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Research Article

On Variant Reflected Backward SDEs, with Applications

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We study a new type of reflected backward stochastic differential equations (RBSDEs), where the reflecting process enters the drift in a nonlinear manner. This type of the reflected BSDEs is based on a variance of the Skorohod problem studied recently by Bank and El Karoui (2004), and is hence named the "Variant Reflected BSDEs" (VRBSDE) in this paper. The special nature of the Variant Skorohod problem leads to a hidden forward-backward feature of the BSDE, and as a consequence this type of BSDE cannot be treated in a usual way. We shall prove that in a small-time duration most of the well-posedness, comparison, and stability results are still valid, although some extra conditions on the boundary process are needed. We will also provide some possible applications where the VRBSDE can be potentially useful. These applications show that the VRBSDE could become a novel tool for some problems in finance and optimal stopping problems where no existing methods can be easily applicable.

1. Introduction

In this paper we study a new type of reflected backward stochastic differential equations based on the notion of *variant Skorohod problem* introduced recently by Bank and El Karoui [1], as an application of a stochastic representation theorem for an optional process. Roughly speaking, the Variant Skorohod Problem states the following.

For a given optional process X of class (D), null at T, find an $\mathbb{F}$-adapted, right-continuous, and increasing process $A = \{A_t\}_{t\geq 0}$ with $A_{0-} = -\infty$, such that

(i) $Y_t \triangleq E\{\int_t^T f(s, A_s)ds|\mathcal{F}_t\} \leq X_t$, $t \in [0,T]$, *P-a.s.;*

(ii) $E\int_0^T |Y_t - X_t| dA_t = 0$.

The condition (ii) above is called the *flat-off* condition. If we assume further that $\mathbb{F}$ is generated by a Brownian motion B, then it is easily seen that the problem is equivalent to:

Finding a pair of processes (A, Z), *where* A *is increasing and* Z *is square integrable, such that*

$$Y_t = \int_t^T f(s, A_s)ds - \int_t^T Z_s dB_s \leq X_t, \quad 0 \leq t \leq T, \tag{1.1}$$

and that the flat-off condition (ii) holds.

We note that the stochastic representation theorem proposed in [1] has already found interesting applications in various areas, such as nonlinear potential theory [2], optimal stopping, and stochastic finance (see, e.g., [3, 4]). However, to date the extension of the Variant Skorohod Problem to the form of an SDE is essentially open, partly due to the highly technical nature already exhibited in its most primitive form.

In this paper we are interested in the following extension of the Variant Skorohod Problem: Let $X = \{X_t\}_{t\geq 0}$ be an optional process of class (D), and let $f : \Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \mapsto \mathbb{R}$ be a random field satisfying appropriate measurability assumptions. Consider the following backward stochastic differential equation (BSDE for short): for $t \in [0,T]$,

$$Y_t = E\left\{X_T + \int_t^T f(s, Y_s, A_s)ds \mid \mathcal{F}_t\right\}, \tag{1.2}$$

where the solution (Y, A) is defined to be such that

(i) $Y_t \leq X_t$, $0 \leq t \leq T$; $Y_T = X_T$;

(ii) $A = \{A_t\}$ is an adapted, increasing process such that $A_{0-} \triangleq -\infty$, and the flat-off condition holds:

$$E\int_0^T |Y_t - X_t| dA_t = 0. \tag{1.3}$$

Again, if the filtration $\mathbb{F}$ is generated by a Brownian motion B, then we can consider an even more general form of BSDE as extension of (1.1):

$$dY_t = -f(t, Y_t, Z_t, A_t)dt + Z_t dW_t, \quad Y_t \leq X_t, \ t \in [0,T], \qquad Y_T = X_T, \tag{1.4}$$

where A is an increasing process satisfying the flat-off condition, and (Y, Z) is a pair of adapted process satisfying some integrable conditions. Hereafter we will call BSDE (1.2) and (1.4) the *Variant Reflected Backward Stochastic Differential Equations* (VRBSDEs for short), for the obvious reasons. We remark that although the "flat-off" condition (iii) looks very similar to the one in the classic Skorohod problem, there is a fundamental difference. That is, the process A cannot be used as a measure to directly "push" the process Y downwards as a reflecting process usually does, but instead it has to act through the drift f, in a sense as a "density" of a reflecting force. Therefore the problem is beyond all the existing frameworks of the reflected SDEs.

Our first task in this paper is to study the well-posedness of the VRBSDE. It is worth noting that the fundamental building block of the nonlinear Skorohod problem is a representation theorem, which in essence is to find an optional process L so that the given optional obstacle process X can be written as

$$X_S = E\left\{\int_S^T f\left(u, \sup_{S\le v\le u} L_v\right)du \mid \mathcal{F}_S\right\}, \tag{1.5}$$

for all stopping time S taking values in $[0,T]$. In fact, the "reflecting" process A is exactly the running maximum of the process L. Consequently, while (1.2) and (1.4) are apparently in the forms of BSDEs, they have a strong nature of a forward-backward SDEs. This brings in some very subtle difficulties, which will be reflected in our results. We would like to mention that the main difficulty here is to find a control for the reflecting process A. In fact, unlike the classic Skorohod problem, the characterization of reflecting process A is far more complicated, and there is no simple way to link it with the solution process Y. We will prove, nevertheless, that the SDE is well-posed over a small-time duration, and a certain continuous dependance and comparison theorems are still valid.

The second goal of this paper is to present some possible applications where the VRBSDE could play a role that no existing methods are amenable. In fact, the form of the VRBSDE (1.2) suggests that the process Y can be viewed as a stochastic recursive intertemporal utility (see, e.g., [5]). We will show that if we consider the utility optimization problem with Hindy-Kreps-Huang type preference (see, e.g., [1, 6, 7]), and the goal is minimizing such a utility while trying to keep it aloft, then the optimal solution will be given by solving a VRBSDE with the given lower boundary. To our best knowledge, such a result is novel. Another possible application of the VRBSDE that will be explored in the paper is a class of optimal stopping problems. We show that the solution to our VRBSDE can be used to describe the value function of a family of optimal stopping problems, and the corresponding reflecting process can be used as a universal signal of exercise time, which extends a result of Bank-Föllmer [3] to an SDE setting.

The rest of the paper is organized as follows. In Section 2 we revisit the stochastic representation theorem, and give the detailed formulation of the VRBSDE. In Section 3 we study the well-posedness of the equation. In Sections 4 and 5 we study the comparison theorem and the continuous dependence results. Finally we present some possible applications of VRBSDEs in the utility minimization problems and a class of optimal stopping problems in Section 6.

2. Formulation of the Variant RBSDE

Throughout this paper we assume that $(\Omega, \mathcal{F}, P; \mathbb{F})$ is a filtered probability space, where $\mathbb{F} \triangleq \{\mathcal{F}_t\}_{0\le t\le T}$ is a filtration that satisfies the usual hypothses. For simplicity we assume that $\mathcal{F} = \mathcal{F}_T$. In the case when the filtration $\mathbb{F}$ is generated by a standard Brownian motion B on the space $(\Omega, \mathcal{F}, P)$, we say that $\mathbb{F}$ is "Brownian" and denote it by $\mathbb{F} = \mathbb{F}^B$. We will always assume that $\mathbb{F}^B$ is augmented by all the P-null sets in $\mathcal{F}$.

We will frequently make use the following notations. Let

(i) $\mathbb{L}_T^\infty$ be the space of all $\mathcal{F}_T$ measurable bounded random variables,

(ii) $\mathbb{H}_T^\infty$ the space of all $\mathbb{R}$-valued, progressively measurable, bounded processes,

(iii) $\mathbb{H}_T^2$ the space of all $\mathbb{R}^d$-valued, progressively measurable process Z, such that $E\int_0^T |Z_s^2| ds < \infty$,

(iv) $\mathcal{M}_{0,T}$ the set of all the stopping times taking values in $[0,T]$.

Similar to the Variant Skorohod Problem, a VRBSDE involves two basic elements: (1) a boundary process $X = \{X_t,\ 0 \le t \le T\}$ which is assumed to be an optional process of class (D) (A process X is said to belong to *Class (D)* on $[0,T]$ if the family of random variables $\{X_\tau :\ \tau \in \mathcal{M}_{0,T}\}$ is uniformly integrable), and is lower-semicontinuous in expectation; and (2) a drift coefficient f. In this paper we will focus only on the case where f is independent of z, and we assume that it satisfies the following *Standing Assumptions:*

(H1) the coefficient $f : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ enjoys the following properties:

(i) for fixed $\omega \in \Omega$, $t \in [0,T]$, and $y \in \mathbb{R}$, the function $f(\omega, t, y, \cdot) : \mathbb{R} \mapsto \mathbb{R}$ is continuous and strictly decreasing from $+\infty$ to $-\infty$,

(ii) for fixed $y, l \in \mathbb{R}^3$, the process $f(\cdot, \cdot, y, l)$ is progressively measurable with

$$E\int_0^T \left|f(t,y,l)\right| dt \le +\infty, \tag{2.1}$$

(iii) there exists a constant $L > 0$, such that for all fixed t, ω, l it holds that

$$\left|f(t,\omega,y',l) - f(t,\omega,y,l)\right| \le L\left|y' - y\right|, \quad \forall y', y \in \mathbb{R}, \tag{2.2}$$

(iv) there exist two constants $k > 0$ and $K > 0$, such that for all fixed t, ω, y it holds that

$$k\left|l' - l\right| \le \left|f(t,y,l') - f(t,y,l)\right| \le K\left|l' - l\right|, \quad \forall\ l', l \in \mathbb{R}. \tag{2.3}$$

We remark that the assumption (iv) in (H1) amounts to saying that the derivative of f with respect to l, if exists, should be bounded from below. While this is merely technical, it also indicates that we require a certain sensitivity of the solution process Y with respect to the reflection process A. This is largely due to the nonlinearity between the solution and the reflecting process, which was not an issue in the classical Skorohod problem.

We now introduce our variant reflected BSDE. Note that we do not assume that the filtration $\mathbb{F}$ is Brownian at this point.

Definition 2.1. Let $\xi \in \mathbb{L}^\infty_T$ and the boundary process X be given. A pair of processes (Y, A) is called a solution of Variant Reflected BSDE with terminal value ξ and boundary X if

(i) Y and A are $\mathbb{F}$-adapted processes with càdlàg paths;

(ii) $Y_t = E\{\xi + \int_t^T f(s, Y_s, A_s) dt \mid \mathcal{F}_t\}$;

(ii) $Y_t \le X_t$, $0 \le t \le T$; $Y_T = X_T = \xi$;

(iv) the process A is $\mathbb{F}$-adapted, increasing, càdlàg, and $A_{0-} \triangleq -\infty$, such that the "flat-off" condition holds:

$$E\int_0^T |Y_t - X_t| dA_t = 0. \tag{2.4}$$

Remark 2.2. The assumption $A_{0-} = -\infty$ has an important implication: the solution Y must satisfy $Y_0 = X_0$. This can be deduced from the flat of condition (2.4), and the fact that $dA_0 > 0$ always holds. Such a fact was implicitly, but frequently, used in [1], and will be crucial in some of our arguments below.

We note that if we denote $M_t = E\{\int_0^T f(t, Y_t, A_t)dt \mid \mathcal{F}_t\}$, $t \in [0, T]$ then M is a martingale on $[0, T]$, and the VRBSDE will read

$$Y_t = \xi + \int_t^T f(s, Y_s, A_s)dt - (M_T - M_t), \quad 0 \le t \le T. \tag{2.5}$$

Thus if we assume further that the filtration is Brownian, than we can consider the more general form of VRBSDE.

Definition 2.3. Assume that the filtration $\mathbb{F} = \mathbb{F}^B$, that is, it is generated by a standard Brownian motion B, with the usual augmentation. Let $\xi \in \mathbb{L}_T^\infty$ and the boundary process X be given. A triplet of processes $\{(Y_t, Z_t, A_t),\ 0 \le t \le T\}$ is called a solution of Variant Reflected BSDE with terminal value ξ and boundary X if

(i) $Y \in \mathbb{H}_T^\infty$, $Z \in \mathbb{H}_T^2$,

(ii) $Y_t = \xi + \int_t^T f(s, Y_s, Z_s, A_s)ds - \int_t^T Z_s\, dB_s$, $0 \le t \le T$,

(iii) $Y_t \le X_t$, $0 \le t \le T$; $Y_T = X_T = \xi$,

(iv) the process $\{A_t\}$ is $\mathbb{F}$-adapted, increasing, càdlàg, and $A_{0-} = -\infty$, such that the flat-off condition holds: $E\int_0^T |Y_t - X_t| dA_t = 0$.

Our study of VRBSDE is based on a *Stochastic Representation Theorem* of Bank and El Karoui [1]. We summarize the stochastic representation and some related fact in the following theorem, which is slightly modified to suit our situation.

Theorem 2.4 (see, Bank-El Karoui [1]). *Assume (H1)-(i), (ii). Then every optional process X of class (D) which is lower semicontinuous in expectation admits a representation of the form*

$$X_S = E\left\{ X_T + \int_S^T f\left(u, \sup_{S \le v \le u} L_v\right) du \mid \mathcal{F}_S \right\} \tag{2.6}$$

for any stopping time $S \in \mathcal{M}_{0,T}$, where L is an optional process taking values in $\mathbb{R} \cup \{-\infty\}$, and it can be characterized as follows:

(i) $f(u, \sup_{S \le v \le u} L_v) \in L^1(\mathbb{P} \otimes dt)$ *for any stopping time S,*

(ii) $L_S = \operatorname{ess\,inf}_{\tau > S} l_{S,\tau}$, *where the "ess inf" is taken over all stopping times $S \in \mathcal{M}_{0,T}$ such that $S < T$, a.s.; and $l_{S,\tau}$ is the unique $\mathcal{F}_S$-measurable random variable satisfying:*

$$E\{X_S - X_\tau \mid \mathcal{F}_S\} = E\left\{ \int_S^\tau f(u, l_{S,\tau}) du \mid \mathcal{F}_S \right\}, \tag{2.7}$$

(iii) *(Gittin Index) if* $V(t,l) \overset{\Delta}{=} \operatorname{ess\,inf}_{\tau\geq t} E\{E\int_t^\tau f(u,l)du + X_\tau | \mathcal{F}_t\}$, $t \in [0,T]$, *is the value functions of a family of optimal stopping problems indexed by* $l \in \mathbb{R}$, *then*

$$L_t = \sup\{l : V(t,l) = X_t\}, \quad t \in [0,T]. \tag{2.8}$$

We should note here, unlike the original stochastic representation theorem in [1] where it assumed that $X_T = 0$, we allow arbitrary terminal value for X_T. This can be obtained easily by considering a new process $\widetilde{X}_t \overset{\Delta}{=} X_t - E[\xi \mid \mathcal{F}_t]$, $t \geq 0$. A direct consequence of the stochastic representation theorem is the following Variant Skorohod Problem, which is again slightly adjusted to our non-zero terminal value case.

Theorem 2.5. *Assume (H1)-(i), (ii). Then for every optional process* X *of class (D) which is lower semicontinuous in expectation, there exists a unique pair of adapted processes* (Y, A), *where* Y *is continuous and* A *is increasing, such that*

$$Y_t = E\left\{X_T + \int_t^T f(s, A_s)ds \mid \mathcal{F}_t\right\}, \quad t \in [0,T]. \tag{2.9}$$

Furthermore, the process A *can be expressed as* $A_t = \sup_{0\leq s\leq t+} L_s$, *where* L *is the process in Theorem 2.4.*

We conclude this section by making following observations. First, the random variable $l_{S,\tau}$, defined by (2.7) is $\mathcal{F}_S$-measrable for any stopping time $\tau > S$, thus the process $s \mapsto L_s$ is $\mathbb{F}$-adapted. However, the running maximum process $A_t \overset{\Delta}{=} \sup_{0\leq u\leq t+} L_u$ depends on the whole path of process L, whence X. Thus, although the variant Skorohod problem (2.9) looks quite similar to a standard *backward* stochastic differential equation, it contains a strong "*forward-backward*" nature. These facts will be important in our future discussions.

3. Existence and Uniqueness

In this section we study the well-posedness of the VRBSDE (2.4). We note that in this case we do not make any restriction on the filtration, as long as it satisfies the *usual hypotheses*.

We will follow the usual technique, namely the contraction mapping theorem, to attack the existence and uniqueness of the solution. It is worth noting that due to the strong forward-backward structure as well as the fundamental non-Markovian nature of the problem, a general result with arbitrary duration is not clear at this point. The results presented in this section will provide the first look at some basic features of such an equation.

We will make use of the following extra assumptions on the boundary process X and the drift coefficient f:

(H2) there exists a constant $\Gamma > 0$, such that

(i) for any $S \in \mathcal{M}_{0,T}$, it holds that

$$\operatorname*{ess\,sup}_{\substack{\tau>S \\ \tau\in\mathcal{M}_{0,T}}} \left| \frac{E\{X_\tau - X_S \mid \mathcal{F}_S\}}{E\{\tau - S \mid \mathcal{F}_S\}} \right| \leq \Gamma, \quad \text{a.s.} \tag{3.1}$$

(ii) $|f(t,0,0)| \leq \Gamma, t \in [0,T]$.

Remark 3.1. The assumption (3.1) is merely technical. It is motivated by the "Gittin indices" studied in [8], and it essentially requires a certain "path regularity" on the boundary process X. However, one should note that it by no means implies the continuity of the paths of $X(!)$. In fact, a semimartingale with absolutely continuous bounded variation part can easily satisfy (3.1), but this does not prevent jumps from the martingale part.

We begin by considering the following mapping $\mathfrak{T}$ on $\mathbb{H}_T^\infty$: for a given process y we define $\mathfrak{T}(y)_t \stackrel{\Delta}{=} Y_t, t \in [0,T]$, where Y is the unique solution of the Variant Skorohod problem:

$$\begin{gathered} Y_t = E\left\{\xi + \int_t^T f(s, y_s, A_s)ds \mid \mathcal{F}_s\right\}, \quad t \in [0,T], \\ E\int_0^T [X_t - Y_t]dA_t = 0, \quad t \in [0,T]. \end{gathered} \tag{3.2}$$

We are to prove that the mapping $\mathfrak{T}$ is a contraction from $\mathbb{H}_T^\infty$ to itself. It is not hard to see, by virtue of Theorems 2.4 and 2.5, that the reflecting process A is determined by y in the following way: $A_t = \sup_{0\leq v\leq t+} L_v$, and L is the solution to the Stochastic Representation:

$$X_t = E\left\{\xi + \int_t^T f\left(s, y_s, \sup_{t\leq v\leq s} L_v\right)ds \mid \mathcal{F}_t\right\}, \quad t \in [0,T]. \tag{3.3}$$

We should note, however, that the contraction mapping argument does not completely solve the existence and uniqueness issue for the Variant BSDE. In fact, it only gives the existence of the fixed point Y, and we will have to argue the uniqueness of the process A separately.

We now establish some a priori estimates that will be useful in our discussion. To begin with, let us consider the stochastic representation

$$X_t = E\left\{\xi + \int_t^T f\left(s, 0, \sup_{t\leq v\leq s} L_v^0\right)ds \mid \mathcal{F}_t\right\}. \tag{3.4}$$

Denote $A_t^0 \stackrel{\Delta}{=} \sup_{0\leq s\leq t+} L_t^0$. We have the following estimate for A^0.

Lemma 3.2. *Assume (H1) and (H2). Then it holds that $\|A^0\|_\infty \leq 2\Gamma/k$, where k and Γ are the constants appearing in (H1) and (H2).*

Proof. For fixed $s \in [0,T]$ and any stopping time $\tau > s$, let $l_{s,\tau}^0$ be the $\mathcal{F}_s$ measurable random variable such that

$$E\{X_s - X_\tau \mid \mathcal{F}_s\} = E\left\{\int_s^\tau f\left(t, 0, l_{s,\tau}^0\right)dt \mid \mathcal{F}_s\right\}. \tag{3.5}$$

Then by Theorem 2.4 we have $L_s^0 = \operatorname{ess\,inf}_{\tau>s} l_{s,\tau}^0$, and $A_t^0 = \sup_{0\leq s\leq t+} L_s^0$. □

Now consider the set $\{\omega : l^0_{s,\tau}(\omega) < 0\}$. Since $f(t,0,\cdot)$ is decreasing, we have

$$
\begin{aligned}
E\{X_s - X_\tau \mid \mathcal{F}_s\} - E\left\{\int_s^\tau f(t,0,0)dt \mid \mathcal{F}_s\right\} &= E\left\{\int_s^\tau f\left(t,0,l^0_{s,\tau}\right) - f(t,0,0)dt \mid \mathcal{F}_s\right\} \\
&\geq E\left\{\int_s^\tau k\left|l^0_{s,\tau}\right|dt \mid \mathcal{F}_s\right\} \geq k\left|l^0_{s,\tau}\right|E\{\tau - s \mid \mathcal{F}_s\}.
\end{aligned}
\tag{3.6}
$$

In other words we have

$$
\left|l^0_{s,\tau}\right| \leq \frac{1}{k}\left\{\frac{E\{X_s - X_\tau \mid \mathcal{F}_s\}}{E\{\tau - s \mid \mathcal{F}_s\}} - \frac{E\{\int_s^\tau f(t,0,0)dt \mid \mathcal{F}_s\}}{E\{\tau - s \mid \mathcal{F}_s\}}\right\}, \quad \text{on } \left\{l^0_{s,\tau} < 0\right\}. \tag{3.7}
$$

Similarly, one can show that on the set $\{l^0_{s,\tau} \geq 0\}$ it holds that

$$
l^0_{s,\tau} \leq \frac{1}{k}\left\{-\frac{E\{X_s - X_\tau \mid \mathcal{F}_s\}}{E\{\tau - s \mid \mathcal{F}_s\}} + \frac{E\{\int_s^\tau f(t,0,0)dt \mid \mathcal{F}_s\}}{E\{\tau - s \mid \mathcal{F}_s\}}\right\}. \tag{3.8}
$$

Consequently, we have

$$
\left|l^0_{s,\tau}\right| \leq \frac{1}{k}\left\{\left|\frac{E\{X_\tau - X_s \mid \mathcal{F}_s\}}{E\{\tau - s \mid \mathcal{F}_s\}}\right| + \frac{E\{\int_s^\tau |f(t,0,0)|dt \mid \mathcal{F}_s\}}{E\{\tau - s \mid \mathcal{F}_s\}}\right\}. \tag{3.9}
$$

Now note that

$$
\left|A^0_t\right| = \left|\sup_{0\leq s\leq t+} L^0_s\right| \leq \sup_{0\leq s\leq t+}\left|L^0_s\right| = \sup_{0\leq s\leq t+}\left\{\operatorname*{ess\,inf}_{\tau>s}\left|l^0_{s,\tau}\right|\right\}, \tag{3.10}
$$

we derive from (3.9) and (H2) that

$$
\left|A^0_t\right| \leq \sup_{0\leq s\leq t+}\left\{\operatorname*{ess\,sup}_{\tau>s}\left|l^0_{s,\tau}\right|\right\} \leq \sup_{0\leq s\leq t+}\left\{\frac{\Gamma+\Gamma}{k}\right\} = \frac{2\Gamma}{k}, \tag{3.11}
$$

proving the lemma.

Clearly, a main task in proving that $\mathcal{T}$ is a contraction mapping is to find the control on the difference of two reflecting processes. To see this let $y, y' \in \mathbb{H}^\infty_T$ be given, and consider the two solutions of the variant Skorohod problem: (Y, A) and (Y', A'). We would like to control $|A'_s - A_s|$ in terms of $|y'_s - y_s|$. The following lemma is crucial.

Lemma 3.3. *Assume (H1) and (H2). Then, for any $t \in [0,T]$, it holds almost surely that*

$$
|A'_t - A_t| \leq \frac{L}{k}\|y' - y\|_\infty. \tag{3.12}
$$

Proof. Again, we fix s and let $\tau \in \mathcal{M}_{0,T}$ be such that $\tau > s$, a.s. Recalling Theorem 2.4, we let $l_{s,\tau}$ and $l'_{s,\tau}$ be two $\mathcal{F}_s$-measurable random variables such that

$$E\{X_s - X_\tau \mid \mathcal{F}_s\} = E\left\{\int_s^\tau f(u, y_u, l_{s,\tau})du \mid \mathcal{F}_s\right\} = E\left\{\int_s^\tau f(u, y'_u, l'_{s,\tau})du \mid \mathcal{F}_s\right\}. \tag{3.13}$$

Define $D_s^\tau = \{\omega \mid l'_{s,\tau}\ (\omega) > l_{s,\tau}(\omega)\}$, then $D_s^\tau \in \mathcal{F}_s$, for any stopping time $\tau > s$.

Now, from (3.13) and noting that $1_{D_s^\tau}$ is $\mathcal{F}_s$-measurable, we deduce that

$$\begin{aligned} &E\left\{\int_s^\tau [f(u, y_u, l_{s,\tau}) - f(u, y_u, l'_s)]1_{D_s^\tau}du \mid \mathcal{F}_s\right\} \\ &\quad = E\left\{\int_s^\tau [f(u, y'_u, l'_{s,\tau}) - f(u, y_u, l'_{s,\tau})\ ]1_{D_s^\tau}du \mid \mathcal{F}_s\right\}. \end{aligned} \tag{3.14}$$

Now, by (H1)-(iv), the left-hand side of (3.14) satisfies

$$E\left\{\int_s^\tau [f(u, y_u, l_{s,\tau}) - f(u, y_u, l'_{s,\tau})]1_{D_s^\tau}du \mid \mathcal{F}_s\right\} \geq k\ \ \left|l_{s,\tau} - l'_{s,\tau}\right| E\{\tau - s \mid \mathcal{F}_s\}1_{D_s^\tau}. \tag{3.15}$$

On the other hand, by (H1)-(iii) we see that the right-hand side of (3.14) satisfies

$$\begin{aligned} &E\left\{\int_s^\tau [f(u, y'_{\mathrm{u}}, l'_{s,\tau}) - f(u, y_u, l'_{s,\tau})]1_{D_s^\tau}du \mid \mathcal{F}_s\right\} \\ &\quad \leq E\left\{\int_s^\tau \left|f(u, y'_u, l'_{s,\tau}) - f(u, y_u, l'_{s,\tau})\right|1_{D_s^\tau}du \mid \mathcal{F}_s\right\} \leq LE\{\|y' - y\|_\infty(\tau - s) \mid \mathcal{F}_s\}1_{D_s^\tau}. \end{aligned} \tag{3.16}$$

Combining above we obtain that

$$k\left|l_{s,\tau} - l'_{s,\tau}\right| E\{\tau - s \mid \mathcal{F}_s\} \leq L\|y' - y\|_\infty E\{\tau - s \mid \mathcal{F}_s\}, \quad \text{on } D_s^\tau. \tag{3.17}$$

Thus $|l_{s,\tau} - l'_{s,\tau}| \leq (L/k)\|y' - y\|_\infty$, on D_s^τ, since $\tau > s$, a.s.

Similarly, one shows that the inequality holds on the complement of D_s^τ as well. It follows that

$$\left|l_{s,\tau} - l'_{s,\tau}\right| \leq \frac{L}{k}\|y' - y\|_\infty. \tag{3.18}$$

Next, recall from Theorem 2.4 that $L_s = \operatorname{ess\,inf}_{\tau>s} l_{s,\tau}$, $L'_s = \operatorname{ess\,inf}_{\tau>s} l'_{s,\tau}$, $A_t = \sup_{0\le s\le t} L_s$, and $A'_t = \sup_{0\le s\le t} L'_s$. We conclude from (3.18) that, for any $t \in [0,T]$,

$$\begin{aligned}\left|A'_t - A_t\right| &= \left|\sup_{0\le s\le t+} L'_s - \sup_{0\le s\le t+} L_s\right| \le \sup_{0\le s\le t+}\left|\operatorname*{ess\,inf}_{\tau>s} l'_{s,\tau} - \operatorname*{ess\,inf}_{\tau>s} l_{s,\tau}\right| \\ &\le \sup_{0\le s\le t+} \operatorname*{ess\,sup}_{\tau>s}\left|l'_{s,\tau} - l_{s,\tau}\right| \le \frac{L}{k}\left\|y' - y\right\|_\infty, \quad P\text{-a.s.}\end{aligned} \tag{3.19}$$

The proof is now complete. □

Remark 3.4. We observe that the step from (3.16) to (3.17) is seemingly rough. It would be more desirable if some more delicate estimates, such as

$$E\left\{\int_s^\tau \left|y_u - y'_u\right| du \mid \mathcal{F}_s\right\} \le CE\{\tau - s \mid \mathcal{F}_s\} E\left\{\sup_{0\le u\le T}\left|y_u - y'_u\right| \mid \mathcal{F}_s\right\} \tag{3.20}$$

could hold for some constant C, so that one can at least remove the boundedness requirement on the solution. But unfortunately (3.20) is not true in general, unless some conditional independence is assumed. Here is a quick example: let $T = 1$ and let τ be a binomial random variable that takes value 1 with probability p and $1/n$ with probability $1 - p$. Define two processes: $y_t = 1_{\{\tau=1\}}$, $t \in [0,1]$; $h_t = 1_{\{\tau\le t\}}$, $t \in [0,1]$; and define $\mathcal{F}_t = \sigma\{(y_u, h_u) : 0 \le u \le t\}$ with $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,1]}$. Then τ is an $\mathbb{F}$-stopping time and y is an $\mathbb{F}$-adapted continuous process.

It is easy to check that $E\{\int_0^\tau |y_u| du\} = p$ and $E\{\tau\}E\{\sup_{0\le u\le 1}|y_u|\} = (p + (1/n)(1-p))p$. Thus if we choose p, n, and a constant $c \ge 1$ such that

$$p < \frac{n-c}{(n-1)c} < 1, \tag{3.21}$$

then (3.20) will fail at $s = 0$, with $C = c$.

We are now ready to prove the main result of this section, the existence and uniqueness of the solution to the Variant RBSDE.

Theorem 3.5. *Assume (H1) and (H2). Assume further that* $(L + K(L/k)T < 1$, *then the Variant reflected BSDE* (1.2) *admits a unique solution* (Y, A).

Proof. We first show that the mapping $\mathfrak{T}$ defined by (3.2) is from $\mathbb{H}_t^\infty$ to itself. To see this, we note that by using assumption (H1) and Lemmas 3.2 and 3.3, one has

$$\begin{aligned}\left|f(s, y_s, A_s)\right| &\le \left|f\left(s, 0, A_s^0\right)\right| + L\left|y_s\right| + K\left|A_s - A_s^0\right| \\ &\le \left|f(s,0,0)\right| + K\left|A_s^0\right| + L\|y\|_\infty + K\frac{L}{k}\|y\|_\infty \\ &\le \Gamma + K\frac{2\Gamma}{k} + L\|y\|_\infty + K\frac{L}{k}\|y\|_\infty.\end{aligned} \tag{3.22}$$

Since $\xi \in L^\infty$ by assumption, we can then easily deduce that $Y = \mathfrak{T}(y) \in \mathbb{H}_T^\infty$.

To prove that $\mathfrak{T}$ is a contraction, we take $y, y' \in \mathbb{H}_T^\infty$, and denote $\mathfrak{T}(y) = Y$ and $\mathfrak{T}(y') = Y'$. Then, for any $t \in [0, T]$, applying Lemma 3.3 we have

$$\begin{aligned}\left|\mathfrak{T}(y)_t - \mathfrak{T}(y')_t\right| &\leq \left|E\left\{\int_t^T [f(s, y_s, A_s) - f(s, y'_s, A_s)]ds \mid \mathcal{F}_t\right\}\right| \\ &\leq T\left(L\|y - y'\|_\infty + K\|A - A'\|_\infty\right) \leq T\left(L + K\frac{L}{k}\right)\|y - y'\|_\infty.\end{aligned} \tag{3.23}$$

Since $T(L + K(L/k)) < 1$ by assumption, we see that $\mathfrak{T}$ is a contraction.

Now, let $Y \in \mathbb{H}_T^\infty$ be the (unique) fixed point of $\mathfrak{T}$, and let A be the corresponding reflecting process defined by $A_t = \sup_{0 \leq v \leq t+} L_v$, where L satisfies the representation

$$X_t = E\left\{\xi + \int_t^T f\left(s, Y_s, \sup_{t \leq v \leq s} L_v\right)ds \mid \mathcal{F}_t\right\}. \tag{3.24}$$

We now show that (Y, A) is the solution to the Variant RBSDE (1.2). To see this, note that (3.24), the definition of A, and the monotonicity of the function f (on the variable l) tell us that, for $t \in [0, T]$,

$$Y_t = E\left\{\xi + \int_t^T f(s, Y_s, A_s)ds \mid \mathcal{F}_t\right\} \leq E\left\{\xi + \int_t^T f\left(s, Y_s, \sup_{t \leq v \leq s} L_v\right)ds \mid \mathcal{F}_t\right\} = X_t. \tag{3.25}$$

Thus it remains to show that the flat-off condition holds. But by the properties of optional projections and definition of L and A, we have

$$\begin{aligned}E\int_0^T [X_t - Y_t]dA_t &= E\int_0^T\left\{\int_t^T\left[f\left(u, Y_u, \sup_{t \leq v \leq u} L_v\right) - f\left(u, Y_u, \sup_{0 \leq v \leq u+} L_v\right)\right]du\right\}dA_t \\ &= E\int_0^T\left\{\int_0^u\left[f\left(u, Y_u, \sup_{t \leq v \leq u+} L_v\right) - f\left(u, Y_u, \sup_{0 \leq v \leq u+} L_v\right)\right]dA_t\right\}du,\end{aligned} \tag{3.26}$$

here the last equality follows from the Fubini theorem and the fact that the Lebesgues measure does not charge the discontinuities of the paths $u \mapsto \sup_{t \leq v \leq u} L_v$, which are only countably many.

Finally, note that on the set $\{(t, \omega) : dA_t(\omega) > 0\}$, t must be a point of increase of $A \cdot (\omega)$. Since A is the running supreme of L we conclude that $\sup_{0 \leq v \leq t+\delta} L_v > \sup_{0 \leq v \leq t-} L_v$, for all $\delta > 0$. This yields that

$$\sup_{t \leq v \leq u+} L_v = \sup_{0 \leq v \leq u+} L_v, \quad \text{on } \{(\mathrm{t}, \omega) : \mathrm{dA_t}(\omega) > 0\}. \tag{3.27}$$

Thus the right side of (3.26) is identically zero, and the flat-off condition holds. This proves the existence of the solution (Y, A).

The uniqueness of the solution can be argued as follows. Suppose that there is another solution (Y', A') to the VRBSDE such that $Y_t \le X_t$, $Y'_t \le X_t$, $t \in [0,T]$, and

$$Y_t = E\left\{ \xi + \int_t^T f(u, Y_u, A_u)du \mid \mathcal{F}_t \right\}, \qquad E\int_0^T |X_u - Y_u| dA_u = 0;$$
$$Y'_t = E\left\{ \xi + \int_t^T f(u, Y'_u, A'_u)du \mid \mathcal{F}_t \right\}, \qquad E\int_0^T \left|X_u - Y'_u\right| dA'_u = 0. \tag{3.28}$$

Since both Y and Y' are the fixed points of the mapping $\mathfrak{T}$, it follows that $Y_t = Y'_u$, $t \in [0,T]$, P-a.s. Now consider the Variant Skorohod Problem

$$\widetilde{Y}_t = E\left\{ \xi + \int_t^T f^Y\left(u, \widetilde{A}_u\right)du \mid \mathcal{F}_t \right\},$$
$$\widetilde{Y}_t \le X_t, \qquad \widetilde{Y}_T = X_T = \xi, \tag{3.29}$$
$$E\int_0^T \left|\widetilde{Y}_t - X_t\right| d\widetilde{A}_t = 0,$$

where $f^Y(u,l) \triangleq f(u, Y_u, l)$. Then there exists a unique pair of process $(\widetilde{Y}, \widetilde{A})$ that solves the Variant Skorohold problem, thanks to Theorem 2.5. But since both (Y, A) and (Y, A') are the solutions to the Variant RBSDE (3.29), it follows that $Y_t = \widetilde{Y}_t$ and $A_t = A'_t = \widetilde{A}_t$, $t \in [0,T]$, a.s., proving the uniqueness, whence the theorem. □

We remark that our existence and uniqueness proof depends heavily on the well-posedness result of the stochastic representation theorem in [1], which requires that $A_{0-} = -\infty$ so that $t = 0$ must be a point of increase of process A. A direct consequence is then $Y_0 = X_0$, by the flat-off condition, as we pointed out in Remark 2.2. The following corollary shows that this is not the only reason that solution of VRBSDE is actually a "bridge" with respect to the boundary process X.

Corollary 3.6. *Suppose that Y is a solution to VRBSDE with generator f and upper boundary X. Then $Y_0 = X_0$.*

Proof. Since Y is a fixed point of the mapping $\mathfrak{T}$ defined by (3.2), we see that Y_0 and X_0 satisfy the following equalities:

$$X_0 = E\left\{ \xi + \int_0^T f\left(s, Y_s, \sup_{0 \le v \le s} L_v\right) ds \right\},$$
$$Y_0 = E\left\{ \xi + \int_0^T f(s, Y_s, A_s) ds \right\} = E\left\{ \xi + \int_0^T f\left(s, Y_s, \sup_{0 \le v \le s+} L_v\right) ds \right\}, \tag{3.30}$$

but as we argued before that the paths of the increasing process $u \mapsto \sup_{0 \le v \le u} L_v$ has only countably many discontinuities, which are negligible under the Lebesgue measure, we conclude that $Y_0 = X_0$. □

4. Comparison Theorems

In this section we study the comparison theorem of the Variant RBSDE, one of the most useful tools in the theory of the BSDEs. We should note that the method that we will employ below follows closely to the uniqueness argument used in [1], which was more or less hidden in the proof of Theorem 3.5 as we applied the uniqueness of the Variant Skorohod problem. As we will see below, such a method is quite different from all the existing arguments in the BSDE context.

We begin by considering two VRBSDEs for $i = 1, 2$,

$$Y_t^i = E\left\{\xi^i + \int_t^T f^i\left(u, Y_u^i, A_u^i\right) du \mid \mathcal{F}_t\right\},$$
$$Y_t^i \le X_t^i, \qquad Y_T^i = X_T^i = \xi^i, \tag{4.1}$$
$$E\int_0^T \left|Y_t^i - X_t^i\right| dA_t^i = 0.$$

In what follows we call (f^i, X^i), $i = 1, 2$, the "parameters" of the VRBSDE (4.1), $i = 1, 2$, respectively. Define two stopping times:

$$s \triangleq \inf\left\{t \in [0, T) \mid A_t^2 > A_t^1 + \varepsilon\right\} \wedge T;$$
$$\tau \triangleq \inf\left\{t \in [s, T) \mid A_t^1 > A_t^2 - \frac{\varepsilon}{2}\right\} \wedge T. \tag{4.2}$$

The following statements are similar to the solutions to Variant Skorohod problems (see [1]). We provide a sketch for completeness.

Lemma 4.1. *The stopping times s and τ defined by* (4.2) *have the following properties:*

(i) *s, τ are points of increase for A^2 and A^1, respectively. In other words, for any $\delta > 0$, it holds that $A_{s-}^2 < A_{s+\delta}^2$ and $A_{\tau-}^1 < A_{\tau+\delta}^1$,*

(ii) *$P\{s < \tau\} = 1$; and $A_t^1 \le A_t^2 - \varepsilon/2$, for all $t \in [s, \tau]$, P-a.s.,*

(iii) *it holds that $Y_s^2 = X_s^2$ and $Y_\tau^1 = X_\tau^1$, P-a.s.*

Proof. Since (ii) is obvious by the definition of s and τ and (iii) is a direct consequence of (i) and the flat-off condition, we need only check property (i).

Let ω be fixed. By the right continuity of A^2 and A^1, as well as the definition of s, we can find a decreasing sequence of stopping times $\{s_n\}$ such that $s_n \searrow s$, and $A_{s_n}^2 > A_{s_n}^1 + \varepsilon$, for n sufficiently large (may assume for all n). Since A^1 is increasing, we have

$$A_{s_n}^2 > A_{s_n}^1 + \varepsilon \ge A_s^1 + \varepsilon \ge A_{s-}^1 + \varepsilon. \tag{4.3}$$

Note that s is the first time A^2 goes above $A^1 + \varepsilon$, one has $A_{s-}^2 \le A_{s-}^1 + \varepsilon$. Thus, $A_{s_n}^2 > A_{s-}^2$, for all n. Now for any $\delta > 0$, one can choose n large enough such that $s_n < s + \delta$ and it follows that $A_{s+\delta}^2 \ge A_{s_n}^2 > A_{s-}^2$, that is, s is a point of increase of A^2.

That τ is a point of increase of A^1 can be proved using a similar argument. □

We now give a simple analysis that would lead to the comparison theorem. Let (Y^i, A^i), $i = 1, 2$ be the solutions to two VRBSDEs with boundaries X^1 and X^2, respectively. Define s and τ as in (4.2). By Lemma 4.1, $s < \tau$, P-a.s., with $Y_s^2 = X_s^2$ and $Y_\tau^1 = X_\tau^1$. To simplify notations let us denote $\delta\Theta = \Theta^1 - \Theta^2$, $\Theta = X, Y, A$, and ξ. Furthermore, let us define two martingales $M_t^i \triangleq E\{\int_0^T f^i(u, Y_u^1, A_u^1)du \mid \mathcal{F}_t\}$, $t \in [0, T]$, $i = 1, 2$, then on the set $\{s < T\}$ we can write

$$\begin{aligned} \delta Y_s &= \delta Y_\tau + \int_s^\tau \left[f^1\left(u, Y_u^1, A_u^1\right) - f^2\left(u, Y_u^2, A_u^2\right)\right] du + (\delta M_T - \delta M_s), \\ &= \delta Y_\tau + \int_s^\tau \nabla_y f_u^1 \delta Y_u du + \int_s^\tau \left[\delta_a f_u^1 + \delta_2 f_u\right] du + (\delta M_T - \delta M_s), \end{aligned} \tag{4.4}$$

where $\delta M \triangleq M^1 - M^2$, and

$$\begin{aligned} \nabla_y f_u^1 &\triangleq \frac{f^1(u, Y_u^1, A_u^1) - f^1(u, Y_u^2, A_u^1)}{Y_u^1 - Y_u^2} 1_{\{Y_u^1 \neq Y_u^2\}}, \\ \delta_a f_u^1 &\triangleq f^1\left(u, Y_u^2, A_u^1\right) - f^1\left(u, Y_u^2, A_u^2\right), \\ \delta_2 f_u &\triangleq f^1\left(u, Y_u^2, A_u^2\right) - f^2\left(u, Y_u^2, A_u^2\right). \end{aligned} \tag{4.5}$$

Now, by (H1) we see that $\nabla_y f^1$ is a bounded process, and by the definition of s, τ, and the monotonicity of f in the variable l, we have $\delta_a f^1 > 0$ on the interval $[s, \tau]$. As usual, we now define $\Gamma_t = e^{\int_0^t \nabla_y f_u^1 du}$, $t \in [0, T]$, and apply Itô's formula to obtain that

$$\Gamma_s \delta Y_s - \Gamma_\tau \delta Y_\tau = \int_s^\tau \Gamma_u \left(\delta_a f_u^1 + \delta_2 f_u\right) du - \int_s^\tau \Gamma_u d(\delta M_u). \tag{4.6}$$

Therefore, if we assume that $f^1 \geq f^2$, then $\delta_2 f \geq 0$, $dP \otimes dt$-a.s., and consequently, taking conditional expectation on both sides of (4.6) we have

$$E\{\Gamma_s \delta Y_s - \Gamma_\tau \delta Y_\tau \mid \mathcal{F}_s\} = E\left\{ \int_s^\tau \Gamma_u \left(\delta_a f_u^1 + \delta_2 f_u\right) du \mid \mathcal{F}_s \right\} > 0. \tag{4.7}$$

On the other hand by the flat-off condition and Lemma 4.1-(iii), one can check that $Y_s^1 - Y_s^2 \leq X_s^1 - X_s^2$ and $Y_\tau^1 - Y_\tau^2 \geq X_\tau^1 - X_\tau^2$,

$$E\{\Gamma_s \delta Y_s - \Gamma_\tau \delta Y_\tau \mid \mathcal{F}_s\} \leq E\{\Gamma_s \delta X_s - \Gamma_\tau \delta X_\tau \mid \mathcal{F}_s\}. \tag{4.8}$$

It is now clear that if the right hand above is nonpositive, then (4.8) contradicts (4.7), and consequently one must have $P\{s < T\} = 0$. In other words, $A_t^2 \leq A_t^1 + \varepsilon$, for all $t \in [0, T]$, P-a.s. Since ε is arbitrary, this would entail that

$$A_t^1 \geq A_t^2, \quad t \in [0, T], \qquad P\text{-a.s.} \tag{4.9}$$

We summarize the arguments into the following comparison theorem.

Theorem 4.2. *Suppose that the parameters of the VRBSDEs* (4.1) (f^i, X^i), $i = 1, 2$, *satisfy (H1) and (H2). Suppose further that*

(i) $f^1(t, y, a) - f^2(t, y, a) \geq 0$, $dP \times dt$ *a.s.,*

(ii) $X_t^1 \leq X_t^2$, $0 \leq t \leq T$, *a.s.,*

(iii) $\delta X_s \leq E[e^{L(t-s)}\delta X_t \mid \mathcal{F}_s]$ *a.s. for all* s *and* t *such that* $s < t$.

Then it holds that $A_t^1 \geq A_t^2$, $t \in [0, T]$, *P-a.s.*

We remark that the assumption (iii) in Theorem 4.2 amounts to saying that the process $e^{Ls}\delta X_s$ is a submartingale. This is a merely technical condition required for the comparison theorem, and it does not add restriction on the regularity of the boundary processes X^1 and X^2 themselves, which are only required to be optional processes satisfying (H2).

Proof of Theorem 4.2. We need only show that the right hand side of (4.8) is nonpositive. To see this, note that since $\delta X_\tau \leq 0$ by assumption (ii), we derive from (4.8) that

$$\begin{aligned} E\{\Gamma_s \delta Y_s - \Gamma_\tau \delta Y_\tau \mid \mathcal{F}_s\} &\leq \Gamma_s E\left\{\delta X_s - e^{\int_s^\tau \nabla_y f_u^1 du}\delta X_\tau \mid \mathcal{F}_s\right\} \\ &\leq \Gamma_s E\left[\delta X_s - e^{L(\tau - s)}\delta X_\tau \mid \mathcal{F}_s\right] \leq 0. \end{aligned} \tag{4.10}$$

The last inequality is due to Assumption 3(iii) and optional sampling. This proves the theorem. □

We should point out that Theorem 4.2 only gives the comparison between the reflecting processes A^1 and A^2, thus it is still one step away from the comparison between Y^1 and Y^2, which is much desirable for obvious reasons. Unfortunately, the latter is not necessarily true in general, due to the "opposite" monotonicity on f^i's on the variable l. We nevertheless have the following corollaries of Theorem 4.2.

Corollary 4.3. *Suppose that all the assumptions of Theorem 4.2 hold. Assume further that* $f^1 = f^2$, *then* $Y_t^1 \leq Y_t^2$, *for all* $t \in [0, T]$, *P-a.s.*

Proof. Let $f = f^1 = f^2$. Define two random functions: $\tilde{f}^i(t, \omega, y) \triangleq f(t, \omega, y, A_t^i(\omega))$, for $(t, \omega, y) \in [0, T] \times \Omega \times \mathbb{R}$. Then, Y^1 and Y^2 can be viewed as the solutions of BSDEs

$$Y_t^i = E\left\{\xi^i + \int_t^T \tilde{f}^i\left(s, Y_u^i\right) du \mid \mathcal{F}_t\right\}, \quad t \in [0, T], \ i = 1, 2. \tag{4.11}$$

Note that $\tilde{f}^1(t, \omega, y) = f(t, \omega, y, A_t^1(\omega)) \leq f(t, \omega, y, A_t^2(\omega)) = \tilde{f}^2(t, \omega, y)$, here the inequality holds due to the fact $A^1 \geq A^2$. Since $\xi^1 = X_T^1 \leq X_T^2 = \xi^2$, by the comparison theorem of BSDEs, we have $Y_t^1 \leq Y_t^2$, for all $t \in [0, T]$, P-a.s. □

Finally, we point out that Theorem 4.2 and Corollary 4.3 provide another proof of the uniqueness of VRBSDE. Namely, $f^1 = f^2$ and $X^1 = X^2$ imply $A^1 = A^2$ and $Y^1 = Y^2$.

5. Continuous Dependence Theorems

In this section we study another important aspect of well-posedness of the VRBSDE, namely the continuous dependence of the solution on the boundary process (whence the terminal as well).

To begin with, let us denote, for any optional process X and any stopping time s and τ such that $s < \tau$,

$$m_{s,\tau}(X) = \frac{E\{X_\tau - X_s \mid \mathcal{F}_s\}}{E\{\tau - s \mid \mathcal{F}_s\}}. \tag{5.1}$$

As we pointed out in Remark 3.1, the random variable $m_{s,\tau}(X)$ in a sense measures the path regularity of the "nonmartingale" part of the boundary process X. We will show that this will be a major measurement for the "closeness" of the boundary processes, as far as the continuous dependence is concerned.

Let $\{X^n\}_{n=1}^{\infty}$ be a sequence optional processes satisfying (H2). We assume that $\{X^n\}$ converge to X_t^0 in $\mathbb{H}_T^\infty$, and that that X^0 satisfies (H2) as well.

Let (Y^n, A^n) be the solutions to the VRBSDE's with parameters (f, X^n), for $n = 0, 1, 2, \ldots$. To be more precise, for $i = 0, 1, 2, \ldots$, we have

$$\begin{gathered}
X_t^n = E\left\{\xi^n + \int_t^T f\left(s, Y_s^n, \operatorname*{Sup}_{t\le v\le s} L_v^n\right) ds \mid \mathcal{F}_t\right\}, \\
A_t^n = \sup_{0\le v\le t+} L_v^n, \\
Y_t^n = E\left\{\xi^n + \int_t^T f(s, Y_s^n, A_s^n) ds \mid \mathcal{F}_t\right\}.
\end{gathered} \tag{5.2}$$

We now follow the similar arguments as in Theorem 3.5 to obtain the following obvious estimate:

$$\left|Y_t^n - Y_t^0\right| \le \left\|\xi^n - \xi^0\right\|_\infty + T\left(L\left\|Y^n - Y^0\right\|_\infty + K\left\|A_u^n - A_u^0\right\|\right). \tag{5.3}$$

Again, we need the following lemma that provides the control of $|A_u^n - A_u^0|$.

Lemma 5.1. *Assume (H1) and (H2). Then for all $t \in [0, T]$, it holds that*

$$\left|A_t^n - A_t^0\right| \le \sup_{s\in[0,T]} \operatorname*{ess\,sup}_{\tau > s} \frac{1}{k}\left|m_{s,\tau}^n - m_{s,\tau}^0\right| + \frac{L}{k}\left\|Y^n - Y^0\right\|_\infty, \tag{5.4}$$

where $m^n = m(X^n)$, for $n = 0, 1, 2, \ldots$.

Proof. The proof is very similar to that of Lemma 3.3. Let $l^n_{s,\tau}$, $n = 0, 1, 2, \dots$ be the $\mathcal{F}_s$ random variables such that

$$E\{X^n_s - X^n_\tau \mid \mathcal{F}_s\} = E\left\{\int_s^\tau f(u, Y^n_u, l^n_{s,\tau})du \mid \mathcal{F}_s\right\}. \tag{5.5}$$

Then

$$E\left\{\int_s^\tau f(u, Y^n_u, l^n_{s,\tau}) - f\left(u, Y^0_u, l^0_{s,\tau}\right)du \mid \mathcal{F}_s\right\} = E\{X^n_s - X^n_\tau \mid \mathcal{F}_s\} - E\left\{X^0_s - X^0_\tau \mid \mathcal{F}_s\right\}. \tag{5.6}$$

Then on the set $D^\tau_s = \{l^n_{s,\tau} < l^0_{s,\tau}\} \in \mathcal{F}_s$ we have

$$\begin{aligned} &1_{D^\tau_s}\left\{E\{X^n_s - X^n_\tau \mid \mathcal{F}_s\} - E\left\{X^0_s - X^0_\tau \mid \mathcal{F}_s\right\}\right\} \\ &\quad = E\left\{1_{D^\tau_s}\int_s^\tau \left[f(u, Y^n_u, l^n_{s,\tau}) - f\left(s, Y^0_u, l^n_{s,\tau}\right) + f\left(s, Y^0_u, l^n_{s,\tau}\right) - f\left(u, Y^0_u, l^0_{s,\tau}\right)\right]du \mid \mathcal{F}_s\right\}. \end{aligned} \tag{5.7}$$

Since $f(s, Y^0_u, l^n_{s,\tau}) > f(u, Y^0_u, l^0_{s,\tau})$ on D^τ_s, we have by (H1) that $f(s, Y^0_u, l^n_{s,\tau}) - f(u, Y^0_u, l^0_{s,\tau}) \geq k|l^n_{s,\tau} - l^0_{s,\tau}|$ on D^τ_s and hence

$$\begin{aligned} 1_{D^\tau_s}k\left|l^n_{s,\tau} - l^0_{s,\tau}\right|E\{\tau - s \mid \mathcal{F}_s\} &\leq 1_{D^\tau_s}\left\{E\{X^n_s - X^n_\tau \mid \mathcal{F}_s\} - E\left\{X^0_s - X^0_\tau \mid \mathcal{F}_s\right\}\right\} \\ &\quad + 1_{D^\tau_s}E\left\{\int_s^\tau L\left|Y^n_u - Y^0_u\right|du \mid \mathcal{F}_s\right\}. \end{aligned} \tag{5.8}$$

We thus conclude that

$$\left|l^n_{s,\tau} - l^0_{s,\tau}\right| \leq \frac{1}{k}\left|m^n_{s,\tau} - m^0_{s,\tau}\right| + \frac{L}{k}\left\|Y^n - Y^0\right\|_\infty, \qquad P\text{-a.s. on } D^\tau_s. \tag{5.9}$$

A similar argument also shows that (5.9) holds on $(D^\tau_s)^c$. Hence (5.9) holds almost surely.

Finally, using the facts that $|L^n_s - L^0_s| = |\operatorname{ess\,inf}_{\tau>s} l^n_{s,\tau} - \operatorname{ess\,inf}_{\tau>s} l^0_{s,\tau}| \leq \operatorname{ess\,sup}_{\tau>s}|l^n_{s,\tau} - l^0_{s,\tau}|$, we conclude that, for any $t \in [0, T]$, it holds P-almost surely that

$$\begin{aligned} \left|A^n_t - A^0_t\right| &= \left|\sup_{0\leq s\leq t+} L^n_s - \sup_{0\leq s\leq t+} L^0_s\right| \leq \sup_{0\leq s\leq T}\left|L^n_s - L^0_s\right| \\ &\leq \sup_{0\leq s\leq T} \operatorname{ess\,sup}_{\tau>s} \frac{1}{k}\left|m^n_{s,\tau} - m^0_{s,\tau}\right| + \frac{L}{k}\left\|Y^n - Y^0\right\|_\infty, \end{aligned} \tag{5.10}$$

proving the lemma. □

Combining (5.3) and Lemma 5.1 we have the following theorem.

Theorem 5.2. *Assume (H1) and (H2). Assume further that* $(L + K(L/k))T < 1$. *Then it holds that*

$$\left|Y_t^n - Y_t^0\right| \leq \frac{1}{1-(1+(K/k))LT}\left\{\left\|\xi^n - \xi^0\right\|_\infty + \frac{KT}{k}\left\|\sup_{s\in[0,T]} \operatorname{ess\,sup}_{\tau>s}|m_{s,\tau}^n - m_{s,\tau}^0|\right\|_\infty\right\}. \tag{5.11}$$

6. Applications of Variant Reflected BSDEs

In this section we consider some possible applications of VRBSDEs. We should note that while these problems are more or less ad hoc, we nevertheless believe that they are novel in that they cannot be solved by standard (or "classical") techniques, and the theory of Variant RBSDEs seems to provide exactly the right solution.

6.1. A Recursive Intertemporal Utility Minization Problem

As one of the main applications of the stochastic representation theorem, Bank and Riedel studied both utility maximization problems and stochastic equilibrium problems with Hindy-Huang-Kreps type of preferences (cf. [6, 9]). We will consider a slight variation of these problems, and show that the VRBSDE is the natural solution.

The main idea of Hindy-Huang-Kreps utility functional is as follows. Instead of considering utility functionals depending directly on the consumption rate, one assumes that that the utilities are derived from the current *level of satisfaction*, defined as a weighted average of the accumulated consumptions:

$$A_t = A(C)_t \triangleq \eta_t + \int_0^t \theta(t,s)dC_s, \quad t \in [0,T], \tag{6.1}$$

where $\eta : [0,T] \mapsto \mathbb{R}$ represents the exogenously given level of satisfaction at time t; $\theta : [0,T]^2 \mapsto \mathbb{R}$ are the instantaneous weights assigned to consumptions made up to time t; and $t \mapsto C_t$ is the accumulated consumption up to time t (hence $C = \{C_t : t \geq 0\}$ is an increasing process, called a *consumption plan*). The Hindy-Huang-Kreps utility is then defined by (cf. [7])

$$EU(C) \triangleq E\left\{V(C_T) + \int_0^T u(t, A(C)_t)dt\right\}, \tag{6.2}$$

here both $V(\cdot)$ and $u(t,\cdot)$ are concave and increasing (utility) functions.

It is now natural to extend the problem to the recursive utility setting. In fact, in [9] it was indicated that, following the similar argument of Duffie-Epstein [5], the recursive utility

$$U_t(C) = E\left\{V_T + \int_t^T u(r, U_r(C), A(C)_r)dr \mid \mathcal{F}_t\right\}, \quad t \in [0,T] \tag{6.3}$$

is well-defined for each consumption plan C. Here $u(r,y,a) : [0,T] \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ denotes a *felicity function* which is continuous, increasing and concave in a; and $A(C)$ is the corresponding level of satisfaction defined by (6.1). In what follows we will denote $U = U(C)$ and $A = A(C)$ for simplicity.

Let us now consider the following optimization problem. Let us assume that η and θ in (6.1) are chosen so that for any consumption plan C, $A(C)$ is an increasing process, and that for a given increasing process A, there is a unique consumption plan C satisfying (6.1). Furthermore, we assume that there is an exogenous lower bound of the utility at each time t (e.g., the minimum cost to execute any consumption plan). We denote it by $\widetilde{X}$, and assume that it is an optional process of Class (D) so that $U_t \geq \widetilde{X}_t$ at each time t. Let us define the set of *admissible consumption plans*, denoted by $\mathcal{A}$, to be the set of all right-continuous increasing processes C, such that the corresponding recursive utility $U_t = U_t(C) \geq \widetilde{X}_t$, $t \in [0,T]$, P-a.s. Our goal is then to find $C^* \in \mathcal{A}$ that minimizes the expected utility (or cost)

$$EU_0 \triangleq E\left\{ \widetilde{X}_T + \int_0^T u(r, U_r, A_r)dr \right\}, \tag{6.4}$$

where $A = A(C)$ is determined by C via (6.1). A consumption plan C^* is *optimal* if the associated recursive utility U^* satisfies $EU_0^* = \min_{C \in \mathcal{A}} E\{U_0(C)\}$.

We remark that the set of admissible consumption plans $\mathcal{A}$ is not empty. In fact, let $Y_t = -U_t$, $Z_t = -\widetilde{Z}_t$, $X_t = -\widetilde{X}_t$ and define $f(t,y,l) \triangleq -u(t,-y,l)$. Then we can write the recursive utility as

$$Y_t = E\left\{ Y_T + \int_t^T f(s, Y_s, A_s)ds \mid \mathcal{F}_t \right\}, \quad t \in [0,T]. \tag{6.5}$$

Let us now assume further that the function f and the process X satisfy (H1) and (H2), then we can solve the VRBSDE with parameters (f, X), to obtain a unique solution (Y^0, A^0). Rewriting $U^0 = -Y^0$, then $(-U^0, A^0)$ satisfies the following VRBSDE:

$$\begin{gathered} U_t^0 = E\left\{ \widetilde{X}_T + \int_t^T u\left(r, U_r^0, A_r^0\right)dr \mid \mathcal{F}_t \right\}, \quad U_t^0 \geq \widetilde{X}_t, \ t \in [0,T], \\ E\int_0^T \left| U_t^0 - \widetilde{X}_t \right| dA_t^0 = 0. \end{gathered} \tag{6.6}$$

Clearly, this implies that $A^0 \in \mathcal{A}$. Furthermore, for any $\varepsilon > 0$, define $A_t^\varepsilon = A_t^0 + \varepsilon$, and let U^ε be the solution to the BSDE $U_t^\varepsilon = E\{\widetilde{X}_T + \int_t^T u(r, U_r^\varepsilon, A_r^\varepsilon)dr \mid \mathcal{F}_T\}$. By the comparison theorem of BSDEs, the utility $U_t^\varepsilon \geq U_t^0 \geq \widetilde{X}_t$, thus $A^\varepsilon \in \mathcal{A}$ as well. In other words, the set $\mathcal{A}$ contains infinitely many elements if it is not empty.

Intuitively, the best choice of the consumption plan would be the one whose corresponding level of satisfaction A is such that the associated utility U coincides with the lower boundary $\widetilde{X}$. But this amounts to saying that the boundary process X must satisfy a backward SDE, which is clearly not necessarily true in general.

The second best guess is then that the optimal level A^* allows its associated recursive utility U^* follow the VRBSDE with the exogenous lower bound $\widetilde{X}$. This turns out to be exactly the case: recall from Corollary 3.6 that the solution $U^0 = -Y^0$ of the VRBSDE (6.6) must satisfy $U_0^0 = -Y_0 = -X_0 = \widetilde{X}_0 \le U(C)$, P-a.s., for all $C \in \mathcal{A}$. Thus A^0 is indeed the optimal level of satisfaction. The following theorem is thus essentially trivial.

Theorem 6.1. *Assume that* $(-U^0, A^0)$ *is the solution to VRBSDE* (6.6)*, then for any admissible consumption plan* $C \in \mathcal{A}$*, it holds that* $U_0^0 \le U_0(C)$ *almost surely. Consequently,* A^0 *is the optimal level of satisfaction.*

Finally, we note that the Theorem 4.2 also leads to the comparison between different recursive utilities corresponding to different lower boundaries. Namely, if $\widetilde{X}^i$, $i = 1,2$ are two lower utility boundaries satisfying the conditions in Theorem 4.2, and U^i, $i = 1,2$ are the corresponding minimal recursive utilities satisfying (6.6), then $\widetilde{X}_t^1 \ge \widetilde{X}_t^2$, $0 \le t \le T$, a.s., implies that $U_t^1 \ge U_t^2$ and $A_t^1 \ge A_t^2$, $0 \le t \le T$, a.s. In particular, it holds that $E[U_0^1] \ge E[U_0^2]$.

6.2. VRBSDE and Optimal Stopping Problems

We now look at a possible extension of the so-called *multiarmed bandits* problem proposed by El Karoui and Karatzas [10]. To be more precise, let us consider a family of optimal stopping problems, parameterized by a given process $Y \in \mathbb{H}_T^\infty$:

$$V(t,l;Y) \triangleq \operatorname*{ess\,inf}_{\tau \ge t} E\left\{ \int_t^\tau f(u, Y_u, l)du + X_\tau \mid \mathcal{F}_t \right\}. \tag{6.7}$$

Here l could be either a constant or a random variable. We note that by choosing the stopping time $\tau \equiv t$, we deduce the natural upper boundary of the value function

$$V(t,l;Y) \le X_t, \quad t \in [0,\mathrm{T}], \quad P\text{-a.s.} \tag{6.8}$$

The following result characterize the relation between the VRBSDE and the value of the optimal stopping problem.

Theorem 6.2. *Assume that the parameters* (f, X) *in* (6.7) *satisfies (H1) and (H2). Then a pair of processes* (Y, A) *is a solution to the VRBSDE* (1.2) *if and only if they solve the following optimal stopping problems:*

(i) $Y_t = V(t, A_t; Y)$, $0 \le t \le T$,

(ii) $A_t = \sup_{0 \le s \le t+} L_s$ *and* $L_s = \sup\{l \in \mathbb{R} : V(s,l;Y) = X_s\}$,

(iii) *it holds that*

$$Y_t = \operatorname*{ess\,inf}_{\tau \ge t} E\left\{ \int_t^\tau f(u, Y_u, A_u)du + X_\tau \mid \mathcal{F}_t \right\}, \quad t \in [0,T]. \tag{6.9}$$

Furthermore, the stopping time $\tau_t^* = \inf\{t \le u \le T : Y_u = X_u\}$ *is optimal.*

Proof. We first asssume that (Y, A) is a solution to the variant RBSDE with parameter (f, X). Note that for any stopping time $\tau \geq t$, we have

$$Y_t = E\left\{ Y_\tau + \int_t^\tau f(u, Y_u, A_u)du \mid \mathcal{F}_t \right\}, \quad t \in [0, T]. \tag{6.10}$$

Since A is increasing, we have $A_u \geq A_t$, for all $u \in [t, \tau]$. Thus by using the monotonicity of f one has

$$Y_t \leq E\left\{ X_\tau + \int_t^\tau f(u, Y_u, A_t)du \mid \mathcal{F}_t \right\}. \tag{6.11}$$

Note that this holds for all stopping times $\tau \geq t$, we conclude that

$$Y_t \leq \operatorname*{ess\,inf}_{\tau \geq t} E\left\{ X_\tau + \int_t^\tau f(u, Y_u, A_t)du \mid \mathcal{F}_t \right\} = V(t, A_t; Y), \qquad P\text{-a.s.} \tag{6.12}$$

Next, define $\tau_t^* \stackrel{\Delta}{=} \inf\{t \leq u \leq T; \ \ Y_u = X_u\} \wedge T$. Then τ_t^* is a stopping time, and the flat-off condition implies that $E\int_t^{\tau_t^*} |Y_u - X_u| dA_u = 0$, and therefore $A_u = A_t$, for all $u \in [t, \tau_t^*)$. Consequently,

$$\begin{aligned} Y_t &= E\left\{ Y_{\tau_t^*} + \int_t^{\tau_t^*} f(u, Y_u, A_u)du \mid \mathcal{F}_t \right\} \\ &= E\left\{ X_{\tau_t^*} + \int_t^{\tau_t^*} f(u, Y_u, A_t)du \mid \mathcal{F}_t \right\} \\ &\geq V(t, A_t; Y), \quad P\text{-a.s.} \end{aligned} \tag{6.13}$$

Combining (6.12) and (6.13) we obtain (i) and (iii).

To prove (ii), we note that by the uniqueness the VRBSDE, we have the solution (Y, A) of VRBSDE must satisfy

$$\begin{aligned} X_t &= E\left\{ \xi + \int_t^T f\left(s, Y_s, \sup_{\tau \leq v \leq s} L_v \right) ds \mid \mathcal{F}_t \right\}, \\ A_t &= \sup_{0 \leq v \leq t+} L_v. \end{aligned} \tag{6.14}$$

As Bank and El Karoui have shown in [1], if we define $V(t, l; Y)$ as (6.7), then the level process L in the stochastic representation in (6.14) satisfies

$$L_t = \sup\{ l \in \mathbb{R} \mid V(t, l; Y) = X_t \}, \quad P\text{-a.s.}, \tag{6.15}$$

hence (Y, A) is the solution to (i)–(iii).

We now prove the converse, that is, any solution (Y, A) of (i)–(iii) must be the solution to the VRBSDE (1.2) with parameters (f, X). The uniqueness of the solution to problem (i)–(iii) will then follow from Theorem 3.5.

To see this, let (Y, A) be the solution to (i)–(iii). By using the Stochastic Representation of [1], one can check that

$$X_\tau = E\left\{\xi + \int_\tau^T f\left(u, Y_u, \sup_{\tau \le v \le u} L_v\right) du \mid \mathcal{F}_\tau\right\}, \tag{6.16}$$

for any stopping time $\tau \ge t$.

Next, we define $U_t \triangleq Y_t + \int_0^t f(u, Y_u, A_u)du$. Then by definition of the optimal stopping problem we see that U_t is the value function of an optimal stopping problem with payoff $H_t = \int_0^t f(u, Y_u, A_u)du + X_t$, that is, $U_t = \operatorname{ess\,inf}_{\tau \ge t} E[H_\tau \mid \mathcal{F}_t]$. It then follows that $-U$ is the Snell envelope of $-H$, that is, $-U$ is the smallest supermartingale that dominates $-H$.

Now denote

$$\tau_t^* \triangleq \inf\{t \le s \le T : -U_s = -H_s\} \wedge T = \inf\{t \le t \le T : Y_s = X_s\} \wedge T. \tag{6.17}$$

By the theory of Snell envelope (cf., e.g., [11]), we know that $-U_t = E\{-H_{\tau_t^*} \mid \mathcal{F}_t\}$, or equivalently

$$\begin{aligned} Y_t &= E\left\{\int_t^{\tau_t^*} f(u, Y_u, A_u)du + X_{\tau_t^*} \mid \mathcal{F}_t\right\} \\ &= E\left\{\xi + \int_t^{\tau_t^*} f(u, Y_u, A_u)du + \int_{\tau_t^*}^T f\left(u, Y_u, \sup_{\widehat{*}_t \le v \le u} L_u\right) du \mid \mathcal{F}_t\right\}. \end{aligned} \tag{6.18}$$

The last equality is due to the Stochastic representation (6.16). From definition (ii) we see that A is the running supreme of L and by assumption the mapping $l \mapsto f(u, Y_u, l)$ is decreasing, we have

$$Y_t \ge E\left\{\xi + \int_t^T f(u, Y_u, A_u)du \mid \mathcal{F}_t\right\}. \tag{6.19}$$

But on the other hand the definition (iii) implies that the reverse direction of the above inequality also holds, thus (Y, A) satisfies (1.2). Finally, following the same argument as that in Theorem 3.5 by using the definition (ii) it's easy to check that the flat-off condition holds. Namly (Y, A) is a solution to the VRBSDE (1.2). The proof is now complete. $\square$

We now consider a special case where VRBSDE is *linear*, in the sense that $f(t, y, a) = \varphi_t + \beta_t y + \gamma_t a$, where φ, β, and γ are bounded, adapted processes. In particular, let us assume that $|\beta_t|, |\varphi_t| \le L$ and $-K \le \gamma_t \le -k < 0$, for all $t \in [0, T]$, P-a.s. Here k, K, and L are some given positive constants.

Suppose that the linear VRBSDE (f, X) has a solution (Y, A). Then, we define a martingale $M_t = E\{\int_0^T f(s, Y_s, A_s)ds \mid \mathcal{F}_t\}$, $t \in [0,T]$ and write the VRBSDE as

$$Y_t = X_T + \int_t^T f(s, Y_s, A_s)ds - (M_T - M_t), \quad t \in [0,T]. \tag{6.20}$$

Next, we define $\Gamma_t \triangleq e^{\int_0^t \beta_s ds}$, and denote $\tilde{\xi}_t = \Gamma_t \xi_t$, for $\xi = X, Y, \varphi, \gamma$, respectively. An easy application of Itô's formula then leads to that

$$\widetilde{Y}_t = E\left\{\widetilde{X}_T + \int_t^T [\tilde{\varphi}_s + \tilde{\gamma}_s A_s]ds \mid \mathcal{F}_t\right\}, \quad t \in [0,T]. \tag{6.21}$$

Furthermore, one also has $\widetilde{Y}_t \le \widetilde{X}_t$, $t \in [0,T]$; and

$$E\left\{\int_0^T \left|\widetilde{Y}_t - \widetilde{X}_t\right| dA_t\right\} \le \|\Gamma\|_\infty E\left\{\int_0^T |Y_t - X_t| dA_t\right\} = 0. \tag{6.22}$$

Namely, the flat-off condition holds.

Summarizing, if we define $\widetilde{V}(t,l) \triangleq \operatorname{ess\,inf}_{\tau \ge t} E\{\int_t^\tau [\tilde{\varphi}_s + \tilde{\gamma}_s l]ds + \widetilde{X}_\tau \mid \mathcal{F}_t\}$. We then have the following corollary of Theorem 6.2.

Corollary 6.3. *The linear variant RBSDE has unique solution of the form*

$$\begin{gathered}
Y_t = \Gamma_t^{-1} \operatorname*{ess\,inf}_{\tau \ge t} E\left\{\int_t^\tau \Gamma_s \varphi_s + \Gamma_s \gamma_s A_t ds + \Gamma_\tau X_\tau \mid \mathcal{F}_t\right\}, \\
A_t = \sup_{0 \le s \le t+} L_t, \\
L_t = \sup\left\{l \mid \widetilde{V}(t,l) = \Gamma_t X_t\right\}.
\end{gathered} \tag{6.23}$$

6.3. Universal Signal for a Family of Optimal Stopping Problems

Continuing from the previous subsection, we conclude by considering the so-called *universal exercise signal* for a family of optimal stopping problems, in the spirit of the "universal exercise time" for the family of American options proposed by Bank-Föllmer [3]. To be more precise, let (Y, A) be the solution to our VRBSDE with generator f and lower bound X, consider the following family of optimal stopping problems indexed by l:

$$\min_{\tau \in \mathcal{S}[0,T]} E\left\{\int_0^\tau f(u, Y_u, l)du + X_\tau\right\}, \quad l \in \mathbb{R}. \tag{6.24}$$

A standard approach for solving such a problem could be to find the Snell envelope for each l. But this is obviously tedious, and often becomes unpractical when l ranges in a large family. Instead, in [3] it was noted that a universal exercise signal for the whole family of optimal stopping problems (6.24) could be determined by the process A, which we present in the following theorem.

Theorem 6.4. *Suppose that* (Y, A) *is a solution to the VRBSDE* (1.2). *For each* $l \in \mathbb{R}$, *define*

$$\tau_l^* \triangleq \inf\{u \geq 0 \mid A_u > l\} \wedge T. \tag{6.25}$$

Then τ_l^* *is the optimal stopping time for the problem (l) in* (6.24). *Namely, it holds that*

$$E\left\{\int_0^{\tau_l^*} f(u, Y_u, l)du + X_{\tau_l^*}\right\} = \inf_{\tau\in\mathcal{S}[0,T]} E\left\{\int_0^{\tau} f(u, Y_u, l)du + X_\tau\right\}, \quad l \in \mathbb{R}. \tag{6.26}$$

Proof. Let τ be any stopping time in $\mathcal{S}[0, T]$. By the definition of A we have

$$\begin{aligned} E\left\{\int_0^{\tau} f(u, Y_u, l)du + X_\tau\right\} &= E\left\{\int_0^{\tau} [f(u, Y_u, l) - f(u, Y_u, A_u)]du\right\} \\ &\quad + E\left\{\int_0^{\tau} f(u, Y_u, A_u)du + X_\tau\right\} = I_1 + I_2, \end{aligned} \tag{6.27}$$

where I_1 and I_2 are the two integrals, respectively. Note that we can further decompose I_1 as follows

$$\begin{aligned} I_1 &= E\left\{\int_0^{\tau} [f(u, Y_u, l) - f(u, Y_u, A_u)]du 1_{\{\tau\leq\tau_l^*\}}\right\} \\ &\quad + E\left\{\int_0^{\tau} [f(u, Y_u, l) - f(u, Y_u, A_u)]du 1_{\{\tau>\tau_l^*\}}\right\} = I_1^1 + I_1^2. \end{aligned} \tag{6.28}$$

Since on the set $\{\tau \leq \tau_l^*\}$, we have $A_u \leq l$, for all $u \in [\tau, \tau_l^*]$, almost surely. The monotonicity of f then yields that

$$\begin{aligned} I_1^1 &= E\left\{\left(\int_0^{\tau_l^*} - \int_\tau^{\tau_l^*}\right) [f(u, Y_u, l) - f(u, Y_u, A_u)]du 1_{\{\tau\leq\tau_l^*\}}\right\} \\ &\geq E\left\{\int_0^{\tau_l^*} [f(u, Y_u, l) - f(u, Y_u, A_u)]du 1_{\{\tau\leq\tau_l^*\}}\right\}. \end{aligned} \tag{6.29}$$

On the other hand, since A is an increasing process, thus $A_u \geq l$ for all $u \geq \tau_l^*$, In particular, on the set $\{\tau > \tau_l^*\}$, it must hold that $f(u, Y_u, l) - f(u, Y_u, A_u) \geq 0$ for all $u \in [\tau_l^*, \tau]$. In other words, we have

$$\begin{aligned} I_1^2 &= E\left\{\left(\int_0^{\tau_l^*} + \int_{\tau_l^*}^{\tau}\right)\left[f(u,Y_u,l) - f(u,Y_u,A_u)\right]du 1_{\{\tau>\tau_l^*\}}\right\} \\ &\geq E\left\{\int_0^{\tau_l^*}\left[f(u,Y_u,l) - f(u,Y_u,A_u)\right]du 1_{\{\tau>\tau_l^*\}}\right\} \geq 0, \end{aligned} \tag{6.30}$$

Combining (6.29) and (6.30) we obtain that

$$I_1 \geq E\left\{\int_0^{\tau_l^*}\left[f(u,Y_u,l) - f(u,Y_u,A_u)\right]du\right\}. \tag{6.31}$$

We now analyze I_2. First note that since X is the upper boundary, one must have

$$I_2 = E\left\{\int_0^{\tau} f(u,Y_u,A_u)du + X_\tau\right\} \geq E\left\{\int_0^{\tau} f(u,Y_u,A_u)du + Y_\tau\right\}. \tag{6.32}$$

But the right hand side above is equal to EY_0, since (Y, A) solve the VRBSDE (1.2), and for the same reason we can deduce (replacing τ by τ_λ^*) that

$$I_2 \geq E\left\{\int_0^{\tau} f(u,Y_u,A_u)du + Y_\tau\right\} = EY_0 = E\left\{\int_0^{\tau_l^*} f(u,Y_u,A_u)du + Y_{\tau_l^*}\right\}. \tag{6.33}$$

We now claim that, P-almost surely, τ_l^* is either a point of increase of A or $\tau_l^* = T$. Indeed, for each fixed ω, let us assume without loss of generality that $\tau_l^*(\omega) < T$. Then, we show that $A_{\tau_l^*-} < A_{\tau_l^*+\varepsilon}$ for any $\varepsilon > 0$ as long as $\tau_l^* + \varepsilon \leq T$. To see this we first recall that by definition of τ_l^*, and the fact that A is an increasing process we must have $A_u \geq l$ for all $u \in [\tau_l^*, T]$. We are to show that for any given $\varepsilon > 0$, there exists $t_0 = t_0(\varepsilon) \in [\tau_l^*, \tau_l^* + \varepsilon]$ such that $A_{t_0} > l$. In fact, if not, then $A_u = l$ for all $u \in [\tau_l^*, \tau_l^* + \varepsilon]$, and this will easily lead to a contradiction to the definition of τ_l^*. It then follows that $A_{\tau_l^*+\varepsilon} \geq A_{t_0} > l \leq A_{\tau_l^*-}$, proving the claim.

The direct consequence of the above claim is that $Y_{\tau_l^*} = X_{\tau_l^*}$, thanks to the flat-off and the terminal conditions. We then derive from (6.33) that

$$I_2 \geq E\left\{\int_0^{\tau_l^*} f(u,Y_u,A_u)du + Y_{\tau_l^*}\right\} = E\left\{\int_0^{\tau_l^*} f(u,Y_u,A_u)du + X_{\tau_l^*}\right\}. \tag{6.34}$$

This, together with (6.27) and (6.31), shows that

$$E\left\{\int_0^{\tau} f(u,Y_u,l)du + X_\tau\right\} \geq E\left\{\int_0^{\tau_l^*} f(u,Y_u,l)du + X_{\tau_l^*}\right\}. \tag{6.35}$$

Namely, τ_l^* it the optimal stopping time, proving the theorem. □

Theorem 6.4 shows that the "reflecting process" in the solution of VRBSDE can be used as a universal signal for exercise, and the optimal exercise time for each problem (l) is exactly the time when process A crosses level l. A further extension of such an idea is to consider a combination of Variant Reflected BSDE with a traditional reflecting boundary, which would have the potential to be applied to study the family of callable and convertible bonds with different interest rates. We hope to address this issue in our future publications.

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