

Research Article

Diffusion Approximations of the Geometric Markov Renewal Processes and Option Price Formulas

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We consider the geometric Markov renewal processes as a model for a security market and study this processes in a diffusion approximation scheme. Weak convergence analysis and rates of convergence of ergodic geometric Markov renewal processes in diffusion scheme are presented. We present European call option pricing formulas in the case of ergodic, double-averaged, and merged diffusion geometric Markov renewal processes.

1. Introduction

Let $N(t)$ be a standard Poisson process and $(Y_k)_{k \in \mathbb{Z}_+}$ be i.i.d. random variables which are independent of $N(t)$ and $S_0^* > 0$. The geometric compound Poisson processes

$$S_t^* = S_0^* \prod_{k=1}^{N(t)} (1 + Y_k), \quad t > 0, \quad (1.1)$$

is a trading model in many financial applications with pure jumps [1, page 214]. Motivated by the geometric compound Poisson processes (1.1), Swishchuk and Islam [2] studied the Geometric Markov renewal processes (2.5) (see Section 2) for a security market in a series scheme. The geometric Markov renewal processes (2.5) are also known as a switched-switching process. Averaging and diffusion approximation methods are important

approximation methods for a switched-switching system. Averaging schemes of the geometric Markov renewal processes (2.5) were studied in [2].

The singular perturbation technique of a reducible invertible-operator is one of the techniques for the construction of averaging and diffusion schemes for a switched-switching process. Strong ergodicity assumption for the switching process means that the singular perturbation problem has a solution with some additional nonrestrictive conditions. Averaging and diffusion approximation schemes for switched-switching processes in the form of random evolutions were studied in [3, page 157] and [1, page 41]. In this paper, we introduce diffusion approximation of the geometric Markov renewal processes. We study a discrete Markov-modulated (B, S) -security market described by a geometric Markov renewal process (GMRP). Weak convergence analysis and rates of convergence of ergodic geometric Markov renewal processes in diffusion scheme are presented. We present European call option pricing formulas in the case of ergodic, double-averaged, and merged diffusion geometric Markov renewal processes.

The paper is organized as follows. In Section 2 we review the definition of the geometric Markov renewal processes (GMRP) from [2]. Moreover we present notation and summarize results such as random evolution of GMRP, Markov renewal equation for GMRP, infinitesimal operator of GMRP, and martingale property of GMRP. In Section 3 we present diffusion approximation of GMRP in ergodic, merged, and double-averaging schemes. In Section 4 we present proofs of the above-mentioned results. Section 4 contains solution of martingale problem, weak convergence, rates of convergence for GMRP, and characterization of the limit measure. In Section 5 we present merged diffusion GMRP in the case of two ergodic classes. European call option pricing formula for ergodic, merged, and diffusion GMRP are presented in Section 6.

2. The Geometric Markov Renewal Processes (GMRP)

In this section we present the Geometric Markov renewal processes. We closely follow [2].

Let $(\Omega, \mathcal{B}, \mathcal{F}_t, \mathbb{P})$ be a standard probability space with complete filtration \mathcal{F}_t and let $(x_k)_{k \in \mathbb{Z}_+}$ be a Markov chain in the phase space (X, \mathcal{X}) with transition probability $P(x, A)$, where $x \in X, A \in \mathcal{X}$. Let $(\theta_k)_{k \in \mathbb{Z}_+}$ be a renewal process which is a sequence of independent and identically distributed (i.i.d.) random variables with a common distribution function $F(x) := \mathbb{P}\{\omega : \theta_k(\omega) \leq x\}$. The random variables $(\theta_k)_{k \in \mathbb{Z}_+}$ can be interpreted as lifetimes (operating periods, holding times, renewal periods) of a certain system in a random environment. From the renewal process $(\theta_k)_{k \in \mathbb{Z}_+}$ we can construct another renewal process $(\tau_k)_{k \in \mathbb{Z}_+}$ defined by

$$\tau_k := \sum_{n=0}^k \theta_n. \quad (2.1)$$

The random variables τ_k are called renewal times (or jump times). The process

$$v(t) := \sup\{k : \tau_k \leq t\} \quad (2.2)$$

is called the counting process.

Definition 2.1 (see [1, 4]). A homogeneous two-dimensional Markov chain $(x_n, \theta_n)_{n \in \mathbb{Z}_+}$ on the phase space $X \times \mathbb{R}_+$ is called a Markov renewal process (MRP) if its transition probabilities are given by the semi-Markov kernel

$$Q(x, A, t) = \mathbb{P}\{x_{n+1} \in A, \theta_{n+1} \leq t \mid x_n = x\}, \quad \forall x \in X, A \in \mathcal{X}, t \in \mathbb{R}_+. \quad (2.3)$$

Definition 2.2. The process

$$x(t) := x_{v(t)} \quad (2.4)$$

is called a *semi-Markov process*.

The ergodic theorem for a Markov renewal process and a semi-Markov process respectively can be found in [3, page 195], [1, page 66], and [4, page 113].

Let $(x_n, \theta_n)_{n \in \mathbb{Z}_+}$ be a Markov renewal process on the phase space $X \times \mathbb{R}_+$ with the semi-Markov kernel $Q(x, A, t)$ defined in (2.3), and let $x(t) := x_{v(t)}$ be a semi-Markov process where the counting process $v(t)$ is defined in (2.2). Let $\rho(x)$ be a bounded continuous function on X such that $\rho(x) > -1$. We define the geometric Markov renewal process (GMRP) $\{S_t\}_{t \in \mathbb{R}_+}$ as a stochastic functional S_t defined by

$$S_t := S_0 \prod_{k=1}^{v(t)} (1 + \rho(x_k)), \quad t \in \mathbb{R}_+, \quad (2.5)$$

where $S_0 > 0$ is the initial value of S_t . We call this process $(S_t)_{t \in \mathbb{R}_+}$ a geometric Markov renewal process by analogy with the geometric compound Poisson processes

$$S_t^* = S_0^* \prod_{k=1}^{N(t)} (1 + Y_k), \quad (2.6)$$

where $S_0^* > 0$, $N(t)$ is a standard Poisson process, $(Y_k)_{k \in \mathbb{Z}_+}$ are i.i.d. random variables. The geometric compound Poisson processes $\{S_t^*\}_{t \in \mathbb{R}_+}$ in (2.6) is a trading model in many financial applications as a pure jump model [5, 6]. The geometric Markov renewal processes $\{S_t\}_{t \in \mathbb{R}_+}$ in (2.5) will be our main trading model in further analysis.

Jump semi-Markov random evolutions, infinitesimal operators, and Martingale property of the GMRP were presented in [2]. For the convenience of readers we repeat them again in the following.

2.1. Jump Semi-Markov Random Evolutions

Let $C_0(\mathbb{R}_+)$ be the space of continuous functions on \mathbb{R}_+ vanishing at infinity, and let us define a family of bounded contracting operators $D(x)$ on $C_0(\mathbb{R}_+)$ as follows:

$$D(x)f(s) := f(s(1 + \rho(x))), \quad x \in X, s \in \mathbb{R}_+. \quad (2.7)$$

With these contraction operators $D(x)$ we define the following jump semi-Markov random evolution (JSMRE) $V(t)$ of the geometric Markov renewal processes $\{S_t\}_{t \in \mathbb{R}_+}$ in (2.5):

$$V(t) = \prod_{k=1}^{v(t)} D(x_k) := D(x_{v(t)}) \circ D(x_{v(t)-1}) \circ \cdots \circ D(x_1). \quad (2.8)$$

Using (2.7) we obtain from (2.8)

$$V(t)f(s) = \prod_{k=1}^{v(t)} D(x_k)f(s) = f\left(s \prod_{k=1}^{v(t)} (1 + \rho(x_k))\right) = f(S_t), \quad (2.9)$$

where S_t is defined in (2.5) and $S_0 = s$. Let $Q(x, A, t)$ be a semi-Markov kernel for Markov renewal process $(x_n; \theta_n)_{n \in \mathbb{Z}_+}$, that is, $Q(x, A, t) = P(x, A)G_x(t)$, where $P(x, A)$ is the transition probability of the Markov chain $(x_n)_{n \in \mathbb{Z}_+}$ and $G_x(t)$ is defined by $G_x(t) := \mathbb{P}(\theta_{n+1} \leq t \mid x_n = x)$. Let

$$u(t, x) := E_x[V(t)g(x(t))] := E[V(t)g(x(t)) \mid x(0) = x] \quad (2.10)$$

be the mean value of the semi-Markov random evolution $V(t)$ in (2.9).

The following theorem is proved in [1, page 60] and [4, page 38].

Theorem 2.3. *The mean value $u(t, x)$ in (2.10) of the semi-Markov random evolution $V(t)$ given by the solution of the following Markov renewal equation (MRE):*

$$u(t, x) - \int_0^t \int_X Q(x, dy, ds) D(y) u(t-s, y) = \bar{G}_x(t) g(x), \quad (2.11)$$

where $\bar{G}_x(t) = 1 - G_x(t)$, $G_x(t) := \mathbb{P}(\theta_{n+1} \leq t \mid x_n = x)$, $g(x)$ is a bounded and continuous function on X .

2.2. Infinitesimal Operators of the GMRP

Let

$$1\rho_T(x) := \frac{\rho(x)}{T}, \quad T > 0, \quad (2.12)$$

$$S_t^T := S_0 \prod_{k=1}^{v(tT)} (1 + \rho_T(x_k)) = S_0 \prod_{k=1}^{v(tT)} \left(1 + T^{-1}\rho(x_k)\right). \quad (2.13)$$

A detailed information about $\rho_T(x)$ and S_t^T can be found in Section 4 of [2]. It can be easily shown that

$$\ln \frac{S_t^T}{S_0} = \sum_{k=1}^{v(tT)} \ln \left(1 + \frac{\rho(x_k)}{T}\right). \quad (2.14)$$

To describe martingale properties of the GMRP $(S_t)_{t \in \mathbb{R}_+}$ in (2.5) we need to find an infinitesimal operator of the process

$$\eta(t) := \sum_{k=1}^{v(t)} \ln(1 + \rho(x_k)). \quad (2.15)$$

Let $\gamma(t) := t - \tau_{v(t)}$ and consider the process $(x(t), \gamma(t))$ on $X \times \mathbb{R}_+$. It is a Markov process with infinitesimal operator

$$\widehat{Q}f(x, t) := \frac{df}{dt} + \frac{g_x(t)}{\overline{G}_x(t)} \int_X [P(x, dy) f(y, 0) - f(x, t)], \quad (2.16)$$

where $g_x(t) := dG_x(t)/dt$, $\overline{G}_x(t) = 1 - G_x(t)$, where $f(x, t) \in C(X \times \mathbb{R}_+)$. The infinitesimal operator for the process $\ln S(t)$ has the form:

$$\widehat{A}f(z, x) = \frac{g_x(t)}{\overline{G}_x(t)} \int_X P(x, dy) [f(z + \ln(1 + \rho(y)), x) - f(z, x)], \quad (2.17)$$

where $z := \ln S_0$. The process $(\ln S(t), x(t), \gamma(t))$ is a Markov process on $\mathbb{R}_+ \times X \times \mathbb{R}_+$ with the infinitesimal operator

$$\widehat{L}f(z, x, t) = \widehat{A}f(z, x, t) + \widehat{Q}f(z, x, t), \quad (2.18)$$

where the operators \widehat{A} and \widehat{Q} are defined in (2.17) and (2.18), respectively. Thus we obtain that the process

$$\widehat{m}(t) := f(\ln S(t), x(t), \gamma(t)) - f(z, x, 0) - \int_0^t (\widehat{A} + \widehat{Q})f(\ln S(u), x(u), \gamma(u)) du \quad (2.19)$$

is an $\widehat{\mathcal{F}}_t$ -martingale, where $\widehat{\mathcal{F}}_t := \sigma(x(s), \gamma(s); 0 \leq s \leq t)$. If $x(t) := x_{v(t)}$ is a Markov process with kernel

$$Q(x, A, t) = P(x, A) \left(1 - e^{-\lambda(x)t}\right), \quad (2.20)$$

namely, $G_x(t) = 1 - e^{-\lambda(x)t}$, then $g_x(t) = \lambda(x)e^{-\lambda(x)t}$, $\widehat{G}_x(t) = e^{-\lambda(x)t}$, $g_x(t)/\widehat{G}_x(t) = \lambda(x)$, and the operator \widehat{A} in (2.17) has the form:

$$\widehat{A}f(z) = \lambda(x) \int_X P(x, dy) [f(z + \ln(1 + \rho(y))) - f(z)]. \quad (2.21)$$

The process $(\ln S(t), x(t))$ on $\mathbb{R}_+ \times X$ is a Markov process with infinitesimal operator

$$\widehat{L}f(z, x) = \widehat{A}f(z, x) + Qf(z, x), \quad (2.22)$$

where

$$Qf(z, x) = \lambda(x) \int_X P(x, dy) (f(y) - f(x)). \quad (2.23)$$

It follows that the process

$$m(t) := f(\ln S(t), x(t)) - f(z, x) - \int_0^t (\widehat{A} + Q)f(\ln S(u), x(u)) du \quad (2.24)$$

is an \mathcal{F}_t -martingale, where $\mathcal{F}_t := \sigma(x(u); 0 \leq u \leq t)$.

2.3. Martingale Property of the GMRP

Consider the geometric Markov renewal processes $(S_t)_{t \in \mathbb{R}_+}$

$$S_t = S_0 \prod_{k=1}^{v(t)} (1 + \rho(x_k)). \quad (2.25)$$

For $t \in [0, T]$ let us define

$$L_t := L_0 \prod_{k=1}^{v(t)} h(x_k), \quad EL_0 = 1, \quad (2.26)$$

where $h(x)$ is a bounded continuous function such that

$$\int_X h(y) P(x, dy) = 1, \quad \int_X h(y) P(x, dy) p(y) = 0. \quad (2.27)$$

If $EL_T = 1$, then geometric Markov renewal process S_t in (2.25) is an (\mathcal{F}_t, P^*) -martingale, where measure P^* is defined as follows:

$$\frac{dP^*}{dP} = L_T, \quad (2.28)$$

$$\mathcal{F}_t := \sigma(x(s); 0 \leq s \leq t).$$

In the discrete case we have

$$S_n = S_0 \prod_{k=1}^n (1 + \rho(x_k)). \quad (2.29)$$

Let $L_n := L_0 \prod_{k=1}^n h(x_k)$, $EL_0 = 1$, where $h(x)$ is defined in (2.27). If $EL_N = 1$, then S_n is an (\mathcal{F}_t, P^*) -martingale, where $dP^*/dP = L_N$, and $\mathcal{F}_n := \sigma(x_k; 0 \leq k \leq n)$.

3. Diffusion Approximation of the Geometric Markov Renewal Process (GMRP)

Under an additional balance condition, averaging effect leads to diffusion approximation of the geometric Markov renewal process (GMRP). In fact, we consider the counting process $v(t)$ in (2.5) in the new accelerated scale of time tT^2 , that is, $v \equiv v(tT^2)$. Due to more rapid changes of states of the system under the balance condition, the fluctuations are described by a diffusion processes.

3.1. Ergodic Diffusion Approximation

Let us suppose that balance condition is fulfilled for functional $S_t^T = S_0 \prod_{k=1}^{v(tT)} (1 + \rho_T(x_k))$:

$$\hat{\rho} = \frac{\int_X p(dx) \int_X P(x, dy) \rho(y)}{m} = 0, \quad (3.1)$$

where $p(x)$ is ergodic distribution of Markov chain $(x_k)_{k \in \mathbb{Z}_+}$. Then $\hat{S}(t) = S_0$, for all $t \in \mathbb{R}^+$. Consider S_t^T in the new scale of time tT^2 :

$$S_T(t) := S_{tT^2}^T = S_0 \prod_{k=1}^{v(tT^2)} (1 + T^{-1} \rho(x_k)). \quad (3.2)$$

Due to more rapid jumps of $v(tT^2)$ the process $S_T(t)$ will be fluctuated near the point S_0 as $T \rightarrow +\infty$. By similar arguments similar to (4.3)–(4.5) in [2], we obtain the following expression:

$$\ln \frac{S_T(t)}{S_0} = T^{-1} \sum_{k=1}^{v(tT^2)} \rho(x_k) - \frac{1}{2} T^{-2} \sum_{k=1}^{v(tT^2)} \rho^2(x_k) + T^{-2} \sum_{k=1}^{v(tT^2)} r(T^{-1} \rho(x_k)) \rho^2(x_k). \quad (3.3)$$

Algorithms of ergodic averaging give the limit result for the second term in (3.3) (see [1, page 43] and [4, page 88]):

$$\lim_{T \rightarrow +\infty} \frac{1}{2} T^{-2} \sum_{k=1}^{v(tT^2)} \rho^2(x_k) = \frac{1}{2} t \hat{\rho}_2, \quad (3.4)$$

where $\hat{\rho}_2 := \int_X p(dx) \int_X P(x, dy) \rho^2(y) / m$. Using algorithms of diffusion approximation with respect to the first term in (3.3) we obtain [4, page 88]:

$$\lim_{T \rightarrow +\infty} T^{-1} \sum_{k=1}^{v(tT^2)} \rho(x_k) = \sigma_\rho w(t), \quad (3.5)$$

where $\sigma_\rho^2 := \int_X p(dx) [1/2 \int_X P(x, dy) \rho^2(y) + \int_X P(x, dy) \rho(y) R_0 P(x, dy) \rho(y)] / m$, R_0 is a potential [3, page 68], of $(x_n)_{n \in \mathbb{Z}_+}$, $w(t)$ is a standard Wiener process. The last term in (3.3)

goes to zero as $T \rightarrow +\infty$. Let $\widehat{S}(t)$ be the limiting process for $S_T(t)$ in (3.3) as $T \rightarrow +\infty$. Taking limit on both sides of (3.3) we obtain

$$\lim_{T \rightarrow +\infty} \ln \frac{S_T(t)}{S_0} = \ln \frac{\widehat{S}(t)}{S_0} = \sigma_\rho w(t) - \frac{1}{2} t \widehat{\rho}_2, \quad (3.6)$$

where σ_ρ^2 and $\widehat{\rho}_2$ are defined in (3.4) and (3.5), respectively. From (3.6) we obtain

$$\widehat{S}(t) = S_0 e^{\sigma_\rho w(t) - (1/2)t\widehat{\rho}_2} = S_0 e^{-(1/2)t\widehat{\rho}_2} e^{\sigma_\rho w(t)}. \quad (3.7)$$

Thus, $\widehat{S}(t)$ satisfies the following stochastic differential equation (SDE):

$$d\widehat{S}(t) = \widehat{S}(t) \left[\frac{1}{2} (\sigma_\rho^2 - \widehat{\rho}_2) dt + \sigma_\rho dw(t) \right]. \quad (3.8)$$

In this way we have the following corollary.

Corollary 3.1. *The ergodic diffusion GMRP has the form*

$$\widehat{S}(t) = S_0 e^{-(1/2)t\widehat{\rho}_2} e^{\sigma_\rho w(t)}, \quad (3.9)$$

and it satisfies the following SDE:

$$\frac{d\widehat{S}(t)}{\widehat{S}(t)} = \frac{1}{2} (\sigma_\rho^2 - \widehat{\rho}_2) dt + \sigma_\rho dw(t). \quad (3.10)$$

3.2. Merged Diffusion Approximation

Let us suppose that the balance condition satisfies the following:

$$\widehat{\rho}(k) = \frac{\int_{X_k} p_k(dx) \int_{X_k} P(x, dy) \rho(y)}{m(k)} = 0, \quad (3.11)$$

for all $k = 1, 2, \dots, r$ where $(x_n)_{n \in \mathbb{Z}_+}$ is the supporting embedded Markov chain, p_k is the stationary density for the ergodic component X_k , $m(k)$ is defined in [2], and conditions of reducibility of X are fulfilled. Using the algorithms of merged averaging [1, 3, 4] we obtain from the second part of the right hand side in (3.3):

$$\lim_{T \rightarrow +\infty} \frac{1}{2} T^{-1} \sum_{k=1}^{v(tT^2)} \rho^2(x_k) = \frac{1}{2} \int_0^t \widehat{\rho}_2(\widehat{x}(s)) ds, \quad (3.12)$$

where

$$\hat{\rho}_2(k) := \frac{\int_{X_k} p_k(dx) \int_{X_k} P(x, dy) \rho^2(y)}{m(k)} \quad (3.13)$$

using the algorithm of merged diffusion approximation that [1, 3, 4] obtain from the first part of the right hand side in (3.3):

$$\lim_{T \rightarrow +\infty} T^{-1} \sum_{k=1}^{v(tT^2)} \rho(x_k) = \int_0^t \hat{\sigma}_\rho(\hat{x}(s)) d\omega(s), \quad (3.14)$$

where

$$\hat{\sigma}_\rho^2(k) := \int_{X_k} p_k(dx) \int_{X_k} P(x, dy) \rho^2(y) + \int_{X_k} P(x, dy) \rho(y) R_0 \int_{X_k} \frac{P(x, dy) \rho(y)}{m(k)}. \quad (3.15)$$

The third term in (3.3) goes to 0 as $T \rightarrow +\infty$. In this way, from (3.3) we obtain:

$$\lim_{T \rightarrow +\infty} \ln \frac{S_T(t)}{S_0} = \ln \frac{\tilde{S}(t)}{S_0} = \int_0^t \hat{\sigma}_\rho(\hat{x}(s)) d\omega(s) - \frac{1}{2} \int_0^t \hat{\rho}_2(\hat{x}(s)) ds, \quad (3.16)$$

where $\tilde{S}(t)$ is the limit $S_T(t)$ as $T \rightarrow +\infty$. From (3.16) we obtain

$$\tilde{S}(t) = S_0 e^{-\frac{1}{2} \int_0^t \hat{\rho}_2(\hat{x}(s)) ds + \int_0^t \hat{\sigma}_\rho(\hat{x}(s)) d\omega(s)}. \quad (3.17)$$

Stochastic differential equation (SDE) for $\tilde{S}(t)$ has the following form:

$$\frac{d\tilde{S}(t)}{\tilde{S}(t)} = \frac{1}{2} \left(\hat{\sigma}_\rho^2(\hat{x}(t)) - \hat{\rho}_2(\hat{x}(t)) \right) dt + \hat{\sigma}_\rho(\hat{x}(t)) d\omega(t), \quad (3.18)$$

where $\hat{x}(t)$ is a merged Markov process.

In this way we have the following corollary.

Corollary 3.2. *Merged diffusion GMRP has the form (3.17) and satisfies the SDE (3.18).*

3.3. Diffusion Approximation under Double Averaging

Let us suppose that the phase space $\hat{X} = \{1, 2, \dots, r\}$ of the merged Markov process $\hat{x}(t)$ consists of one ergodic class with stationary distributions $(\hat{p}_k; k = \{1, 2, \dots, r\})$. Let us also suppose that the balance condition is fulfilled:

$$\sum_{k=1}^r \hat{p}_k \hat{\rho}(k) = 0. \quad (3.19)$$

Then using the algorithms of diffusion approximation under double averaging (see [3, page 188], [1, page 49] and [4, page 93]) we obtain:

$$\lim_{T \rightarrow +\infty} \ln \frac{S_T(t)}{S_0} = \ln \frac{\check{S}(t)}{S_0} = \check{\sigma}_\rho w(t) - \frac{1}{2} \check{\rho}_2 t, \quad (3.20)$$

where

$$\check{\sigma}_\rho^2 := \sum_{k=1}^r \hat{p}_k \hat{\sigma}_\rho^2(k), \quad \check{\rho}_2 := \sum_{k=1}^r \hat{p}_k \hat{\rho}_2(k), \quad (3.21)$$

and $\hat{\rho}_2(k)$ and $\hat{\sigma}_\rho^2(k)$ are defined in (3.13) and (3.15), respectively. Thus, we obtain from (3.20):

$$\check{S}(t) = S_0 e^{-(1/2)\check{\rho}_2 t + \check{\sigma}_\rho w(t)}. \quad (3.22)$$

Corollary 3.3. *The diffusion GMRP under double averaging has the form*

$$\check{S}(t) = S_0 e^{-(1/2)\check{\rho}_2 t + \check{\sigma}_\rho w(t)}, \quad (3.23)$$

and satisfies the SDE

$$\frac{d\check{S}(t)}{\check{S}(t)} = \frac{1}{2} (\check{\sigma}_\rho^2 - \check{\rho}_2) dt + \check{\sigma}_\rho d\omega(t). \quad (3.24)$$

4. Proofs

In this section we present proofs of results in Section 3. All the above-mentioned results are obtained from the general results for semi-Markov random evolutions [3, 4] in series scheme. The main steps of proof are (1) weak convergence of S_t^T in Skorokhod space $D_R[0, +\infty)$ [7, page 148]; (2) solution of martingale problem for the limit process $\hat{S}(t)$; (3) characterization of the limit measure for the limit process $\hat{S}(t)$; (4) uniqueness of solution of martingale problem. We also give here the rate of convergence in the diffusion approximation scheme.

4.1. Diffusion Approximation (DA)

Let

$$G_t^T := T^{-1} \sum_{k=0}^{v(tT^2)} \rho(x_k), \quad G_n^T := G_{\tau_n T^{-1}}^T, \quad G_0^T = \ln s, \quad (4.1)$$

and the balance condition is satisfied:

$$\hat{\rho} := \int_{\mathbb{X}} p(dx) \int_{\mathbb{X}} P(x, dy) \rho(y) = 0. \quad (4.2)$$

Let us define the functions

$$\phi^T(s, x) := f(s) + T^{-1}\phi_f^1(s, x) + T^{-2}\phi_f^2(s, x), \quad (4.3)$$

where ϕ_f^1 and ϕ_f^2 are defined as follows:

$$\begin{aligned} (P - I)\phi_f^1(s, x) &= \rho(x)f(s), \\ (P - I)\phi_f^2(s, x) &= [-A(x) + \hat{A}]f(s), \end{aligned} \quad (4.4)$$

where

$$\hat{A} := \int_{\mathcal{X}} p(dx)A(x), \quad (4.5)$$

and $A(x) := [\rho^2(x)/2 + \rho(x)(R_0 - I)\rho(x)]d^2/ds^2$. From the balance condition (4.2) and equality $\Pi(\hat{A} - A(x)) = 0$ it follows that both equations in (4.3) simultaneously solvable and the solutions $\phi_f^i(s, x)$ are bounded functions, $i = 1, 2$.

We note that

$$f(S_{n+1}^T) - f(G_n^T) = \frac{1}{T}\rho(x_n)\frac{df(x_n)}{ds} \quad (4.6)$$

and define

$$\phi^T(s, x) := f(s) + T^{-1}\phi_f^1(s, x) + T^{-2}\phi_f^2(s, x), \quad (4.7)$$

where $\phi_f^1(s, x)$ and $\phi_f^2(s, x)$ are defined in (4.4) and (4.5), respectively. We note, that $G_{n+1}^T - G_n^T = T^{-1}\rho(x_n)$.

4.2. Martingale Problem for the Limiting Problem $G_0(t)$ in DA

Let us introduce the family of functions:

$$\begin{aligned} \psi^T(s, t) &:= \phi^T\left(G_{[tT^2]}^T, x_{[tT^2]}\right) - \phi^T\left(G_{[sT^2]}^T, x_{[sT^2]}\right) \\ &\quad - \sum_{j=[sT^2]}^{[tT^2]-1} E\left[\phi^T\left(G_j^T, x_{j+1}\right) - \phi^T\left(G_j^T, x_j\right) \mid \mathcal{F}_j\right], \end{aligned} \quad (4.8)$$

where ϕ^T are defined in (4.7) and G_j^T is defined by

$$G_{\tau_n/T}^T = \frac{1}{T} \sum_{k=0}^n \rho(x_k). \quad (4.9)$$

Functions $\psi^T(s, t)$ are $\mathcal{F}_{[tT^2]}$ -martingale by t . Taking into account the expression (4.6) and (4.7), we find the following expression:

$$\begin{aligned}
\psi^T(s, t) &= f\left(G_{[tT^2]}^T\right) - f\left(G_{[sT^2]}^T\right) + \epsilon \left[\phi_f^1\left(G_{[tT^2]}^T, x_{[tT^2]}\right) - \phi_f^1\left(G_{[sT^2]}^T, x_{[sT^2]}\right) \right] \\
&\quad + \epsilon^2 \left[\phi_f^2\left(G_{[tT^2]}^T, x_{[tT^2]}\right) - \phi_f^2\left(G_{[sT^2]}^T, x_{[sT^2]}\right) \right] \\
&\quad - T^{-1} \sum_{j=[sT^2]}^{[tT^2]-1} \left\{ \rho(x_j) \frac{df(G_j^T)}{dg} + E\left(\phi_f^1(G_j^T, x_{j+1}) - \phi_f^2(G_j^T, x_j) \mid \mathcal{F}_j\right) \right\} \\
&\quad - T^{-2} \sum_{j=[sT^2]}^{[tT^2]-1} \left\{ 2^{-1} \rho^2(x_j) \frac{df(G_j^T)}{dg} + \rho(x_j) E\left(\frac{d\phi_f^1(G_j^T, x_{j+1})}{dg} \mid \mathcal{F}_j\right) \right. \\
&\quad \quad \left. + E\left[\phi_f^2(G_j^T, x_{j+1}) - \phi_f^2(G_j^T, x_j) \mid \mathcal{F}_j\right] \right\} + o(T^{-2}) \\
&= f\left(G_{[tT^2]}^T\right) - f\left(G_{[sT^2]}^T\right) + \left[\phi_f^1\left(G_{[tT^2]}^T, x_{[tT^2]}\right) - \phi_f^1\left(G_{[sT^2]}^T, x_{[sT^2]}\right) \right] \\
&\quad + T^{-2} \left[\phi_f^2\left(G_{[tT^2]}^T, x_{[tT^2]}\right) - \phi_f^2\left(G_{[sT^2]}^T, x_{[sT^2]}\right) \right] - T^{-2} \sum_{j=[sT^2]}^{[tT^2]-1} \hat{A}f(G_j^T) + O(T^{-2}),
\end{aligned} \tag{4.10}$$

where $O(T^{-2})$ is the sum of terms with T^{-2} nd order. Since $\psi^T(0, t)$ is $\mathcal{F}_{[tT^2]}$ -martingale with respect to measure Q_T , generated by process $G_T(t)$ in (4.1), then for every scalar linear continuous functional η_0^s we have from (4.8)-(4.10):

$$\begin{aligned}
0 &= E^T \left[\left(\psi^T(s, t) \eta_0^s \right) \right] \\
&= E^T \left[\left(f\left(G_{[tT^2]}^T\right) - f\left(G_{[sT^2]}^T\right) - T^{-2} \sum_{j=[sT^2]}^{[tT^2]-1} \hat{A}f(G_j^T) \right) \eta_0^s \right] \\
&\quad - T^{-1} E^T \left[\left(\phi_f^1\left(G_{[tT^2]}^T, x_{[tT^2]}\right) - \phi_f^1\left(G_{[sT^2]}^T, x_{[sT^2]}\right) \right) \eta_0^s \right] \\
&\quad - T^{-2} E^T \left[\left(\phi_f^2\left(G_{[tT^2]}^T, x_{[tT^2]}\right) - \phi_f^2\left(G_{[sT^2]}^T, x_{[sT^2]}\right) \right) \eta_0^s \right] - O(T^{-2}),
\end{aligned} \tag{4.11}$$

where E^T is a mean value by measure Q_T . If the process $G_{[tT^2]}^T$ converges weakly to some process $G_0(t)$ as $T \rightarrow +\infty$, then from (4.11) we obtain

$$0 = E^T \left[\left(f(G_0(t)) - f(G_0(s)) - \int_s^t \hat{A}f(G_0(u))du \right) \right], \quad (4.12)$$

that is, the process

$$f(G_0(t)) - f(G_0(s)) - \int_s^t \hat{A}f(G_0(u))du \quad (4.13)$$

is a continuous Q_T -martingale. Since \hat{A} is the second order differential operator and coefficient σ_1^2 is positively defined, where

$$\sigma_1^2 := \int_X \pi(dx) \left[\frac{\rho^2(x)}{2} + \rho(x)R_0\rho(x) \right], \quad (4.14)$$

then the process $G_0(t)$ is a Wiener process with variance σ_1^2 in (4.14): $G_0(t) = \sigma\omega(t)$. Taking into account the renewal theorem for $v(t)$, namely, $T^{-1}v(tT^2) \rightarrow_{T \rightarrow +\infty} t/m$, and the following representation

$$G_T(t) = T^{-1} \sum_{k=0}^{v(tT^2)} \rho(x_k) = T^{-1} \sum_{k=0}^{[tT^2]} \rho(x_k) + T^{-1} \sum_{k=[tT^2]+1}^{v(tT^2)} \rho(x_k) \quad (4.15)$$

we obtain, replacing $[tT^2]$ by $v(tT^2)$, that process $G_T(t)$ converges weakly to the process $\hat{G}_0(t)$ as $T \rightarrow +\infty$, which is the solution of such martingale problem:

$$f(\hat{G}_0(t)) - f(\hat{G}_0(s)) - \int_s^t \hat{A}_0 f(\hat{G}_0(u))du \quad (4.16)$$

is a continuous Q_T -martingale, where $\hat{A}_0 := \hat{A}/m$, and \hat{A} is defined in (4.5)-(4.5).

4.3. Weak Convergence of the Processes $G_T(t)$ in DA

From the representation of the process $G_T(t)$ it follows that

$$\begin{aligned} \Delta_T(s, t) &:= |G_T(t) - G_T(s)| = \left| T^{-1} \sum_{k=v(sT^2)+1}^{v(tT^2)} \rho(x_k) \right| \\ &\leq T^{-1} \sup_x \rho(x) \left| v(tT^2) - v(sT^2) - 1 \right|. \end{aligned} \quad (4.17)$$

This representation gives the following estimation:

$$|\Delta_T(t_1, t_2)| |\Delta_T(t_2, t_3)| \leq T^{-2} \left(\sup_x \rho(x) \right)^2 \left| v(t_3 T^2) - v(t_1 T^2) \right|^2. \quad (4.18)$$

Taking into account the same reasonings as in [2] we obtain the weak convergence of the processes $G_T(t)$ in DA.

4.4. Characterization of the Limiting Measure Q for Q_T as $T \rightarrow +\infty$ in DA

From Section 4.3 (see also Section 4.1.4 of [2]) it follows that there exists a sequence T_n such that measures Q_{T_n} converge weakly to some measure Q on $D_R[0, +\infty)$ as $T \rightarrow +\infty$, where $D_R[0, +\infty)$ is the Skorokhod space [7, page 148]. This measure is the solution of such martingale problem: the following process

$$m(s, t) := f(\widehat{G}_0(t)) - f(\widehat{G}_0(s)) - \int_s^t \widehat{A}_0 f(\widehat{G}_0(u)) du \quad (4.19)$$

is a Q -martingale for all $f(g) \in C^2(R)$ and

$$Em(s, t)\eta_0^s = 0, \quad (4.20)$$

for scalar continuous bounded functional η_0^s , E is a mean value by measure Q . From (4.19) it follows that $E^T m^T(s, t)\eta_0^s = 0$, and it is necessary to show that the limiting passing in (4.1) goes to the process in (3.12) as $T \rightarrow +\infty$. From equality (4.11) we find that $\lim_{T_n \rightarrow +\infty} E^{T_n} m(s, t)\eta_0^s = Em(s, t)\eta_0^s$. Moreover, from the following expression

$$\left| E^T m(s, t)\eta_0^s - Em(s, t)\eta_0^s \right| \leq \left| (E^T - E)m(s, t)\eta_0^s \right| + E^T \left| m(s, t) - m^T(s, t) \right| \left| \eta_0^s \right| \xrightarrow{T \rightarrow +\infty} 0, \quad (4.21)$$

we obtain that there exists the measure Q on $D_R[0, +\infty)$ which solves the martingale problem for the operator \widehat{A}_0 (or, equivalently, for the process $\widehat{G}_0(t)$ in the form (4.12)). Uniqueness of the solution of the martingale problem follows from the fact that operator \widehat{A}_0 generates the unique semigroup with respects to the Wiener process with variance σ_1^2 in (4.14). As long as the semigroup is unique then the limit process $\widehat{G}_0(t)$ is unique. See [3, Chapter 1].

4.5. Calculation of the Quadratic Variation for GMRP

If $G_n^T = G_{T^{-1}\tau_n}^T$, the sequence

$$m_n^T := G_n^T - G_0^T - \sum_{k=0}^{n-1} E \left[G_{k+1}^T - G_k^T \mid \mathcal{F}_k \right], \quad G_0^T = g, \quad (4.22)$$

is \mathcal{F}_n -martingale, where $\mathcal{F}_n := \sigma\{x_k, \theta_k; 0 \leq k \leq n\}$. From the definition it follows that the characteristic $\langle m_n^T \rangle$ of the martingale m_n^T has the form

$$\langle m_n^T \rangle = \sum_{k=0}^{n-1} E \left[\left(m_{k+1}^T - m_k^T \right)^2 \mid \mathcal{F}_k \right]. \quad (4.23)$$

To calculate $\langle m_n^T \rangle$ let us represent m_n^T in (4.22) in the form of martingale-difference:

$$m_n^T = \sum_{k=0}^{n-1} \left[G_{k+1}^T - E \left(G_{k+1}^T \mid \mathcal{F}_k \right) \right]. \quad (4.24)$$

From representation

$$G_{n+1}^T - G_n^T = \frac{1}{T} \rho(x_n) \quad (4.25)$$

it follows that $E(G_{k+1}^T \mid \mathcal{F}_k) = G_k^T + T^{-1} \rho(x_k)$, that is why

$$G_{k+1}^T - E \left(G_{k+1}^T \mid \mathcal{F}_k \right) = T^{-1} (\rho(x_k) - P\rho(x_k)). \quad (4.26)$$

Since from (4.22) it follows that

$$m_{k+1}^T - m_k^T = G_{k+1}^T - E \left(G_{k+1}^T \mid \mathcal{F}_k \right) = T^{-1} (\rho(x_k) - P\rho(x_k)), \quad (4.27)$$

then substituting (4.27) in (4.23) we obtain

$$\langle m_n^T \rangle = T^{-2} \sum_{k=0}^{n-1} [(I - P)\rho(x_k)]^2. \quad (4.28)$$

In an averaging scheme (see [2]) for GMRP in the scale of time tT we obtain that $\langle m_{[tT]}^T \rangle$ goes to zero as $T \rightarrow +\infty$ in probability, which follows from (4.27):

$$\langle m_{[tT]}^T \rangle = T^{-2} \sum_{k=0}^{[tT]-1} [(I - P)\rho(x_k)]^2 \rightarrow 0 \quad \text{as } T \rightarrow +\infty \quad (4.29)$$

for all $t \in \mathbb{R}_+$. In the diffusion approximation scheme for GMRP in scale of time tT^2 from (4.27) we obtain that characteristic $\langle m_{[tT^2]}^T \rangle$ does not go to zero as $T \rightarrow +\infty$ since

$$\langle m_{[tT^2]}^T \rangle = T^{-2} \sum_{k=0}^{[tT^2]-1} [(I - P)\rho(x_k)]^2 \rightarrow t\sigma_1^2, \quad (4.30)$$

where $\sigma_1^2 := \int_X \pi(dx) [(I - P)\rho(x)]^2$.

4.6. Rates of Convergence for GMRP

Consider the representation (4.22) for martingale m_n^T . It follows that

$$G_n^T = g + m_n^T + \sum_{k=0}^{n-1} E[G_{k+1}^T - G_k^T | \mathcal{F}_k]. \quad (4.31)$$

In diffusion approximation scheme for GMRP the limit for the process $G_{[tT^2]}^T$ as $T \rightarrow +\infty$ will be diffusion process $\widehat{S}(t)$ (see (3.10)). If $m_0(t)$ is the limiting martingale for $m_{[tT^2]}^T$ in (4.22) as $T \rightarrow +\infty$, then from (4.31) and (3.10) we obtain

$$E[G_{[tT^2]}^T - \widehat{S}(t)] = E[m_{[tT^2]}^T - m_0(t)] + T^{-1} \sum_{k=0}^{[tT^2]-1} \rho(x_k) - \widehat{S}(t). \quad (4.32)$$

Since $E[m_{[tT^2]}^T - m_0(t)] = 0$, (because $m_{[tT^2]}^T$ and $m_0(t)$ are zero-mean martingales) then from (4.32) we obtain:

$$\left| E[G_{[tT^2]}^T - \widehat{S}(t)] \right| \leq T^{-1} \left| \sum_{k=0}^{[tT^2]-1} \rho(x_k) - \widehat{S}(t)T \right|. \quad (4.33)$$

Taking into account the balance condition $\int_X \pi(dx) \rho(x) = 0$ and the central limit theorem for a Markov chain [4, page 98], we obtain

$$\left| \sum_{k=0}^{[tT^2]-1} \rho(x_k) - \widehat{S}(t)T \right| = C_1(t_0), \quad (4.34)$$

where $C_1(t_0)$ is a constant depending on $t_0, t \in [0, t_0]$. From (4.33), (4.2), and (4.32) we obtain:

$$\left| E[G_{[tT^2]}^T - \widehat{S}(t)] \right| \leq T^{-1} C_1(t_0). \quad (4.35)$$

Thus, the rates of convergence in diffusion scheme has the order T^{-1} .

5. Merged Diffusion Geometric Markov Renewal Process in the Case of Two Ergodic Classes

5.1. Two Ergodic Classes

Let $P(x, A) := \mathbb{P}\{x_{n+1} \in A \mid x_n = x\}$ be the transition probabilities of supporting embedded reducible Markov chain $\{x_n\}_{n \geq 0}$ in the phase space X . Let us have two ergodic classes X_0 and X_1 of the phase space such that:

$$X = X_0 \cup X_1, \quad X_0 \cap X_1 = \emptyset. \quad (5.1)$$

Let $\{\widehat{X} = \{0, 1\}, \mathcal{U}\}$ be the measurable merged phase space. A stochastic kernel $P_0(x, A)$ is consistent with the splitting (5.1) in the following way:

$$P_0(x, X_k) = 1_k := \begin{cases} 1, & x \in X_k, \\ 0, & x \notin X_k, \end{cases} \quad k = 0, 1. \quad (5.2)$$

Let the supporting embedded Markov chain $(x_n)_{n \in \mathbb{Z}_+}$ with the transition probabilities $P_0(x, A)$ be uniformly ergodic in each class X_k , $k = 0, 1$ and have a stationary distribution $\pi_k(dx)$ in the classes X_k , $k = 0, 1$:

$$\pi_k(A) = \int_{X_k} \pi_k(dx) P_0(x, A), \quad A \subset X_k, \quad k = 0, 1. \quad (5.3)$$

Let the stationary escape probabilities of the embedded Markov chain $(x_n)_{n \in \mathbb{Z}_+}$ with transition probabilities $P(x, A) := \mathbb{P}\{x_{n+1} \in A \mid x_n = x\}$ be positive and sufficiently small, that is,

$$q_k(A) = \int_{X_k} \pi_k(dx) P(x, X \setminus X_k) > 0, \quad k = 0, 1. \quad (5.4)$$

Let the stationary sojourn time in the classes of states be uniformly bounded, namely,

$$0 \leq C_1 \leq m_k := \int_{X_k} \pi_k(dx) m(x) \leq C_2, \quad k = 0, 1, \quad (5.5)$$

where

$$m(x) := \int_0^\infty \overline{G}_x(t) dt. \quad (5.6)$$

5.2. Algorithms of Phase Averaging with Two Ergodic Classes

The merged Markov chain $(\widehat{x}_n)_{n \in \mathbb{Z}_+}$ in merged phase space \widehat{X} is given by matrix of transition probabilities

$$\begin{aligned} \widehat{P} &= (\widehat{p}_{kr})_{k,r=0,1}; \\ \widehat{p}_{01} &= 1 - \widehat{p}_{11} = \int_{X_1} \pi_1(dx) P(x, X_0) = 1 - \int_{X_1} \pi_1(dx) P(x, X_1); \\ \widehat{p}_{10} &= 1 - \widehat{p}_{00} = \int_{X_0} \pi_0(dx) P(x, X_1) = 1 - \int_{X_0} \pi_0(dx) P(x, X_0). \end{aligned} \quad (5.7)$$

As $\hat{p}_{kr} \neq 0$, $k = 0, 1$, then \hat{x}_n has virtual transitions. Intensities $\hat{\Lambda}_k$ of sojourn times $\hat{\theta}_k$, $k = 0, 1$, of the merged MRP are calculated as follows:

$$\hat{\Lambda}_k = \frac{1}{m_k}, \quad m_k = \int_{X_k} \pi_k(dx) m(x), \quad k = 0, 1. \quad (5.8)$$

And, finally, the merged MRP $(\hat{x}_n, \hat{\theta})_{n \in \mathbb{Z}_+}$ in the merged phase space \hat{X} is given by the stochastic matrix

$$\hat{Q}(t) = (\hat{Q}_{kr})_{k,r=0,1} := \hat{p}_{kr} (1 - e^{-\hat{\Lambda}_k t}), \quad k, r = 0, 1. \quad (5.9)$$

Hence, the initial semi-Markov system is merged to a Markov system with two classes.

5.3. Merged Diffusion Approximation in the Case of Two Ergodic Classes

The merged diffusion GMRP in the case of two ergodic classes has the form:

$$\tilde{S}(t) = S_0 e^{-(1/2) \int_0^t \hat{\rho}^2(\hat{x}(s)) ds + \int_0^t \hat{\sigma}_\rho(\hat{x}(s)) dw(s)} \quad (5.10)$$

which satisfies the stochastic differential equation (SDE):

$$\frac{d\tilde{S}(t)}{\tilde{S}(t)} = \frac{1}{2} (\hat{\sigma}_\rho^2(\hat{x}(t)) - \hat{\rho}^2(\hat{x}(t))) dt + \hat{\sigma}_\rho(\hat{x}(t)) dw(t), \quad (5.11)$$

where

$$\begin{aligned} \hat{\rho}^2(1) &:= \int_{X_1} p_1(dx) \int_{X_1} \frac{P(x, dy) \rho^2(y)}{m(1)}, \\ \hat{\rho}^2(0) &:= \int_{X_0} p_0(dx) \int_{X_0} \frac{P(x, dy) \rho^2(y)}{m(0)}, \\ \hat{\sigma}_\rho^2(1) &:= \int_{X_1} p_1(dx) \int_{X_1} P(x, dy) \rho^2(y) + \int_{X_1} P(x, dy) \rho(y) R_0 \int_{X_1} \frac{P(x, dy) \rho(y)}{m(1)}, \\ \hat{\sigma}_\rho^2(0) &:= \int_{X_0} p_0(dx) \int_{X_0} P(x, dy) \rho^2(y) + \int_{X_0} P(x, dy) \rho(y) R_0 \int_{X_0} \frac{P(x, dy) \rho(y)}{m(0)}, \end{aligned} \quad (5.12)$$

$\hat{x}(t)$ is a merged Markov process in $\hat{X} = \{0, 1\}$ with stochastic matrix $\hat{Q}(t)$ in (5.9).

6. European Call Option Pricing Formulas for Diffusion GMRP

6.1. Ergodic Geometric Markov Renewal Process

As we have seen in Section 3, an ergodic diffusion GMRP $\widehat{S}(t)$ satisfies the following SDE (see (3.10)):

$$\frac{d\widehat{S}(t)}{\widehat{S}(t)} = \frac{1}{2}(\sigma_\rho - \widehat{\rho}_2)dt + \sigma_\rho dw(t), \quad (6.1)$$

where

$$\widehat{\rho}_2 = \int_X p(dx) \int_X \frac{P(x, dy)\rho^2(y)}{m}, \quad (6.2)$$

$$\sigma_\rho^2 = \int_X p(dx) \left[\frac{1}{2} \int_X P(x, dy)\rho^2(y) + \int_X \frac{P(x, dy)\rho(y)R_0P(x, dy)\rho(y)}{m} \right]. \quad (6.3)$$

The risk-neutral measure P^* for the process in (6.1) is:

$$\frac{dP^*}{P} = \exp \left\{ -\theta t - \frac{1}{2}\theta^2 w(t) \right\}, \quad (6.4)$$

where

$$\theta = \frac{((1/2)(\sigma_\rho - \widehat{\rho}_2) - r)}{\sigma_\rho}. \quad (6.5)$$

Under P^* , the process $e^{-rt}\widehat{S}_t$ is a martingale and the process $w^*(t) = w(t) + \theta t$ is a Brownian motion. In this way, in the risk-neutral world, the process \widehat{S}_t has the following form

$$\frac{d\widehat{S}(t)}{\widehat{S}(t)} = rdt + \sigma_\rho dw^*(t). \quad (6.6)$$

Using Black-Scholes formula (see [8]) we obtain the European call option pricing formula for our model (6.6):

$$C = S_0\Phi(d_+) - Ke^{-rT}\Phi(d_-), \quad (6.7)$$

where

$$d_+ = \frac{\ln(S_0/K) + (r + (1/2)\sigma_\rho t)}{\sigma_\rho\sqrt{t}}, \quad (6.8)$$

$$d_- = \frac{\ln(S_0/K) + (r - (1/2)\sigma_\rho t)}{\sigma_\rho\sqrt{t}},$$

$\Phi(x)$ is a normal distribution and σ_ρ is defined in (6.3).

6.2. Double Averaged Diffusion GMRP

Using the similar arguments as in (6.1)–(6.7), we can get European call option pricing formula for a double averaged diffusion GMRP in (3.24):

$$\frac{d\check{S}(t)}{\check{S}(t)} = \frac{1}{2} \left(\check{\sigma}_\rho^2 - \check{\rho}_2 \right) dt + \check{\sigma}_\rho dw(t), \quad (6.9)$$

where $\check{\sigma}_\rho^2$ and $\check{\rho}_2$ are defined in (3.21), (see also (3.13)), and (3.15). Namely, the European call option pricing formula for a double averaged diffusion GMRP is:

$$C = S_0 \Phi(d_+) - K e^{-rT} \Phi(d_-), \quad (6.10)$$

where

$$\begin{aligned} d_+ &= \frac{\ln(S_0/K) + (r + (1/2)\check{\sigma}_\rho t)}{\check{\sigma}_\rho \sqrt{t}}, \\ d_- &= \frac{\ln(S_0/K) + (r - (1/2)\check{\sigma}_\rho t)}{\check{\sigma}_\rho \sqrt{t}}, \end{aligned} \quad (6.11)$$

$\Phi(x)$ is a normal distribution and $\check{\sigma}_\rho$ is defined in (3.21).

6.3. European Call Option Pricing Formula for Merged Diffusion GMRP

From Section 3.2, the merged diffusion GMRP has the following form:

$$\frac{d\tilde{S}(t)}{\tilde{S}(t)} = \frac{1}{2} \left(\hat{\sigma}_\rho^2(\hat{x}(t)) - \hat{\rho}_2(\hat{x}(t)) \right) dt + \hat{\sigma}_\rho(\hat{x}(t)) dw(t), \quad (6.12)$$

where $\hat{\sigma}_\rho^2$ and $\hat{\rho}_2$ are defined in Section 3.2 (see (3.18)). Taking into account the result on European call option pricing formula for regime-switching geometric Brownian motion (see [4, page 224, corollary]), we obtain the option pricing formula for the merged diffusion GMRP:

$$C = \int C_T^{\text{BS}} \left(\left(\frac{z}{T} \right)^{2-1}, T, S_0 \right) F_T^x(dz), \quad (6.13)$$

where C_T^{BS} is a Black-Scholes value and $F_T^x(dz)$ is a distribution of the random variable

$$z_T^x = \int_0^T \hat{\sigma}_\rho^2(\hat{x}(t)) ds, \quad (6.14)$$

where $\hat{x}(t)$ is a merged Markov process.

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