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Research Article

Interloss Time in $M/M/1/1$ Loss System

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We consider the interloss times in the $M/M/1/1$ Erlang Loss System. Here we present the explicit form of the probability density function of the time spent between two consecutive losses in the $M/M/1/1$ model. This density function solves a Cauchy problem for the second-order differential equations, which was used to evaluate the corresponding laplace transform. Finally the connection between the Erlang's loss rate and the evaluated probability density function is showed.

1. Introduction

In this paper, we treat the random variable $T_{(i)}$ representing the time spent between the ith and the $(i-1)$th lost unit or ith interloss time, in the $M/M/1/1$ loss system.

The $M/M/1/1$ model is characterized by the Markov property of entering and exiting processes, by one service channel and by the system capacity to accommodate one customer at a time (for an overview see Medhi [1, page. 77]).

Our work has been inspired by the location problem of emergency vehicles (ambulances). Each vehicle can be regarded as an $M/M/1/1$ (or $M/E_k/1/1$) system, because its clients cannot wait in the queue.

In Emergency Medical Systems (EMSs), the nearest ambulance to the accident place is called "district unit", and it assures the best performance to the system. If an emergency call arrives at EMS while its "district unit" is busy, the nearest ambulance among those available is dispatched (see Larson [2] or Larson [3]). The length of interloss times affects the performance of the system and provides an informative support on efficiency of EMS.

For the exponential $M/M/1/1$ loss model it was conducted, in Ferrante [4], a detailed description of the process of losses

$$\{L(t)\}_{t>0}, \tag{1.1}$$

where $L(t)$ represent the random number of losses in the time interval $[0,t)$. Let $P_m^{(k)}(t)$ be the conditional probability to lose m clients in $[0,t)$ with k customers in the system at time $t=0$:

$$P_m^{(k)}(t) = P\{L(t) = m \mid k\}, \quad k = 0,1, \tag{1.2}$$

the main results found in Ferrante [4] are the explicit values of the conditional probabilities of no losses in $[0,t)$:

$$\begin{aligned} P_0^{(1)}(t) &= e^{-(t/2)(2\lambda+\mu)}\left[\cosh\frac{t\sqrt{\mu(4\lambda+\mu)}}{2} + \frac{\mu}{\sqrt{\mu(4\lambda+\mu)}}\sinh\frac{t\sqrt{\mu(4\lambda+\mu)}}{2}\right], \\ P_0^{(0)}(t) &= e^{-(t/2)(2\lambda+\mu)}\left[\cosh\frac{t\sqrt{\mu(4\lambda+\mu)}}{2} + \frac{2\lambda+\mu}{\sqrt{\mu(4\lambda+\mu)}}\sinh\frac{t\sqrt{\mu(4\lambda+\mu)}}{2}\right], \end{aligned} \tag{1.3}$$

and the iterative procedure to determine the distribution of the total number of losses in $[0,t)$.All is obtained by solving the inhomogeneous differential equations

$$P_m^{(k)}(t) = -(2\lambda+\mu)\frac{d}{dt}P_m^{(k)}(t) - \lambda^2 P_m^{(k)}(t) + \lambda\frac{d}{dt}P_{m-1}^{(k)}(t) + \lambda^2 P_{m-1}^{(k)}, \tag{1.4}$$

with $P_{-1}^{(k)}(t) = 0$ and the initial conditions depending on the k $(=0,1)$ customers in the system at $t=0$. Furthermore, the generating probability functions $G_k(s,t)$ of $P_m^{(k)}(t)$ were evaluated and their explicit values are the following:

$$\begin{aligned} G_1(s,t) &= e^{-(t/2)(2\lambda+\mu-\lambda s)} \\ &\times\left[\cosh\frac{t\sqrt{(\lambda s-\mu)^2+4\lambda\mu}}{2} + \frac{\mu+\lambda s}{\sqrt{(\lambda s-\mu)^2+4\lambda\mu}}\sinh\frac{t\sqrt{(\lambda s-\mu)^2+4\lambda\mu}}{2}\right], \\ G_0(s,t) &= e^{-(t/2)(2\lambda+\mu-\lambda s)} \\ &\times\left[\cosh\frac{t\sqrt{(\lambda s-\mu)^2+4\lambda\mu}}{2} + \frac{2\lambda+\mu-\lambda s}{\sqrt{(\lambda s-\mu)^2+4\lambda\mu}}\sinh\frac{t\sqrt{(\lambda s-\mu)^2+4\lambda\mu}}{2}\right]. \end{aligned} \tag{1.5}$$

The aim of this work is to identify the type of the process of interloss times

$$\{T_{(i)}\}_{i\in Z^+} \tag{1.6}$$

for the $M/M/1/1$ loss model and to find the differential equation which governs it, in order to determine the probability density functions

$$f_{T_{(i)}}(t) = \frac{d}{dt} P\{T_{(i)} < t\}, \tag{1.7}$$

with $i = 1, 2, \ldots$ and the related properties.

Our results show unexpected connections among very different branches of probability such as random motion on hyperbolic space and queueing systems. In effect, the probabilities appearing below have a structure quite similar to the Hyperbolic distances of moving particles envisaged in Cammarota and Orsingher [5].

In Section 2, we establish that $\{T_{(i)}\}_{i\in Z^+}$ is a renewal process for the $M(\lambda)/M(\mu)/1/1$ loss system and that the density functions (1.7) solve the second-order linear homogeneous differential equations

$$\frac{d^2}{dt^2} f_{T_{(i)}}(t) = -(2\lambda + \mu) \frac{d}{dt} f_{T_{(i)}}(t) - \lambda^2 f_{T_{(i)}}(t). \tag{1.8}$$

Let $\nu(t)$ be the number of customers in the system at the moment t, and let t_{l_i} be the moment of the ith loss with $t_{l_0} = 0$. The initial conditions for (1.8) depend on $\nu(t_{l_{i-1}})$, and the renewal process $\{T_{(i)}\}_{i\in Z^+}$ has the following property:

$$\begin{aligned} &f_{T_{(1)}}(t) = f_{T_{(i)}}(t), \quad \text{if } \nu(0) = 1, \\ &f_{T_{(1)}}(t) \neq f_{T_{(i)}}(t), \quad \text{if } \nu(0) = 0 \end{aligned} \tag{1.9}$$

for $i > 1$.

In Section 2, we also present the derivation of (1.8) and its solution conditionally by $\nu(0)$.

Let $f_{T_{(1)}}(t; i)$ be the conditional density function of the 1th interloss time with i (= 0,1) customers in the system at time $t = 0$:

$$f_{T_{(1)}}(t; i) = \frac{d}{dt} P\{T_{(1)} < t \mid \nu(0) = i\}. \tag{1.10}$$

For the $M(\lambda)/M(\mu)/1/1$ model, the explicit values obtained for (1.10) are the following:

$$\begin{aligned} f_{T_{(1)}}(t; 1) &= \lambda e^{-(t/2)(2\lambda+\mu)} \left[\cosh \frac{t\sqrt{\mu(4\lambda + \mu)}}{2} - \frac{\mu}{\sqrt{\mu(4\lambda + \mu)}} \sinh \frac{t\sqrt{\mu(4\lambda + \mu)}}{2} \right], \\ f_{T_{(1)}}(t; 0) &= \frac{2\lambda^2 e^{-(t/2)(2\lambda+\mu)}}{\sqrt{\mu(4\lambda + \mu)}} \sinh \frac{t\sqrt{\mu(4\lambda + \mu)}}{2}. \end{aligned} \tag{1.11}$$

In Section 3, we compute the laplace transforms of (1.10):

$$F^*_{T_{(1)}}(s;i) = \int_0^{\infty} e^{-st} f_{T_{(1)}}(t;i)dt \tag{1.12}$$

for $i = 0, 1$, using (1.8).

The explicit values obtained for (1.12) are the following:

$$\begin{aligned} F^*_{T_{(1)}}(s;1) &= \frac{\lambda(\lambda+s)}{(\lambda+s)^2 + s\mu}, \\ F^*_{T_{(1)}}(s;0) &= \frac{\lambda^2}{(\lambda+s)^2 + s\mu}. \end{aligned} \tag{1.13}$$

Finally, let $\Theta_1^{(i)}$ be the conditional means of the 1th interloss time

$$\Theta_1^{(i)} = E\left[T_{(1)} \mid \nu(0) = i\right], \tag{1.14}$$

with i = 0,1; it has been checked that their values are

$$\Theta_1^{(1)} = \frac{1}{r}, \qquad \Theta_1^{(0)} = \frac{1}{r} + \frac{1}{\lambda}, \tag{1.15}$$

where r is the Erlang loss rate, and λ^{-1} is the interarrival mean time.

2. First Interloss Time

In the $M(\lambda)/M(\mu)/1/1$ model, let $\nu(t)$ be the number of customers in the system at the moment t, let $\tau_{(k)}$ be the kth interarrival time, let t_k be the moment when the kth client enters the system, let S_k be the service time of the kth served customer, let $T_{(i)}$ be the ith interloss time. Furthermore, let $l_i(t)$ be the arrival order of ith loss happened in t, starting from the $(i-1)$th loss, and let t_{l_i} be the moment when the ith loss happen, with $t_{l_0} = 0$ and $t_{l_i(t)} = t$.

If we consider that the system is busy at time $t = 0$, the event *"The* 1th *interloss time ist"* is represented by Figure 1.

The random variable $T_{(1)}$ can be expressed as follows:

$$T_{(1)} = \sum_{k=1}^{l_1(t)} \tau_{(k)}, \tag{2.1}$$

where $l_1(t)$ represents the arrival order of the 1th loss happened in $t_{l_1(t)} = t$, starting from zero.

Now, let $P_n^{(1)}(t)$ be the conditional probability that the arrival order of 1th loss happened in t with $\nu(0) = 1$ is equal to n :

$$P_n^{(1)}(t) = P\{l_1(t) = n \mid \nu(0) = 1\}, \tag{2.2}$$

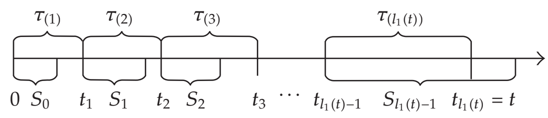

Figure 1: The 1th interloss time is t with $P\{\nu(0) = 1\} = 1$.

and it can be expressed as follows

$$P_n^{(1)}(t) = P\left\{S_0 < \tau_{(1)}, \dots, S_{n-2} < \tau_{(n-1)}, S_{n-1} > t - t_{n-1}\right\} \tag{2.3}$$

and can be computed conditionally by the $(n-1)$ moments when the served customers have arrived at the system

$$\begin{aligned} P_n^{(1)}(t) &= \frac{(n-1)!}{t^{n-1}} \int_0^t dt_1 \cdots \int_{t_{n-2}}^t dt_{n-1} e^{-\mu(t-t_{n-1})} \prod_{i=1}^{n-1} \left[1 - e^{-\mu(t_i - t_{i-1})}\right] \\ &= \frac{(n-1)!}{t^{n-1}} F_{n-1,1}^{(0)}(t), \end{aligned} \tag{2.4}$$

where

$$F_{n-1,1}^{(0)}(t) = \int_0^t dt_1 \cdots \int_{t_{n-2}}^t dt_{n-1} e^{-\mu(t-t_{n-1})} \prod_{i=1}^{n-1} \left[1 - e^{-\mu(t_i - t_{i-1})}\right]. \tag{2.5}$$

Lemma 2.1. *The functions*

$$F_{n,1}^{(s)}(t) = \int_s^t dt_1 \cdots \int_{t_{n-1}}^t dt_n e^{-\mu(t-t_n)} \prod_{i=1}^{n} \left[1 - e^{-\mu(t_i - t_{i-1})}\right] \tag{2.6}$$

with $t_0 = s$ do not depend on t but on the time interval $[s, t)$:

$$F_{n,1}^{(s)}(t) = F_{n,1}^{(0)}(t-s). \tag{2.7}$$

Proof. We proceed by showing that (2.7) is true for $n = 1$:

$$\begin{aligned} F_{1,1}^{(s)}(t) &= \int_s^t dt_1 e^{-\mu(t-t_1)} \left[1 - e^{-\mu(t_1 - s)}\right] \\ &= \frac{1 - e^{-\mu(t-s)}}{\mu} - (t-s) e^{-\mu(t-s)} \\ &= F_{1,1}^{(0)}(t-s). \end{aligned} \tag{2.8}$$

Then, we suppose that it is true for $n-1$, and we obtain that

$$\begin{aligned} F_{n,1}^{(s)}(t) &= \int_s^t dt_1 \left[1 - e^{-\mu(t_1-s)}\right] F_{n-1,1}^{(t_1)}(t) \\ &= \int_s^t dt_1 G_1^{(s)}(t_1) F_{n-1,1}^{(0)}(t-t_1), \end{aligned} \tag{2.9}$$

where

$$G_1^{(s)}(t_1) = \int_s^{t_1} dx \mu e^{-\mu(t_1-x)}. \tag{2.10}$$

Finally, by the Markov property of the exponential distribution, the (2.7) appears

$$F_{n,1}^{(s)}(t) = \int_0^{t-s} dt_1 G_1^{(0)}(t_1) F_{n-1,1}^{(0)}(t-s-t_1). \tag{2.11}$$

□

The conditional density function

$$f_{T_{(1)}}(t;1) = \frac{d}{dt} P\{T_{(1)} < t \mid \nu(0) = 1\} \tag{2.12}$$

can be evaluated as mean of convolution of $l_1(t)$ exponential probability density functions, and thus we have that

$$f_{T_{(1)}}(t;1) = \sum_{n=1}^{\infty} f_{\sum_{k=1}^{n} \tau_{(k)}}(t) P_n^{(1)}(t) = e^{-\lambda t} \sum_{n=0}^{\infty} \lambda^{n+1} F_{n,1}^{(0)}(t). \tag{2.13}$$

At first, we state the following result concerning the evaluation of the integrals $F_{n,1}^{(0)}(t), n \geq 1$.

Lemma 2.2. *The functions*

$$F_{n,1}^{(0)}(t) = \int_0^t dt_1 \cdots \int_{t_{n-1}}^t dt_n e^{-\mu(t-t_n)} \prod_{i=1}^{n} \left[1 - e^{-\mu(t_i - t_{i-1})}\right] \tag{2.14}$$

satisfy the difference-differential equations

$$\frac{d^2}{dt^2} F_{n,1}^{(0)}(t) = -\mu \frac{d}{dt} F_{n,1}^{(0)}(t) + \mu F_{n-1,1}^{(0)}(t), \tag{2.15}$$

where $t_0 = 0, t > 0, n \geq 1$.

Proof. We first note that

$$\begin{aligned}\frac{d}{dt}F_{n,1}^{(0)}(t) &= \frac{d}{dt}\int_0^t dt_1 \cdots \int_{t_{n-1}}^t dt_n e^{-\mu(t-t_n)} \prod_{i=1}^{n}\left[1-e^{-\mu(t_i-t_{i-1})}\right] \\ &= \int_0^t dt_1 \cdots \int_{t_{n-2}}^t dt_{n-1}\left[1-e^{-\mu(t-t_{n-1})}\right] \prod_{i=1}^{n-1}\left[1-e^{-\mu(t_i-t_{i-1})}\right] \\ &\quad - \mu\int_0^t dt_1 \cdots \int_{t_{n-1}}^t dt_n e^{-\mu(t-t_n)} \prod_{i=1}^{n}\left[1-e^{-\mu(t_i-t_{i-1})}\right],\end{aligned} \tag{2.16}$$

and therefore

$$\begin{aligned}\frac{d^2}{dt^2}F_{n,1}^{(0)}(t) &= \mu\int_0^t dt_1 \cdots \int_{t_{n-2}}^t dt_{n-1} e^{-\mu(t-t_{n-1})} \prod_{i=1}^{n-1}\left[1-e^{-\mu(t_i-t_{i-1})}\right] \\ &\quad - \mu\int_0^t dt_1 \cdots \int_{t_{n-2}}^t dt_{n-1}\left[1-e^{-\mu(t-t_{n-1})}\right] \prod_{i=1}^{n-1}\left[1-e^{-\mu(t_i-t_{i-1})}\right] \\ &\quad + \mu^2\int_0^t dt_1 \cdots \int_{t_{n-1}}^t dt_n e^{-\mu(t-t_n)} \prod_{i=1}^{n}\left[1-e^{-\mu(t_i-t_{i-1})}\right] \\ &= -\mu\frac{d}{dt}F_{n,1}^{(0)}(t) + \mu F_{n-1,1}^{(0)}(t).\end{aligned} \tag{2.17}$$

□

In view of Lemma 2.2 we can prove also the following.

Theorem 2.3. *The function* $f_{T_{(1)}}(t;1)$ *satisfies the second-order linear homogeneous differential equation*

$$\frac{d^2}{dt^2}f_{T_{(1)}}(t;1) = -(2\lambda+\mu)\frac{d}{dt}f_{T_{(1)}}(t;1) - \lambda^2 f_{T_{(1)}}(t;1), \tag{2.18}$$

with the initial conditions

$$f_{T_{(1)}}(0;1) = \lambda, \qquad \frac{d}{dt}f_{T_{(1)}}(t;1)|_{t=0} = -\lambda(\lambda+\mu). \tag{2.19}$$

The explicit value of $f_{T_{(1)}}(t;1)$ *is*

$$f_{T_{(1)}}(t;1) = \lambda e^{-(t/2)(2\lambda+\mu)}\left[\frac{\sqrt{\mu(4\lambda+\mu)}-\mu}{2\sqrt{\mu(4\lambda+\mu)}}e^{(t/2)\sqrt{\mu(4\lambda+\mu)}} + \frac{\sqrt{\mu(4\lambda+\mu)}+\mu}{2\sqrt{\mu(4\lambda+\mu)}}e^{-(t/2)\sqrt{\mu(4\lambda+\mu)}}\right]. \tag{2.20}$$

Proof. From (2.13), it follows that

$$\frac{d}{dt} f_{T_{(1)}}(t;1) = -\lambda f_{T_{(1)}}(t;1) + e^{-\lambda t} \sum_{n=0}^{\infty} \lambda^{n+1} \frac{d}{dt} F_{n,1}^{(0)}(t), \tag{2.21}$$

and thus, in view of (2.17) and by letting $F_{-1,1}^{(0)}(t) = 0$, we have that

$$\begin{aligned} \frac{d^2}{dt^2} f_{T_{(1)}}(t;1) &= -\lambda \frac{d}{dt} f_{T_{(1)}}(t;1) - \lambda e^{-\lambda t} \sum_{n=0}^{\infty} \lambda^{n+1} \frac{d}{dt} F_{n,1}^{(0)}(t) + e^{-\lambda t} \sum_{n=0}^{\infty} \lambda^{n+1} \frac{d^2}{dt^2} F_{n,1}^{(0)}(t) \\ &= -2\lambda \frac{d}{dt} f_{T_{(1)}}(t;1) - \lambda^2 f_{T_{(1)}}(t;1) + \lambda\mu f_{T_{(1)}}(t;1) - \mu \left[\frac{d}{dt} f_{T_{(1)}}(t;1) + \lambda f_{T_{(1)}}(t;1) \right] \\ &= -(2\lambda + \mu) \frac{d}{dt} f_{T_{(1)}}(t;1) - \lambda^2 f_{T_{(1)}}(t;1). \end{aligned} \tag{2.22}$$

While the first condition is straightforward to verify, the second one needs some explanations: if we write

$$\frac{d}{dt} f_{T_{(1)}}(t;1)\Big|_{t=0} = \lim_{\Delta t \to 0} \frac{f_{T_{(1)}}(\Delta t;1) - f_{T_{(1)}}(0;1)}{\Delta t}, \tag{2.23}$$

and observe that

$$f_{T_{(1)}}(\Delta t;1) = e^{-\lambda \Delta t} \lambda F_{0,1}^{(0)}(\Delta t) = \lambda e^{-(\lambda+\mu)\Delta t} = \lambda \left[1 - (\lambda + \mu)\Delta t \right] + o(\Delta t), \tag{2.24}$$

by substituting (2.24) in (2.23) the second condition emerges.

The general solution to (2.22) has the form

$$e^{-(t/2)(2\lambda+\mu)} \left[A e^{(t/2)\sqrt{\mu(4\lambda+\mu)}} + B e^{-(t/2)\sqrt{\mu(4\lambda+\mu)}} \right]. \tag{2.25}$$

By imposing the initial conditions (2.19) to (2.18) we obtain (2.20). □

Remark 2.4. By (2.7) derive that the functions $f_{T_{(i)}}(t;1)$ do not depend on t, but on the time interval $[t_{l_{i-1}}, t)$, in fact if $t_{l_{i-1}} = s$, we have that

$$f_{T_{(i)}}(t;1,s) = e^{-\lambda(t-s)} \sum_{n=0}^{\infty} \lambda^{n+1} F_{n,1}^{(s)}(t) = f_{T_{(i)}}(t-s;1), \tag{2.26}$$

for $i = 1, 2, \ldots$.

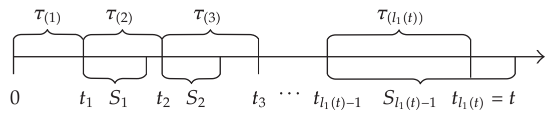

Figure 2: The 1th interloss time is t with $P\{\nu(0) = 0\} = 1$.

Furthermore, by the Markov properties of the $M/M/1/1$ system, the random variables $T_{(1)}, T_{(2)}, \ldots$ are independent, and $\{T_{(i)}\}_{i\in Z^+}$ is a renewal process with

$$f_{T_{(1)}}(t) = f_{T_{(i)}}(t) \quad \text{if } \nu(0) = 1, \qquad f_{T_{(1)}}(t) \neq f_{T_{(i)}}(t) \quad \text{if } \nu(0) = 0. \tag{2.27}$$

Remark 2.5. If $\lambda \to 0$ we get $f_{T_{(1)}}(t;1) = 0$, because the 1th interloss time is greater than t, $\vee t > 0$, when nobody enters.

If $\mu \to 0$, we have that $f_{T_{(1)}}(t;1) = \lambda e^{-\lambda t}$ because without exits and with the system busy at $t = 0$, the 1th interloss time has the same distribution of the interarrival time.

Remark 2.6. The probability density function $f_{T_{(1)}}(t;1)$ can be expressed by the following hyperbolic functions:

$$f_{T_{(1)}}(t;1) = \lambda e^{(-t/2)(2\lambda+\mu)}\left[\cosh\frac{t\sqrt{\mu(4\lambda+\mu)}}{2} - \frac{\mu}{\sqrt{\mu(4\lambda+\mu)}}\sinh\frac{t\sqrt{\mu(4\lambda+\mu)}}{2}\right]. \tag{2.28}$$

Now, if we assume that the system is free at the starting point, the event *"The* 1th *interloss time ist"* is represented by Figure 2.

Let $P_n^{(0)}(t)$ be the conditional probability that the nth entered customer is the 1th lost at time t, when $\nu(0) = 0$,

$$\begin{aligned} P_n^{(0)}(t) &= P\{l_1(t) = n \mid \nu(0) = 0\} \\ &= P\{S_1 < \tau_{(2)}, \ldots, S_{n-2} < \tau_{(n-1)}, S_{n-1} > t - t_{n-1}\}, \end{aligned} \tag{2.29}$$

it can be computed conditionally by the $(n-1)$ moments when the served customers have arrived at the system

$$P_n^{(0)}(t) = \frac{(n-1)!}{t^{n-1}}\int_0^t dt_1 \cdots \int_{t_{n-2}}^t dt_{n-1} e^{-\mu(t-t_{n-1})}\prod_{i=2}^{n-1}\left[1 - e^{-\mu(t_i-t_{i-1})}\right] = \frac{(n-1)!}{t^{n-1}} F_{n-1,0}^{(0)}(t), \tag{2.30}$$

where

$$F_{n-1,0}^{(0)}(t) = \int_0^t dt_1 \cdots \int_{t_{n-2}}^t dt_{n-1} e^{-\mu(t-t_{n-1})}\prod_{i=2}^{n-1}\left[1 - e^{-\mu(t_i-t_{i-1})}\right]. \tag{2.31}$$

The function $f_{T_{(1)}}(t;0)$ can be computed as mean of convolution of $l_1(t)$ exponential probability density functions. So we have that

$$f_{T_{(1)}}(t;0) = \sum_{n=2}^{\infty} f_{\sum_{k=1}^{n}\tau_{(k)}}(t) P_n^{(0)}(t) = e^{-\lambda t}\sum_{n=2}^{\infty}\lambda^n F_{n-1,0}^{(0)}(t). \tag{2.32}$$

Lemma 2.7. *The functions*

$$F_{n,0}^{(0)}(t) = \int_0^t dt_1 \cdots \int_{t_{n-1}}^t dt_n e^{-\mu(t-t_n)} \prod_{i=2}^{n}\left[1 - e^{-\mu(t_i - t_{i-1})}\right] \tag{2.33}$$

satisfy the difference-differential equations

$$\frac{d^2}{dt^2}F_{n,0}^{(0)}(t) = -\mu\frac{d}{dt}F_{n,0}^{(0)}(t) + \mu F_{n-1,0}^{(0)}(t), \tag{2.34}$$

where $t > 0, n \geq 1, F_{0,0}^{(0)}(t) = 0$.

Proof. See proof of Lemma 2.2. □

In view of Lemma 2.7 we can prove also the following.

Theorem 2.8. *The function* $f_{T_{(1)}}(t;0)$ *satisfies the second-order linear homogeneous differential equation*

$$\frac{d^2}{dt^2}f_{T_{(1)}}(t;0) = -(2\lambda + \mu)\frac{d}{dt}f_{T_{(1)}}(t;0) - \lambda^2 f_{T_{(1)}}(t;0), \tag{2.35}$$

with the initial conditions

$$f_{T_{(1)}}(0;0) = 0, \qquad \frac{d}{dt}f_{T_{(1)}}(t;0)|_{t=0} = \lambda^2. \tag{2.36}$$

The explicit value of $f_{T_{(1)}}(t;0)$ *is*

$$f_{T_{(1)}}(t;0) = \frac{\lambda^2}{\sqrt{\mu(4\lambda+\mu)}} e^{-(t/2)(2\lambda+\mu)}\left[e^{(t/2)\sqrt{\mu(4\lambda+\mu)}} - e^{-(t/2)\sqrt{\mu(4\lambda+\mu)}}\right]. \tag{2.37}$$

Proof. By substituting in (2.21) and (2.22) $F_{n,1}^{(0)}(t)$ with $F_{n,0}^{(0)}(t)$, (2.35) emerges. While the first condition is straightforward to verify, the second one needs some explanations: if we write

$$\frac{d}{dt}f_{T_{(1)}}(t;0)|_{t=0} = \lim_{\Delta t \to 0}\frac{f_{T_{(1)}}(\Delta t;0) - f_{T_{(1)}}(0;0)}{\Delta t}, \tag{2.38}$$

and observe that

$$f_{T_{(1)}}(\Delta t;0) = e^{-\lambda\Delta t}\lambda^2 e^{-\mu\Delta t}\int_0^{\Delta t} e^{\mu t_1}dt_1 = \lambda^2\Delta t + o(\Delta t), \tag{2.39}$$

by substituting (2.39) in (2.38), the second condition emerges.

By imposing the initial conditions (2.36) to (2.35), we obtain (2.37). □

Remark 2.9. If $\lambda \to 0$, we get $f_{T_{(1)}}(t;0) = 0$ because the 1th interloss time is greater than t, $\vee t > 0$, when nobody enters.

If $\mu \to 0$, we have that $f_{T_{(1)}}(t;0) = \lambda^2 t e^{-\lambda t}$ because without exits and with the system free at $t = 0$, the 1th interloss time is equal to the sum of 1th and 2th interarrival times.

Remark 2.10. The function $f_{T_{(1)}}(t;0)$ can be expressed by the following hyperbolic function:

$$f_{T_{(1)}}(t;0) = \frac{2\lambda^2 e^{-(t/2)(2\lambda+\mu)}}{\sqrt{\mu(4\lambda+\mu)}}\sinh\frac{t\sqrt{\mu(4\lambda+\mu)}}{2}. \tag{2.40}$$

Remark 2.11. The function $f_{T_{(1)}}(t;0)$ can also be computed as the convolution of the exponential density with rate λ and $f_{T_{(1)}}(t;1)$:

$$f_{T_{(1)}}(t;0) = \int_0^t \lambda e^{-\lambda t_1} f_{T_{(1)}}(t-t_1;1)dt_1. \tag{2.41}$$

In fact, if we observe that

$$F_{n,0}^{(0)}(t) = \int_0^t dt_1\int_{t_1}^t dt_2\cdots\int_{t_{n-1}}^t dt_n e^{-\mu(t-t_n)}\prod_{i=2}^{n}\left[1-e^{-\mu(t_i-t_{i-1})}\right] = \int_0^t dt_1 F_{n-1,1}^{(t_1)}(t), \tag{2.42}$$

by (2.7), (2.32), and (2.42) we obtain that

$$f_{T_{(1)}}(t;0) = e^{-\lambda t}\sum_{n=2}^{\infty}\lambda^n\int_0^t dt_1 F_{n-2,1}^{(0)}(t-t_1) = f_{\tau_{(1)}} * f_{T_{(1)}}(t;1). \tag{2.43}$$

3. Interloss Mean Time

In this section we compute the laplace transforms of the density functions (2.20) and (2.37), and we evaluate the averages of the 1th interloss time, conditionally by $\nu(0)$.

Theorem 3.1. *The laplace transform of* $f_{T_{(1)}}(t;1)$

$$F^*_{T_{(1)}}(s;1) = \int_0^{\infty} e^{-st} f_{T_{(1)}}(t;1)dt \tag{3.1}$$

satisfies the equation

$$s^2 F^*_{T_{(1)}}(s;1) - s\lambda + \lambda(\lambda+\mu) = -(2\lambda+\mu) s F^*_{T_{(1)}}(s;1) + \lambda(2\lambda+\mu) - \lambda^2 F^*_{T_{(1)}}(s;1), \tag{3.2}$$

and its explicit value is

$$F^*_{T_{(1)}}(s;1) = \frac{\lambda(\lambda+s)}{(\lambda+s)^2 + s\mu}. \tag{3.3}$$

Proof. By (2.18), (2.19), and the property

$$\int_0^\infty dt e^{-st} \frac{d}{dt} f_{T_{(1)}}(t;1) = s F^*_{T_{(1)}}(s;1) - f_{T_{(1)}}(0;1), \tag{3.4}$$

the results (3.2) and (3.3) emerge. □

Remark 3.2. The 1th conditional interloss mean time

$$\Theta_1^{(1)} = E\left[T_{(1)} \mid \nu(0) = 1\right] \tag{3.5}$$

can be found by evaluating the derivative of (3.3) with respect to s in $s = 0$, as follows:

$$-\frac{d}{ds} F^*_{T_{(1)}}(s;1)|_{s=0} = \frac{\lambda+\mu}{\lambda^2}. \tag{3.6}$$

In the $M/M/1/1$ model, the interloss mean time (3.5) is equal to the inverse of the Erlang loss rate.

Theorem 3.3. *The laplace transform of* $f_{T_{(1)}}(t;0)$

$$F^*_{T_{(1)}}(s;0) = \int_0^\infty e^{-st} f_{T_{(1)}}(t;0) dt \tag{3.7}$$

satisfies the equation

$$s^2 F^*_{T_{(1)}}(s;0) - \lambda^2 = -(2\lambda+\mu) s F^*_{T_{(1)}}(s;0) - \lambda^2 F^*_{T_{(1)}}(s;0), \tag{3.8}$$

and its explicit value is

$$F^*_{T_{(1)}}(s;0) = \frac{\lambda^2}{(\lambda+s)^2 + s\mu}. \tag{3.9}$$

Proof. By (2.35), (2.36), and (3.4), the results (3.8) and (3.9) emerge. □

Remark 3.4. The conditional interloss mean time

$$\Theta_1^{(0)} = E\left[T_{(1)} \mid \nu(0) = 0\right] \tag{3.10}$$

can be found by evaluating the derivative of (3.9) with respect to s in $s = 0$, as follows:

$$-\frac{d}{ds} F^*_{T_{(1)}}(s;0)\big|_{s=0} = \Theta_1^{(1)} + \frac{1}{\lambda}. \tag{3.11}$$

Remark 3.5. By (2.41), the result (3.9) can be obtained using (3.3), as follows:

$$F^*_{T_{(1)}}(s;0) = \frac{\lambda}{\lambda + s} F^*_{T_{(1)}}(s;1). \tag{3.12}$$

Now, let T_n be the time spent between $t = 0$ and the nth loss, the conditional density functions

$$f_{T_n}(t;i) = \frac{d}{dt} P\{T_n < t; i\}, \tag{3.13}$$

with $i = 0, 1$, can be evaluated by convolutions as follows: if $i = 0$, we have that

$$f_{T_n}(t;0) = f_{\tau_{(1)}} * f_{T_{(1)}}(t;1) * \cdots * f_{T_{(n)}}(t;1), \tag{3.14}$$

while if $i = 1$, we have that

$$f_{T_n}(t;1) = f_{T_{(1)}}(t;1) * \cdots * f_{T_{(n)}}(t;1). \tag{3.15}$$

By the independence between interloss times, the laplace transforms of (3.14) and (3.15) can be expressed as power of (3.3) as follows:

(i) if $i = 0$,

$$F^*_{T_n}(s;0) = F^*_{\tau_{(0)}+\sum_{k=1}^n T_{(k)}}(s;1) = \frac{\lambda}{\lambda + s}\left[\frac{\lambda(\lambda + s)}{(\lambda + s)^2 + \mu s}\right]^n, \tag{3.16}$$

(ii) if $i = 1$,

$$F^*_{T_n}(s;1) = F^*_{\sum_{k=1}^n T_{(k)}}(s;1) = \left[\frac{\lambda(\lambda + s)}{(\lambda + s)^2 + \mu s}\right]^n. \tag{3.17}$$

Remark 3.6. The conditional averages

$$\Theta_n^{(i)} = E[T_n \mid \nu(0) = i], \tag{3.18}$$

with $i = 0, 1$, can be permuted by evaluating the derivatives of (3.16) and (3.17) with respect to s in $s = 0$, thus obtaining

(i) if $i = 0$, we have that

$$-\frac{d}{ds} F^{*}_{T_n}(s;0)\big|_{s=0} = n\Theta_1^{(1)} + \frac{1}{\lambda}, \tag{3.19}$$

(ii) if $i = 1$, we have that

$$-\frac{d}{ds} F^{*}_{T_n}(s;1)\big|_{s=0} = n\Theta_1^{(1)}. \tag{3.20}$$

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