta};h)$ by

$$\phi^{(k)}(-\overline{\theta};h) = k\int_{-\infty}^0 h(k(-\overline{\theta}-\varsigma))\phi(\varsigma)\,d\varsigma, \tag{3.25}$$

where $h:\Re\to\Re_+$ is the mollifier defined by

$$h(\varsigma)=\begin{cases}0, & \text{if } |\varsigma|\geq 1,\\ c\exp\left\{\dfrac{1}{|\varsigma|^2-1}\right\}, & \text{if } |\varsigma|<1,\end{cases}\tag{3.26}$$

and $c>0$ is the constant chosen so that $\int_{-\infty}^{\infty}h(\varsigma)d\varsigma=1$.

It is clear that

$$\lim_{k\to\infty}\Psi\big(\phi(0),\phi^{(k)}(-\overline{\theta};h)\big)=\Psi\big(\phi(0),\phi(-\overline{\theta})\big),\tag{3.27}$$

since $\lim_{k\to\infty}\phi^{(k)}(-\overline{\theta};h)=\phi(-\overline{\theta})$. Moreover,

$$\Gamma\Psi\big(\phi(0),\phi^{(k)}(-\overline{\theta};h)\big)$$
$$=\Psi_y\big(\phi(0),-\phi(\overline{\theta})\big)\left(\phi(0)kh(-k\overline{\theta})-2k^3\int_{-\infty}^{0}\phi(\theta)h\big(k(\overline{\theta}+\theta)\big)\frac{\overline{\theta}+\theta}{\big(k^2(-\overline{\theta}-\theta)^2-1\big)^2}d\theta\right)\tag{3.28}$$

and by the Lebesque dominating convergence theorem, we have

$$\lim_{k\to\infty}\Gamma\Psi\big(\phi(0),\phi^{(k)}(-\overline{\theta};h)\big)=0.\tag{3.29}$$

Therefore, for any finite **G**-stopping time τ, we have from Theorem 3.4 and sample path convergence property of the Ito integrals (see [23, 24]) that

$$\begin{aligned}
e^{-\delta\tau}\Psi\big(S(\tau),S(\tau-\overline{\theta})\big)&=\lim_{k\to\infty}e^{-\delta\tau}\Psi\big(S(\tau),S_\tau^{(k)}(-\overline{\theta};h)\big)\\
&=\lim_{k\to\infty}\bigg[\Psi\big(\psi(0),\psi^{(k)}(-\overline{\theta};h)\big)\\
&\qquad+\int_0^\tau e^{-\delta s}(\mathbf{A}+\Gamma-\delta I)\Psi\big(S(s),S_s^{(k)}(-\overline{\theta};h)\big)ds\\
&\qquad+\int_0^\tau e^{-\delta s}\Psi_x\big(S(s),S_s^{(k)}(-\overline{\theta};h)\big)S(s)f(S_s)dW(s)\bigg]\\
&=\Psi\big(\psi(0),\psi(-\overline{\theta})\big)+\int_0^\tau e^{-\delta s}\Big(\Psi_x\big(S(s),S(s-\overline{\theta})\big)S(s)f(S_s)\\
&\qquad+\frac{1}{2}\Psi_{xx}\big(S(s),S(s-\overline{\theta})\big)S^2(s)g^2(S_s)\Big)ds\\
&\quad+\int_0^\tau e^{-\delta s}\Psi_x\big(S(s),S(s-\overline{\theta})\big)S(s)f(S_s)dW(s).
\end{aligned}\tag{3.30}$$

This proves the theorem. □

3.2. Dynkin's formula for the controlled state process. Combining the above results in this section and results for general jumped processes (see [25, 26]), we have the following Dynkin formula for the controlled (by the admissible strategy π) $\mathcal{S}_\kappa$-valued state process $\{Z(t) = (X(t), N_t, S(t), S_t),\ t \geq 0\}$:

$$E[e^{-\delta\tau}\Phi(Z(\tau))] = \Phi(Z(0-)) + E\left[\int_0^\tau e^{-\delta t}\mathcal{L}^{C(t)}\Phi(Z(t))\,dt\right] + E\left[\sum_{0\leq t\leq\tau} e^{-\delta t}(\Phi(Z(t)) - \Phi(Z(t-)))\right], \tag{3.31}$$

for all $\Phi : \Re\times\mathbf{N}\times\Re\times L^2_\rho \to \Re$ such that $\Phi(\cdot,\xi,\psi(0),\psi) \in C^1(\Re)$ for each $(\xi,\psi(0),\psi) \in \mathbf{N}\times\Re\times L^2_\rho$ and $\Phi(x,\xi,\cdot,\cdot) \in C^{2,2}_{\text{lip}}(\Re\times L^2_\rho)\cap\mathfrak{D}(\Gamma)$ for each $(x,\xi) \in \Re\times\mathbf{N}$, where

$$\mathcal{L}^c\Phi(x,\xi,\psi(0),\psi) = (\mathbf{A}+\Gamma-\delta I+(rx-c)\partial_x)\Phi(x,\xi,\psi(0),\psi), \tag{3.32}$$

and $\mathbf{A}$ and Γ are as defined in (3.12) and (3.15).

Note that $E[\cdots]$ in the above stands for $E^{x,\xi,\psi(0),\psi;\pi}[\cdots]$.

In the case, $\Phi \in C(\Re\times\mathbf{N}\times\Re\times L^2_\rho)$ is such that $\Phi(x,\xi,\cdot,\cdot) : \Re\times L^2_\rho \to \Re$ is a quasi-tame (resp., tame) function on $\Re\times L^2_\rho$ of the form described in (3.17)-(3.18) (resp., (3.22)), then the following Ito formula for the controlled state process $\{Z(t) = (X(t), N_t, S(t), S_t),\ t \geq 0\}$ also holds true.

Theorem 3.6. *If $\Phi \in C(\Re\times\mathbf{N}\times\Re\times L^2_\rho)$ is such that $\Phi(x,\xi,\cdot,\cdot) : \Re\times L^2_\rho \to \Re$ is a quasi-tame function (resp., tame) on $\Re\times L^2_\rho$, then*

$$e^{-\delta\tau}\Phi(Z(\tau)) = \Phi(Z(0-)) + \int_0^\tau e^{-\delta t}\mathcal{L}^{C(t)}\Phi(Z(t))\,dt + \int_0^\tau e^{-\delta t}\partial_{\psi(0)}\Phi(Z(t))S(t)f(S_t)\,dW(t) + \left[\sum_{0\leq t\leq\tau} e^{-\delta t}(\Phi(Z(t)) - \Phi(Z(t-)))\right], \tag{3.33}$$

for every P-a.s. finite $\mathbf{G}$-stopping time τ.

Moreover, if $\Phi(x,\xi,\psi(0),\psi) = \Psi(x,\xi,m(\psi(0),\psi))$ where $\Psi \in C(\Re\times\mathbf{N}\times\Re^{n+1})$ and $m(\psi(0),\psi)$ is given by (3.17)-(3.18) (resp., (3.22)), then

$$\mathcal{L}^c\Phi(x,\xi,\psi(0),\psi) = (\mathbf{A}+\Gamma-kI+(rx-c)\partial_x)\Psi(x,\xi,m(\psi(0),\psi)) \tag{3.34}$$

and $(\mathbf{A}+\Gamma)\Psi(x,\xi,m(\psi(0),\psi))$ is as given in (3.21) (resp., (3.24)) for each fixed $(x,\xi) \in \Re\times\mathbf{N}$ (and of Theorem 3.6)

4. The quasi-variational HJB inequality

The main objective of this section is to derive the dynamic programming equation for the value function in form of an infinite-dimensional quasi variational Hamilton-Jacobi-Bellman (HJB) inequality (or QVHJBI) (see (QVHJBI $(*)$) in Section 4.3.4).

4.1. The dynamic programming principle.

The following Bellman-type dynamic programming principle (DPP) was established in [6] and still holds true in our problem by combining it with that obtained in [27–29] for optimal classical control of stochastic functional differential equations with a bounded memory. For the sake of saving space, we take the following result as the starting point without proof for deriving our dynamic principle equation.

Proposition 4.1. *Let $(x,\xi,\psi(0),\psi) \in \mathcal{S}_\kappa$ be given and let $\mathbb{O}$ be an open subset of $\mathcal{S}_\kappa$ containing $(x,\xi,\psi(0),\psi)$. For $\pi = (C,\mathcal{T}) \in \mathcal{U}_\kappa(x,\xi,\psi(0),\psi)$, let $\{(X(t),N_t,S(t),S_t),\ t \geq 0\}$ be given by (2.5), and (2.32)–(2.36). Define*

$$\tau = \inf\{t \geq 0 \mid (X(t),N_t,S(t),S_t) \notin \overline{\mathbb{O}}\}, \quad \text{where } \overline{\mathbb{O}} \text{ is the closure of } \mathbb{O}. \tag{4.1}$$

Then, for each $t \in [0,\infty)$, the following optimality equation holds:

$$V_\kappa(x,\xi,\psi(0),\psi) = \sup_{\pi \in \mathcal{U}_\kappa(x,\xi,\psi(0),\psi)} E\bigg[\int_0^{t\wedge\tau} e^{-\delta s}\frac{C^\gamma(s)}{\gamma}ds + \mathbf{1}_{\{t\wedge\tau<\infty\}}e^{-\delta(t\wedge\tau)}V_\kappa(X(t\wedge\tau),N_{t\wedge\tau},S(t\wedge\tau),S_{t\wedge\tau})\bigg], \tag{4.2}$$

where the notation $a \wedge b = \min\{a,b\}$ for $a,b \in \mathfrak{R}$ is used.

4.2. Derivation of the QVHJBI.

In this section, we will derive the Hamilton-Jacobi-Bellman (HJB) quasi-variational inequality (see (QVHJBI $(*)$) in Section 4.3.4) based on the dynamic programming principle described in Proposition 4.1. We emphasize here that it is not our intension to rigorously verify every step involved in the derivations since the rigorous verification is to be done in [1], the continuation of this paper. To derive (QVHJBI $(*)$) in Section 4.3.4, we consider the effects on the value function when there is consumption but no transaction and when there is transaction but no consumption.

4.2.1. Consumptions without transaction. Assume first that there is no transaction then the corresponding state process $\{Z(t) = (X(t),N_t,S(t),S_t),\ t \geq 0\}$ satisfies the following set of equations:

$$dX(t) = [rX(t) - C(t)]dt, \quad t \geq 0, \frac{dS(t)}{S(t)} = f(S_t)dt + g(S_t)dW(t), \quad t \geq 0, \tag{4.3}$$

$$N_t = \xi, \quad t \geq 0, \tag{4.4}$$

with the initial state $(X(0-),N_{0-},S(0),S_0) = (x,\xi,\psi(0),\psi) \in \mathcal{S}_\kappa$.

In this case, $V_\kappa(X(t),N_t,S(t),S_t) = V_\kappa(X(t-),N_{t-},S(t),S_t)$ for all $t \geq 0$, since there is no jump transaction. Assume that the value function $V_\kappa : \mathcal{S}_\kappa \to \Re_+$ is sufficiently smooth. From Proposition 4.1 and (4.4), we have

$$\begin{aligned}
0 &\geq \lim_{t\downarrow 0} \frac{E\left[e^{-\delta t} V_\kappa(X(t),N_t,S(t),S_t) - V_\kappa(x,\xi,\psi(0),\psi)\right]}{t} + \lim_{t\downarrow 0} \frac{1}{t} E\left[\int_0^t e^{-\delta s} \frac{C^\gamma(s)}{\gamma} ds\right] \\
&= \lim_{t\downarrow 0} \frac{E\left[e^{-\delta t}\left(V_\kappa(X(t),N_t,S(t),S_t) - V_\kappa(x,\xi,\psi(0),\psi)\right)\right]}{t} \\
&\quad + \lim_{t\downarrow 0} \frac{\left[\left(e^{-\delta t}-1\right) V_\kappa(x,\xi,\psi(0),\psi)\right]}{t} + \lim_{t\downarrow 0} \frac{1}{t} E\left[\int_0^t e^{-\delta s} \frac{C^\gamma(s)}{\gamma} ds\right] \\
&= \lim_{t\downarrow 0} \frac{E\left[e^{-\delta t} \int_0^t \left(\mathbf{A} + \Gamma - (rX(t) - C(t))\partial_x\right) V_\kappa(X(t),N_t,S(t),S_t)\, dt\right]}{t} \\
&\quad - \delta V_\kappa(x,\xi,\psi(0),\psi) + \lim_{t\downarrow 0} \frac{1}{t} E\left[\int_0^t e^{-\delta s} \frac{C^\gamma(s)}{\gamma} ds\right] \\
&= \left(\mathbf{A} + \Gamma + (rx - c)\partial_x - \delta\right) V_\kappa(x,\xi,\psi(0),\psi) + \frac{c^\gamma}{\gamma}, \quad \forall c \geq 0.
\end{aligned} \tag{4.5}$$

This shows that

$$\begin{aligned}
0 \geq \mathcal{A} V_\kappa(x,\xi,\psi(0),\psi) &\equiv \sup_{c\geq 0}\left(\mathcal{L}^c V_\kappa(x,\xi,\psi(0),\psi) + \frac{c^\gamma}{\gamma}\right) \\
&= \left(\mathbf{A} + \Gamma + rx\partial_x - \delta\right) V_\kappa(x,\xi,\psi(0),\psi) + \sup_{c\geq 0}\left(\frac{c^\gamma}{\gamma} - c\partial_x V_\kappa(x,\xi,\psi(0),\psi)\right) \\
&= \left(\mathbf{A} + \Gamma + rx\partial_x - \delta\right) V_\kappa(x,\xi,\psi(0),\psi) + \frac{1-\gamma}{\gamma}\left(\partial_x V_\kappa\right)^{\gamma/(\gamma-1)}(x,\xi,\psi(0),\psi),
\end{aligned} \tag{4.6}$$

since the maximum of the above expression is achieved at

$$c^* = \left(\partial_x V_\kappa\right)^{1/(\gamma-1)}(x,\xi,\psi(0),\psi). \tag{4.7}$$

Note that the Fréchet differential operators $\mathbf{A}$ and Γ are defined in (3.12) and (3.15), respectively.

4.2.2. Transactions without consumption. We next consider the case where there are transactions but no consumption. For each locally bounded $\Phi : \mathcal{S}_\kappa \to \Re_+$ and each $(x,\xi,\psi(0),\psi) \in \mathcal{S}_\kappa$ define the *intervention operator*

$$\mathcal{M}_\kappa \Phi(x,\xi,\psi(0),\psi) = \sup\left\{\Phi(\hat{x},\hat{\xi},\hat{\psi}(0),\hat{\psi}) \mid \zeta \in \mathcal{R}(\xi) - \{\mathbf{0}\}, (\hat{x},\hat{\xi},\hat{\psi}(0),\hat{\psi}) \in \mathcal{S}_\kappa\right\}, \tag{4.8}$$

where $(\hat{x},\hat{\xi},\hat{\psi}(0),\hat{\psi})$ are as defined as follows:

$$\begin{aligned}\hat{x} = x - \kappa - \big(m(0) + \mu\,|\,m(0)\,|\,\big)\psi(0)\\ - \sum_{k=1}^{\infty}\big[(1+\mu)m(-k)\psi(0) - \beta m(-k)\big(\psi(0) - \psi(\tau(-k))\big)\big]\\ \cdot \mathbf{1}_{\{n(-k)<0,0\le m(-k)\le -n(-k)\}}\\ - \sum_{k=1}^{\infty}\big[(1-\mu)m(-k)\psi(0) - \beta m(-k)\big(\psi(0) - \psi(\tau(-k))\big)\big]\\ \cdot \mathbf{1}_{\{n(-k)>0,-n(-k)\le m(-k)\le 0\}},\end{aligned} \tag{4.9}$$

and for all $\theta \in (-\infty,0]$,

$$\begin{aligned}\hat{\xi}(\theta) = (\xi \oplus \zeta)(\theta) = m(0)\mathbf{1}_{\{\tau(0)\}}(\theta)\\ + \sum_{k=1}^{\infty}\big(n(-k) + m(-k)\big[\mathbf{1}_{\{n(-k)<0,0\le m(-k)\le -n(-k)\}}\\ + \mathbf{1}_{\{n(-k)>0,-n(-k)\le m(-k)\le 0\}}\big]\big)\mathbf{1}_{\{\tau(-k)\}}(\theta),\end{aligned} \tag{4.10}$$

and again

$$(\hat{\psi}(0),\hat{\psi}) = (\psi(0),\psi). \tag{4.11}$$

If $(\hat{x},\hat{\xi},\hat{\psi}(0),\hat{\psi}) \notin \mathcal{S}_\kappa$ for all $\zeta \in \mathcal{R}(\xi) - \{\mathbf{0}\}$, we set $\mathcal{M}_\kappa\Phi(x,\xi,\psi(0),\psi) = 0$. If for all $(x,\xi,\psi(0),\psi) \in \mathcal{S}_\kappa$ there exists $(\hat{x},\hat{\xi},\hat{\psi}(0),\hat{\psi}) \in \mathcal{S}_\kappa$ such that

$$\mathcal{M}_\kappa\Phi(x,\xi,\psi(0),\psi) = \Phi(\hat{x},\hat{\xi},\hat{\psi}(0),\hat{\psi}), \tag{4.12}$$

then we set

$$\hat{\zeta}(x,\xi,\psi(0),\psi) = \hat{\zeta}_\Phi(x,\xi,\psi(0),\psi) = (\hat{x},\hat{\xi},\hat{\psi}(0),\hat{\psi}) \in \mathcal{R}(\xi). \tag{4.13}$$

Note we let $\hat{\zeta}(x,\xi,\psi(0),\psi)$ denote a measurable selection of the map

$$(x,\xi,\psi(0),\psi) \longmapsto (\hat{x},\hat{\xi},\hat{\psi}(0),\hat{\psi}) \tag{4.14}$$

defined in (4.13).

We make the following technical assumption regarding the existence of a measurable selection:

$$\hat{\zeta}(x,\xi,\psi(0),\psi) = \hat{\zeta}_{V_\kappa}(x,\xi,\psi(0),\psi) \tag{4.15}$$

for the value function $V_\kappa : \mathcal{S}_\kappa \to \Re$, that is, there exists a measurable function $\hat{\zeta}_{V_\kappa} : \mathcal{S}_\kappa \to \Re$ such that

$$V_\kappa(\hat{\zeta}(x,\xi,\psi(0),\psi)) = \mathcal{M}_\kappa V_\kappa(x,\xi,\psi(0),\psi) \quad \forall (x,\xi,\psi(0),\psi) \in \mathcal{S}_\kappa. \tag{4.16}$$

Assumption 4.2. For each $(x,\xi,\psi(0),\psi) \in \mathcal{S}_\kappa$, there exists a measurable function $\hat{\zeta}_{V_\kappa} : \mathcal{S}_\kappa \to \Re$ such that (4.16) is satisfied for every $(x,\xi,\psi(0),\psi) \in \mathcal{S}_\kappa$.

Assume without loss of generality that the *investor's* portfolio immediately prior to time t is $(X(t-),N_{t-},S(t),S_t) = (x,\xi,\psi(0),\psi) \in \mathcal{S}_\kappa$. An immediate transaction of the amount $\zeta \in \mathcal{R} - \{\mathbf{0}\}$ without consumption at time t (i.e., $C(t) = 0$) yields that $(X(t),N_t, S(t),S_t) = (\hat{x},\hat{\xi},\hat{\psi}(0),\hat{\psi})$, where $\hat{x}$, $\hat{\xi}$, and $\hat{\psi}(0)$, $\hat{\psi}$ are as given in (4.9)–(4.11). It is clear that

$$V_\kappa(x,\xi,\psi(0),\psi) \geq \mathcal{M}_\kappa V_\kappa(x,\xi,\psi(0),\psi), \quad \forall (x,\xi,\psi(0),\psi) \in \mathcal{S}_\kappa. \tag{4.17}$$

Combining Sections 4.2.1 and 4.2.2, we have the following inequality:

$$\max\{\mathcal{A}V_\kappa, \mathcal{M}_\kappa V_\kappa - V_\kappa\} \leq 0 \quad \text{on } \mathcal{S}_\kappa^\circ, \tag{4.18}$$

where $\mathcal{S}_\kappa^\circ$ denotes the interior of the *solvency region* $\mathcal{S}_\kappa$.

Using a standard technique in deriving the variational HJB inequality for stochastic classical-singular and classical-impulse control problems (see [30, 31] for stochastic impulse controls, [2, 7] for stochastic classical-impulse controls, and [28, 29] for classical and singular controls of stochastic delay equations), one can show that on the set

$$\{(x,\xi,\psi(0),\psi) \in \mathcal{S}_\kappa^\circ \mid \mathcal{M}_\kappa V_\kappa(x,\xi,\psi(0),\psi) < V_\kappa(x,\xi,\psi(0),\psi)\} \tag{4.19}$$

we have $\mathcal{A}V_\kappa = 0$ and on the set

$$\{(x,\xi,\psi(0),\psi) \in \mathcal{S}_\kappa^\circ \mid \mathcal{A}V_\kappa(x,\xi,\psi(0),\psi) < 0\} \tag{4.20}$$

we have $\mathcal{M}_\kappa V_\kappa = V_\kappa$. Therefore, we have the following QVHJBI on $\mathcal{S}_\kappa^\circ$:

$$\max\{\mathcal{A}V_\kappa, \mathcal{M}_\kappa V_\kappa - V_\kappa\} = 0 \quad \text{on } \mathcal{S}_\kappa^\circ, \tag{4.21}$$

where

$$\mathcal{A}\Phi = (\mathbf{A} + \Gamma + rx\partial_x - \delta)\Phi + \sup_{c\geq 0}\left(\frac{c^\gamma}{\gamma} - c\partial_x\Phi\right), \tag{4.22}$$

$\mathcal{M}_\kappa\Phi$ is as given in (4.8), and the operators $\mathbf{A}$ and Γ are given as follows.

4.3. Boundary values of the QVHJBI.

4.3.1. The solvency region for $\kappa = 0$ and $\mu > 0$. When there is no fixed transaction cost (i.e., $\kappa = 0$ and $\mu > 0$), the solvency region can be written as

$$\mathcal{S}_0 = \{(x,\xi,\psi(0),\psi) \mid G_0(x,\xi,\psi(0),\psi) \geq 0\}, \tag{4.23}$$

where G_0 is the liquidating function given in (2.26) with $\kappa = 0$. This is because

$$x \geq 0, \qquad n(-i) \geq 0 \quad \forall i = 0,1,2,\ldots \Longrightarrow G_0(x,\xi,\psi(0),\psi) \geq 0. \tag{4.24}$$

In this case, all shares of the *stock* owned or owed can be liquidated due the absence of a fixed transaction cost $\kappa = 0$.

4.3.2. Decomposition of $\partial\mathcal{S}_\kappa$. For $I \subset \aleph \equiv \{0,1,2,\ldots\}$, the boundary $\partial\mathcal{S}_\kappa$ of $\mathcal{S}_\kappa$ can be decomposed as follows:

$$\partial\mathcal{S}_\kappa = \bigcup_{I\subset\aleph} \left(\partial_{-,I}\mathcal{S}_\kappa \cup \partial_{+,I}\mathcal{S}_\kappa\right), \tag{4.25}$$

where

$$\partial_{-,I}\mathcal{S}_\kappa = \partial_{-,I,1}\mathcal{S}_\kappa \cup \partial_{-,I,2}\mathcal{S}_\kappa,$$

$$\partial_{+,I}\mathcal{S}_\kappa = \partial_{+,I,1}\mathcal{S}_\kappa \cup \partial_{+,I,2}\mathcal{S}_\kappa,$$

$$\begin{aligned}
\partial_{+,I,1}\mathcal{S}_\kappa &= \big\{(x,\xi,\psi(0),\psi) \mid G_\kappa(x,\xi,\psi(0),\psi) = 0,\ x \geq 0,\\
&\qquad n(-i) < 0\ \forall i \in I,\ n(-i) \geq 0\ \forall i \notin I\big\},\\
\partial_{+,I,2}\mathcal{S}_\kappa &= \big\{(x,\xi,\psi(0),\psi) \mid G_\kappa(x,\xi,\psi(0),\psi) < 0,\ x \geq 0,\\
&\qquad n(-i) = 0\ \forall i \in I,\ n(-i) \geq 0\ \forall i \notin I\big\},\\
\partial_{-,I,1}\mathcal{S}_\kappa &= \big\{(x,\xi,\psi(0),\psi) \mid G_\kappa(x,\xi,\psi(0),\psi) = 0,\ x < 0,\\
&\qquad n(-i) < 0\ \forall i \in I,\ n(-i) \geq 0\ \forall i \notin I\big\},\\
\partial_{-,I,2}\mathcal{S}_\kappa &= \big\{(x,\xi,\psi(0),\psi) \mid G_\kappa(x,\xi,\psi(0),\psi) < 0,\ x = 0,\\
&\qquad n(-i) = 0\ \forall i \in I,\ n(-i) \geq 0\ \forall i \notin I\big\}.
\end{aligned} \tag{4.26}$$

The interface (intersection) between $\partial_{+,I,1}\mathcal{S}_\kappa$ and $\partial_{+,I,2}\mathcal{S}_\kappa$ is denoted by

$$\begin{aligned}
Q_{+,I} = \big\{(x,\xi,\psi(0),\psi) \mid G_\kappa(x,\xi,\psi(0),\psi) = 0,\ x \geq 0,\\
n(-i) = 0\ \forall i \in I,\ n(-i) \geq 0\ \forall i \notin I\big\}.
\end{aligned} \tag{4.27}$$

Whereas the interface between $\partial_{-,I,1}\mathcal{S}_\kappa$ and $\partial_{-,I,2}\mathcal{S}_\kappa$ is denoted by

$$\begin{aligned}
Q_{-,I} = \big\{(0,\xi,\psi(0),\psi) \mid G_\kappa(0,\xi,\psi(0),\psi) = 0,\ x = 0,\\
n(-i) = 0\ \forall i \in I,\ n(-1) \geq 0\ \forall i \notin I\big\}.
\end{aligned} \tag{4.28}$$

For example, if $I = \aleph$, then $n(-i) < 0\ \forall i = 0,1,2,\ldots$ and

$$G_\kappa(x,\xi,\psi(0),\psi) \geq 0 \Longrightarrow x \geq \kappa. \tag{4.29}$$

In this case, $\partial_{-,\aleph}\mathcal{S}_\kappa = \emptyset$ (the empty set),

$$\begin{aligned}
\partial_{+,\aleph,1}\mathcal{S}_\kappa &= \{(x,\xi,\psi(0),\psi) \mid G_\kappa(x,\xi,\psi(0),\psi) = 0,\ x \geq 0,\ n(-i) < 0\ \forall i \in \aleph\},\\
\partial_{+,\aleph,2}\mathcal{S}_\kappa &= \{(x,\xi,\psi(0),\psi) \mid G_\kappa(x,\xi,\psi(0),\psi) < 0,\ x \geq 0,\ n(-i) = 0\ \forall i \in \aleph\}\\
&= \{(x,\mathbf{0},\psi(0),\psi) \mid 0 \leq x \leq \kappa\}.
\end{aligned} \tag{4.30}$$

On the other hand, if $I = \emptyset$ (the empty set), that is, $n(-i) \geq 0$ for all $i \in \aleph$, then

$$\begin{aligned}
\partial_{+,\emptyset,1}\mathcal{S}_\kappa &= \{(x,\xi,\psi(0),\psi) \mid G_\kappa(x,\xi,\psi(0),\psi) = 0,\ x \geq 0,\ n(-i) \geq 0\ \forall i \in \aleph\},\\
\partial_{+,\emptyset,2}\mathcal{S}_\kappa &= \{(x,\xi,\psi(0),\psi) \mid G_\kappa(x,\xi,\psi(0),\psi) < 0,\ x \geq 0,\ n(-i) \geq 0\ \forall i \in \aleph\},\\
\partial_{-,\emptyset,1}\mathcal{S}_\kappa &= \{(x,\xi,\psi(0),\psi) \mid G_\kappa(x,\xi,\psi(0),\psi) = 0,\ x < 0,\ n(-i) \geq 0\ \forall i \in \aleph\},\\
\partial_{-,\emptyset,2}\mathcal{S}_\kappa &= \{(x,\xi,\psi(0),\psi) \mid G_\kappa(x,\xi,\psi(0),\psi) < 0,\ x = 0,\ n(-i) \geq 0\ \forall i \in \aleph\}.
\end{aligned} \tag{4.31}$$

4.3.3. Boundary conditions for the value function. Let us now examine the conditions of the value function $V_\kappa : \mathcal{S}_\kappa \to \Re_+$ on the boundary $\partial\mathcal{S}_\kappa$ of the *solvency region* $\mathcal{S}_\kappa$ defined in (4.25)-(4.26).

We make the following observations regarding the behavior of the value function V_κ on the boundary $\partial\mathcal{S}_\kappa$.

Lemma 4.3. *Let $(x,\xi,\psi(0),\psi) \in \mathcal{S}_\kappa$, and let $\hat{x}$, $\hat{\xi}$, and $(\hat{\psi}(0),\hat{\psi})$ be as defined in (4.9)–(4.11). Then*

$$G_0(\hat{x},\hat{\xi},\hat{\psi}(0),\hat{\psi}) = G_0(x,\xi,\psi(0),\psi) - \kappa. \tag{4.32}$$

Proof. Suppose the *investor's* current portfolio is $(x,\xi,\psi(0),\psi) \in \mathcal{S}_\kappa$ then an instantaneous transaction of the quantity $\zeta = \{m(-k),\ k = 0,1,2,\ldots\} \in \mathcal{R}(\xi)$ will facilitate an instantaneous jump of the state from $(x,\xi,\psi(0),\psi)$ to the new state $(\hat{x},\hat{\xi},\hat{\psi}(0),\hat{\psi})$.

The result follows immediately by substituting $(\hat{x},\hat{\xi},\hat{\psi}(0),\hat{\psi})$ into G_0 defined by (2.26). This proves the lemma. □

Lemma 4.4. *If there is no fixed transaction cost (i.e., $\kappa = 0$ and $\mu > 0$) and if $(x,\xi,\psi(0),\psi) \in \partial_{I,1}\mathcal{S}_0$, that is,*

$$G_0(x,\xi,\psi(0),\psi) = 0, \tag{4.33}$$

then the only admissible strategy is to do no consumption but to close all open positions in order to bring his portfolio to $\{0\} \times \{\mathbf{0}\} \times \mathbf{M}^2_{\rho,+}$ after paying proportional transaction costs and capital gain taxes, and so forth.

Proof. For a fixed $(x,\xi,\psi(0),\psi) \in \mathcal{S}_0$, let $I \subset \aleph \equiv \{0,1,2\ldots\}$ be such that

$$i \in I \Longrightarrow n(-i) < 0, \qquad i \notin I \Longrightarrow n(-i) \geq 0. \tag{4.34}$$

To guarantee that $(X(t),N_t,S(t),S_t) \in \mathcal{S}_0$, we require that

$$G_0(X(t),N_t,S(t),S_t) \geq 0 \quad \forall t \geq 0. \tag{4.35}$$

Applying Theorem 3.5 to the process $\{e^{-rt}G_0(X(t),N_t,S(t),S_t),\ t \geq 0\}$ we obtain

$$\begin{aligned} e^{-r\tau}G_0(X(\tau),N_\tau,S(\tau),S_\tau) &= G_0(x,\xi,\psi(0),\psi) \\ &\quad + \int_0^\tau (\partial_t + \mathbf{A} + \Gamma)[e^{-rt}G_0(X(t),N_t,S(t),S_t)]dt \\ &\quad + \int_0^\tau \partial_{\psi(0)}[e^{-rt}G_0(X(t),N_t,S(t),S_t)]S(t)f(S_t)dW(t) \\ &\quad + \int_0^\tau \partial_x[e^{-rt}G_0(X(t),N_t,S(t),S_t)](rX(t) - C(t))dt \\ &\quad + \sum_{0\leq t\leq \tau} e^{-rt}[G_0(X(t),N_t,S(t),S_t) - G_0(X(t-),N_{t-},S(t),S_t)], \end{aligned} \tag{4.36}$$

for every almost surely finite **G**-stopping time τ, where $X(t)$ and N_t are given in (2.32)–(2.36) with $\kappa = 0$.

Taking into the account of (2.5) and (2.32)–(2.36) and substituting them into the function G_0, we have

$$G_0(X(t),N_t,S(t),S_t) = G_0(X(t-),N_{t-},S(t),S_t). \tag{4.37}$$

Intuitively, this is also because of the invariance of liquidated value of the assets without increase of stock value. Hence (4.36) becomes the following by grouping the terms $n(Q(t) - i)$ according to $i \in I$ and $i \notin I$:

$$\begin{aligned} &d[e^{-rt}G_0(X(t),N_t,S(t),S_t)] \\ &\quad = e^{-rt}\Big[- C(t) + \sum_{i\in I}(1+\mu-\beta)n(Q(t)-i)S(t)(f(S_t) - r) \\ &\qquad + \sum_{i\notin I}(1-\mu-\beta)n(Q(t)-i)S(t)(f(S_t)-r) \\ &\qquad - r\beta\sum_{i\in I} n(Q(t)-i)S(\tau(Q(t)-i)) \\ &\qquad - r\beta\sum_{i\notin I} n(Q(t)-i)S(\tau(Q(t)-i))\Big]dt \\ &\quad + e^{-rt}\Big[\sum_{i\in I}(1+\mu-\beta)n(Q(t)-i) \\ &\qquad + \sum_{i\notin I}(1-\mu-\beta)n(Q(t)-i)\Big]S(t)g(S_t)dW(t). \end{aligned} \tag{4.38}$$

Now fix the first exit time $\hat{\tau}$ ($\hat{\tau}$ is a **G**-stopping time) defined by

$$\begin{aligned}\hat{\tau} \equiv 1 \wedge \inf\{t \geq 0 \mid & n(Q(t)-i)S(\tau(Q(t)-i)) \notin \\ & \text{the interval } (n(-i)\psi(\tau(-i)) - 1, 0) \text{ for } i \in I, \\ & n(Q(t)-i)S(\tau(Q(t)-i)) \notin \\ & \text{the interval } (0, n(-i)\psi(\tau(-i)) + 1) \text{ for } i \notin I\}.\end{aligned} \tag{4.39}$$

We can integrate (4.38) from 0 to $\hat{\tau}$, keeping in mind that $(x,\xi,\psi(0),\psi) \in \partial_{I,1}\mathcal{S}_0$ (or equivalently, $G_0(x,\xi,\psi(0),\psi) = 0$), to obtain

$$\begin{aligned}0 &\leq e^{-r\hat{\tau}} G_0(X(\hat{\tau}), N_{\hat{\tau}}, S(\hat{\tau}), S_{\hat{\tau}}) \\ &= \int_0^{\hat{\tau}} e^{-rs}\Big[-C(s) + \sum_{i\in I}(1+\mu-\beta)n(Q(s)-i)S(s)(f(S_s)-r) \\ &\quad + \sum_{i\notin I}(1-\mu-\beta)n(Q(s)-i)S(s)(f(S_s)-r) \\ &\quad - r\beta\sum_{i\in I} n(Q(s)-i)S(\tau(Q(s)-i)) \\ &\quad - r\beta\sum_{i\notin I} n(Q(s)-i)S(\tau(Q(s)-i))\Big]ds \\ &\quad + \int_0^{\hat{\tau}} e^{-rs}\Big[\sum_{i\in I}(1+\mu-\beta)n(Q(s)-i) \\ &\quad + \sum_{i\notin I}(1-\mu-\beta)n(Q(s)-i)\Big]S(s)g(S_s)\,dW(s).\end{aligned} \tag{4.40}$$

Now use the facts that $0 < \mu+\beta < 1$, $C(t) \geq 0$, $\alpha \geq f(S_t) > r > 0$, $n(-i) < 0$ for $i \in I$ and $n(-i) \geq 0$ for $i \notin I$ and Rule 6 to obtain the following inequality:

$$\begin{aligned}0 &\leq e^{-r\hat{\tau}} G_0(X(\hat{\tau}), N_{\hat{\tau}}, S(\hat{\tau}), S_{\hat{\tau}}) \\ &\leq \int_0^{\hat{\tau}} e^{-rs}\Big[\sum_{i\notin I}(1-\mu-\beta)n(Q(s)-i)S(t)(\alpha-r)\Big]dt \\ &\quad + \int_0^{\hat{\tau}} e^{-rs}\Big[\sum_{i\in I}(1+\mu-\beta)n(Q(t)-i) \\ &\quad + \sum_{i\notin I}(1-\mu-\beta)n(Q(s)-i)\Big]S(s)g(S_s)\,dW(s).\end{aligned} \tag{4.41}$$

It is clear that

$$E\left[\int_t^{\hat{\tau}} e^{-rs}\left(\sum_{i\in I}(1+\mu-\beta)n(Q(s)-i)\right)S(s)g(S_s)\,dW(s)\right]=0. \tag{4.42}$$

Now define the following process:

$$\widetilde{W}(t)=\frac{\alpha-r}{g(S_t)}t+W(t),\quad t\geq 0. \tag{4.43}$$

Then by the Girsanov transformation (see [23, 24]), $\{\widetilde{W}(t),\ t\geq 0\}$ is a Brownian motion defined on a new probability space $(\Omega,\mathscr{F},\widetilde{P};\mathbf{F})$, where $\widetilde{P}$ and P are equivalent probability measures, and hence

$$\begin{aligned} E\Bigg[\int_0^{\hat{\tau}} e^{-rs}&\left(\sum_{i\notin I}(1-\mu-\beta)n(Q(s)-i)S(s)(\alpha-r)\right)ds\\ &+\int_0^{\hat{\tau}} e^{-rs}\left(\sum_{i\notin I}(1-\mu-\beta)n(Q(s)-i)S(s)g(S_s)\,dW(s)\right)\Bigg]\\ &=E^{\widetilde{P}}\left[\int_0^{\hat{\tau}} e^{-rs}\left(\sum_{i\notin I}(1-\mu-\beta)n(Q(s)-i)\right)S(s)g(S_s)\,d\widetilde{W}(s)\right]=0. \end{aligned} \tag{4.44}$$

Therefore

$$\left[\int_0^{\hat{\tau}} e^{-rs}\left(\sum_{k\notin I}(1-\nu-\beta)n(Q(s)-k)S(s)g(S_s)\,d\widetilde{W}(s)\right)\right]=0\quad \widetilde{P}\text{-a.s.}, \tag{4.45}$$

by Assumptions 2.1–2.3.

Since $G_0(X(t),N_t,S(t),S_t)\geq 0$ for all $t\geq 0$, this implies that $\hat{\tau}=0$ a.s., that is,

$$(X(\hat{\tau}),N_{\hat{\tau}},S(\hat{\tau}),S_{\hat{\tau}})=(x,\xi,\psi(0),\psi)\in\partial_{I,1}\mathscr{S}_0. \tag{4.46}$$

We need to determine the conditions under which the exit time occurred.

Let k be the index of the shares of the stock where the state process violated the condition for the stopping time $\hat{\tau}$. In other words, if $k\in I$, then

$$n(Q(\hat{\tau})-k)S(\tau(Q(\hat{\tau})-k))\notin((n(-k)\psi(-k))-1,0) \tag{4.47}$$

or

$$\text{if } k\notin I,\quad \text{then } n(Q(\hat{\tau})-k)S(\tau(Q(\hat{\tau})-k))\notin((0,n(-k)\psi(-k))+1). \tag{4.48}$$

We will examine both cases separately.

Case 1. Suppose $k\in I$. Then

$$\begin{aligned} &\text{either } n(Q(\hat{\tau})-k)S(\tau(Q(\hat{\tau})-k))\leq n(-k)\psi(\tau(-k))-1\\ &\text{or } n(Q(\hat{\tau})-k)S(\tau(Q(\hat{\tau})-k))\geq 0. \end{aligned} \tag{4.49}$$

We have established that

$$(X(\hat{\tau}), N_{\hat{\tau}}, S(\hat{\tau}), S_{\hat{\tau}}) \in \partial_{I,1}\mathcal{S}_0, \tag{4.50}$$

and this is inconsistent with

$$\begin{aligned} &\text{both } n(Q(\hat{\tau}) - k)S(\tau(Q(\hat{\tau}) - k)) \le n(-k)\psi(-k) - 1 \\ &\text{and } n(Q(\hat{\tau}) - k)S(\tau(Q(\hat{\tau}) - k)) > 0. \end{aligned} \tag{4.51}$$

Therefore, we know $n(Q(\hat{\tau} - k)) = 0$.

Case 2. Suppose $k \notin I$. Then

$$\begin{aligned} &\text{either } n(Q(\hat{\tau}) - k)S(\tau(Q(\hat{\tau}) - k)) \ge n(-k)\psi(-k) + 1 \\ &\text{or } n(Q(\hat{\tau}) - k)S(\tau(Q(\hat{\tau}) - k)) \le 0. \end{aligned} \tag{4.52}$$

Again, since

$$(X(\hat{\tau}), N_{\hat{\tau}}, S(\hat{\tau}), S_{\hat{\tau}}) \in \partial_{I,1}\mathcal{S}_0, \tag{4.53}$$

we see that $n(Q(\hat{\tau}) - k) = 0$. We conclude from both cases that $(X(\hat{\tau}), N_{\hat{\tau}}) = (0, \{\mathbf{0}\})$. This means that the only admissible strategy is to bring the portfolio from $(x, \xi, \psi(0), \psi)$ to $(0, \mathbf{0}, \psi(0), \psi)$ by an appropriate amount of the transaction specified in the lemma. This proves the lemma.

□

We have the following result.

Theorem 4.5. *Let $\kappa > 0$ and $\mu > 0$. On $\partial_{I,1}\mathcal{S}_\kappa$ for $I \subset \aleph$, then the investor should not consume but close all open positions in order to bring his portfolio to $\{0\} \times \{\mathbf{0}\} \times \Re_+ \times L^2_{\rho,+}$. In this case, the value function $V_\kappa : \partial_{I,1}\mathcal{S}_\kappa \to \Re_+$ satisfies the following equation:*

$$(\mathcal{M}_\kappa\Phi - \Phi)(x, \xi, \psi(0), \psi) = 0. \tag{4.54}$$

Proof. Suppose the *investor's* current portfolio is $(x, \xi, \psi(0), \psi) \in \partial_{I,1}\mathcal{S}_\kappa$ for some $I \subset \aleph$. A transaction of the quantity $\zeta = \{m(-k),\ k = 0, 1, 2, \ldots\} \in \mathcal{R}(\xi) - \{\mathbf{0}\}$ will facilitate an instantaneous jump of the state from $(x, \xi, \psi(0), \psi)$ to the new state $(\hat{x}, \hat{\xi}, \hat{\psi}(0), \hat{\psi})$ as given in (4.9)–(4.11).

We observe that, since $\zeta = (m(-k),\ k = 0, 1, 2, \ldots) \in \mathcal{R}(\xi) - \{\mathbf{0}\}$, $n(-k) < 0$ implies $\hat{n}(-k) = n(-k) + m(-k) \le 0$ and $n(-k) > 0$ implies $\hat{n}(-k) = n(-k) + m(-k) \ge 0$ for $k = 0, 1, 2, \ldots$.

Taking into account the new portfolio after a transaction $(\hat{x}, \hat{\xi}, \hat{\psi}(0), \hat{\psi})$, we have from Lemma 4.3 that

$$G_0(\hat{x}, \hat{\xi}, \hat{\psi}(0), \hat{\psi}) = G_0(x, \xi, \psi(0), \psi) - \kappa. \tag{4.55}$$

Therefore if $(x,\xi,\psi(0),\psi)\in\partial_{I,1}\mathcal{S}_\kappa$ for some $I\subset\aleph$, then

$$G_\kappa(x,\xi,\psi(0),\psi)=G_0(x,\xi,\psi(0),\psi)-\kappa=0=G_0(\hat{x},\hat{\xi},\hat{\psi}(0),\hat{\psi}). \tag{4.56}$$

This implies $(\hat{x},\hat{\xi},\hat{\psi}(0),\hat{\psi})\in\partial_{I,1}\mathcal{S}_0$. From Lemma 4.4, we prove that the only admissible strategy is to make no consumption but to make another trading from the new state $(\hat{x},\hat{\xi},\hat{\psi}(0),\hat{\psi})\in\partial_{I,1}\mathcal{S}_0$. Therefore, starting from $(x,\xi,\psi(0),\psi)\in\partial_{I,1}\mathcal{S}_\kappa$ we make two immediate instantaneous transactions (which will be counted as only one transaction) with the total amount specified by the following two equations:

$$\begin{aligned}0=x-\kappa+&\sum_{i\in I^c}\left[n(-i)\psi(0)(1-\mu-\beta)+\beta n(-i)\psi(\tau(-i))\right]\\&+\sum_{i\in I}\left[n(-i)\psi(0)(1+\mu-\beta)+\beta n(-i)\psi(\tau(-i))\right],\end{aligned} \tag{4.57}$$

$$\mathbf{0}=\xi\oplus\zeta, \tag{4.58}$$

to reach the final destination $(0,\mathbf{0},\psi(0),\psi)$. This proves the theorem. □

We conclude from some simple observations and Theorem 4.5 the following.

Boundary condition (i). On the hyper plane

$$\partial_{-,\varnothing,2}\mathcal{S}_\kappa=\left\{(0,\xi,\psi(0),\psi)\in\mathcal{S}_\kappa\mid G_\kappa(0,\xi,\psi(0),\psi)<0,\ n(-i)\ge 0\ \forall i\right\}, \tag{4.59}$$

the only strategy for the *investor* is to do no transaction and no consumption, since $x=0$ and $G_\kappa(0,\xi,\psi(0),\psi)<0$ (hence there is no money to consume and not enough money to pay for the transaction costs, etc.), but to let the stock prices grow according to (2.5). Thus, the value function V_κ on $\partial_{-,\varnothing,2}\mathcal{S}_\kappa$ satisfies the equation

$$\mathcal{L}^0\Phi\equiv(\mathbf{A}+\Gamma-\delta+rx\partial_x)\Phi=0 \tag{4.60}$$

provided that it is smooth enough.

Boundary condition (ii). On $\partial_{I,1}\mathcal{S}_\kappa$ for $I\subset\aleph$, then the *investor* should not consume but buy back $n(-i)$ shares for $i\in I$ and sell $n(-i)$ shares for $i\in I^c$ of the stock in order to bring his portfolio to $\{0\}\times\{\mathbf{0}\}\times\Re_+\times L^2_{\rho,+}$ after paying transaction costs and capital gains taxes, and so forth. In other words, bring his portfolio from the position $(x,\xi,\psi(0),\psi)\in\partial_{I,1}\mathcal{S}_\kappa$ to $(0,\mathbf{0},\psi(0),\psi)$ by the quantity that satisfies (4.57)-(4.58). In this case, the value function $V_\kappa:\partial_{I,1}\mathcal{S}_\kappa\to\Re_+$ satisfies the following equation:

$$(\mathcal{M}_\kappa\Phi-\Phi)(x,\xi,\psi(0),\psi)=0. \tag{4.61}$$

Note that this is a restatement of Theorem 4.5.

Boundary condition (iii). On $\partial_{+,I,2}\mathcal{S}_\kappa$ for $I \subset \aleph$, the only optimal strategy is to make no transaction but to consume optimally according to the optimal consumption rate function $c^*(x,\xi,\psi(0),\psi,) = (\partial V_\kappa/\partial x)^{1/(\gamma-1)}(x,\xi,\psi(0),\psi)$ which is obtained via

$$c^*(x,\xi,\psi(0),\psi) = \arg\max_{c\geq 0}\left\{\mathcal{L}^c V_\kappa(x,\xi,\psi(0),\psi) + \frac{c^\gamma}{\gamma}\right\}, \tag{4.62}$$

where $\mathcal{L}^c$ is the differential operator defined by

$$\mathcal{L}^c\Phi(x,\xi,\psi(0),\psi) \equiv (\mathbf{A}+\Gamma-\delta)\Phi + (rx-c)\partial_x\Phi. \tag{4.63}$$

This is because the cash in his *savings* account is not sufficient to buy back any shares of the *stock* but to consume optimally. In this case, the value function $V_\kappa : \partial_{+,I,2}\mathcal{S}_\kappa \to \Re_+$ satisfies the following equation provided that it is smooth enough:

$$\mathcal{A}\Phi \equiv (\mathbf{A}+\Gamma-\delta)\Phi + rx\partial_x\Phi + \frac{1-\gamma}{\gamma}(\partial_x\Phi)^{\gamma/(\gamma-1)} = 0. \tag{4.64}$$

Boundary condition (iv). On $\partial_{-,I,2}\mathcal{S}_\kappa$, the only admissible consumption-investment strategy is to do no consumption and no transaction but to let the stock price grow as in the boundary condition (i).

Boundary condition (v). On $\partial_{+,\aleph,2}\mathcal{S}_\kappa = \{(x,\xi,\psi(0),\psi) \mid 0 \leq x \leq \kappa,\ n(-i)=0\ \forall i=0,1,\ldots\}$, the only admissible consumption-investment strategy is to do no transaction but to consume optimally like in boundary condition (iii).

Remark 4.6. From boundary conditions (i)–(v), it is clear that the value function V_κ is discontinuous on the interfaces $Q_{+,I}$ and $Q_{-,I}$ for all $I \subset \aleph$.

4.3.4. The QVHJBI with boundary conditions. We conclude from the above subsections that the QVHJBI (together with the boundary conditions) can be expressed as follows:

$$\text{QVHJBI}(*) = \begin{cases} \max\{\mathcal{A}\Phi, \mathcal{M}_\kappa\Phi - \Phi\} = 0 & \text{on } \mathcal{S}_\kappa^\circ, \\ \mathcal{A}\Phi = 0, & \text{on } \bigcup_{I\subset\aleph}\partial_{+,I,2}\mathcal{S}_\kappa, \\ \mathcal{L}^0\Phi = 0, & \text{on } \bigcup_{I\subset\aleph}\partial_{-,I,2}\mathcal{S}_\kappa, \\ \mathcal{M}_\kappa\Phi - \Phi = 0 & \text{on } \bigcup_{I\subset\aleph}\partial_{I,1}\mathcal{S}_\kappa, \end{cases} \tag{QVHJBI (*)}$$

where $\mathcal{A}\Phi$, $\mathcal{L}^0\Phi$ ($\mathcal{L}^c\Phi$ with $c=0$), and $\mathcal{M}_\kappa$ are as defined in (4.64), (4.63), and (4.8), respectively.

As mentioned earlier, the second paper (see [1]) establishes the verification theorem for the optimal consumption-trading strategy. It is also shown there that the value function defined in (2.41) is a viscosity solution of (QVHJBI $(*)$) defined above.

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