*Research Article*

# A Markov Regime-Switching Marked Point Process for Short-Rate Analysis with Credit Risk

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We investigate a Markov, regime-switching, marked point process for the short-term interest rate in a market. The intensity of the marked point process is a bounded, predictable process and is modulated by two observable factors. One is an economic factor described by a diffusion process, and another one is described by a Markov chain. The states of the chain are interpreted as different rating categories of corporate credit ratings issued by rating agencies. We consider a general pricing kernel which can explicitly price economic, market, and credit risks. It is shown that the price of a pure discount bond satisfies a system of coupled partial differential-integral equations under a risk-adjusted measure.

## 1. Introduction

Modeling the dynamics of short-term interest rates, or short rates, has long been a central issue in the theory and practice of banking and finance. In the past three decades or so, numerous quantitative models have been proposed to model short rates. Some classic models for short rates include Merton [1], Vasicek [2], Cox et al. [3], Hull and White [4], Duffie and Kan [5], and others. The common feature of these models is that short rates are modeled by continuous-time diffusion processes, where information flows described by Brownian motions frequently influence stochastic movements of short rates in small amounts. In practice, some information items, such as surprise information and extraordinary market events, may have large economic impact on short rates and cause jumps in short rates. Short rate models based on Brownian information flows may not be appropriate to describe such large movements, or jumps, in short rates. Several authors considered jump-diffusion processes, or related processes, to incorporate large jumps in short rates and developed the corresponding theoretical bond pricing models. Some examples include Ahn and Thompson

[6], Babbs and Webber [7], Backus et al. [8], Chacko [9, 10], Das [11], Das and Foresi [12], and others.

Babbs and Webber [7] and Elliott et al. [13] considered a stochastic interest rate model, where the short rate was modeled by a pure jump process with the jump intensity parameter depending on a diffusion state variable. They formulated their model in a pure exchange economy in a finite-time horizon and incorporated the impact of an observed economic factor on the jump frequency of the short-rate process. The key advantage of their model is to model jumps in short rates attributed to changes in macroeconomic factors such as inflation and economic growth. There is a strong empirical evidence for the relationships between macroeconomic factors and the term structure of interest rates, (see, e.g., Ang and Piazzesi [14], Rudebusch and Wu [15], Dewachtera and Lyrioa [16], and Hördahl et al. [17]). The 2010 European sovereign debt crisis centered on Greek government bonds has appeared in the highlights in many financial news. One of the major causes of this debt crisis is attributed to the fast economic growth in Greece since the new millennium of year 2000. The economy in Greece grew at an annual rate of 4.2% from 2000 to 2007. A strong and rapidly growing economy allowed the government of Greece to run large structural deficits, which, in turn, significantly increased the yields of Greek government bonds.

Bond ratings issued by rating agencies, such as Standard & Poor's and Moody's, are important indicators of the ability and willingness of rated entities, for example , sovereigns and corporations, to fulfill their financial obligations. It is well known that bond ratings have a direct impact on the term structure of interest rates and yield spreads. A downgrade (upgrade) in bond ratings may result in widening (narrowing) of yield spreads. Indeed, the significant impacts of credit ratings on bond yields have been discussed in Chapter 6 of the revised edition of the classic text "The Intelligent Investor" by Graham and Zweig [18], which is known as the stock market Bible. There has been a considerable amount of literature on studying the impact of bond ratings on the term structure of interest rates and the pricing of corporate bonds. Rating-based term structure models represent a popular approach to incorporate rating-related risk in modeling term structure of interest rates and pricing corporate bonds. These models are an extension to the reduced-form, or intensity-based, credit risk models pioneered by Jarrow and Turnbull [19] and Madan and Unal [20] and were studied extensively by Lando [21, 22]. The key idea of the rating-based term structure models was to incorporate the impact of transitions in bond ratings on evaluating the probability of default of a corporate bond in an intensity-based credit risk model. However, it seems that the risk due to ratings migrations was not priced explicitly in the rating-based term structure models. The pricing probability measure was supposed to be given exogenously, and it was not discussed in detail how different sources of risk, for example , market risk due to fluctuations of short rates and credit risk attributed to transitions of credit ratings, are priced explicitly in the specification of a pricing kernel. Further, a simplifying assumption for the independence between the short rate and the default event, or transitions of ratings, was imposed in the rating-based models under both the pricing and physical measures. However, in practice, the short rate and the bond ratings may not be independent of each other. Lastly, the rating-based term structure models do not seem to model explicitly the impact of macroeconomic conditions on the short rate. It seems more realistic to develop a model which can incorporate the impacts of both bond ratings and macroeconomic conditions in modeling the term structure of interest rates and pricing bonds. It is also of scientific interest to develop a finer structure of a pricing kernel with a view to pricing economic risk, interest-rate risk, and rating-related risk explicitly.

In this paper, we investigate a Markov, regime-switching, marked point process for the short-term interest rate, or the short rate, in a market. The intensity of the market point process is a bounded, predictable process and is modulated by two observable factors, one described by a diffusion process and another one described by a Markov chain. The factor described by the diffusion process is interpreted as proxies for some observed *macro-economic* factors. The states of the Markov chain are interpreted as different rating categories of corporate credit ratings issued by rating agencies. The proposed model has three major sources of risks, namely, economic, market, and credit risks. The economic risk is attributed to the uncertainty of the economic factor described by the diffusion process. The market risk is due to random fluctuations of the short rate. The credit risk is attributed to transitions of corporate credit ratings or qualities. We consider a general pricing kernel which can price explicitly the three sources of risk. We show that in the regime-switching environment attributed to transitions of credit ratings, the transformed intensity of the marked point process vanishes when the short rate leaves a predetermined bounded interval. It is also shown that the price of a corporate zero-coupon bond satisfies a system of coupled partial differential-integral equations (PDIEs) under a risk-adjusted measure.

The rest of this paper is organized as follows. The next section presents the proposed short-rate model. Section 3 describes the general pricing kernel and analyzes its properties. In Section 4, we derive the system of coupled PDIEs for the discount bond prices for different rating categories. The final section summarizes the paper.

## 2. A Short-Rate Model with Credit Risk

We consider a continuous-time economy, where economic activities take place continuously over time in a finite-time horizon $\mathcal{T} := [0, T]$, where $T \in (0, \infty)$. To model uncertainty, we consider a complete probability space $(\Omega, \mathcal{F}, P)$, where $P$ is a real-world probability measure. We assume that probability space is rich enough to model economic, market, and credit risks.

Firstly, we describe transitions of sovereign credit ratings over time by a continuous-time, finite-state, observable Markov chain $\mathbf{X} := \{\mathbf{X}(t) \mid t \in \mathcal{T}\}$ on $(\Omega, \mathcal{F}, P)$ with state space $\mathcal{S} := \{\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_N\} \subset \mathfrak{R}^N$. The states of the chain represent different rating categories of a corporate, or bond, credit rating system. Corporate credit ratings are issued by some international rating agencies such as Standard & Poor's and Moody's. These ratings are publicly accessible and may be used as forward-looking estimates of default probabilities of corporations and similar jurisdictions. Different rating agencies may adopt different rating scales. For example, the rating scales adopted by the Standard & Poor's are, from excellent to poor, "AAA", "AA+", "AA", "AA−", "A+", "A", "A−", "BBB+", "BBB", "BBB−", "BB+", "BB", "BB−", "B+", "B", "B−", "CCC+", "CCC", "CCC−", "CC", "C", and "D". Ratings lower than "BBB" are regarded as speculative. The rating scales used by the Moody's are, from excellent to poor, "Aaa", "Aa1", "Aa2", "Aa3", "A1", "A2", "A3", "Baa1", "Baa2", "Baa3", "Ba1", "Ba2", "Ba3", "B1", "B2", "B3", "Caa1", "Caa2", "Caa3", "Ca", and "C". Modeling these rating systems involves the use of a high dimensional Markov chain. One possible way to reduce the dimensionality of the chain is to group those rating scales which have frequent intertransition into single rating scales.

Now, following the convention in Elliott et al. [23], we identify, without loss of generality, the state space of the chain $\mathbf{X}$ with a finite set of standard unit vectors $\mathcal{I} := \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_N\} \in \mathfrak{R}^N$, where the $j$th component of $\mathbf{e}_i$ is the Kronecker delta $\delta_{ij}$, for each $i, j = 1, 2, \dots, N$. To describe the probability law of the chain $\mathbf{X}$ under $P$, we define an intensity

matrix $\mathbf{A} := [a_{ij}]_{i,j=1,2,\ldots,N}$ of the chain $\mathbf{X}$. For each $i, j = 1, 2, \ldots, N$ with $i \neq j$, $a_{ji}$ is the constant transition intensity of the chain from state $\mathbf{e}_i$ to state $e_j$. The transition intensities $a_{ji}, i, j = 1, 2, \ldots, N$, must satisfy the following properties:

(1) $a_{ji} \geq 0$;

(2) $\sum_{j=1}^{N} a_{ji} = 0$, so $a_{ii} \leq 0$.

From now on, we suppose that $a_{ij} > 0$, for $i \neq j$, so $a_{ii} < 0$. We further assume that $A_1 < |a_{ji}| < A_2, \forall i, j = 1, 2, \ldots, N$, for some positive constants $A_1$ and $A_2$ with $A_1 < A_2$.

Write $F^{\mathbf{X}} := \{\mathcal{F}^{\mathbf{X}}(t) \mid t \in \mathcal{T}\}$ for the $P$-completed, right-continuous filtration generated by the chain $\mathbf{X}$. With the canonical state space $\mathcal{O}$ of the chain $\mathbf{X}$, Elliott et al. [23] gave the following semimartingale dynamics for the chain $\mathbf{X}$ under $P$:

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \mathbf{A}\mathbf{X}(u)du + \mathbf{M}(t). \tag{2.1}$$

Here, $\mathbf{M} := \{\mathbf{M}(t) \mid t \in \mathcal{T}\}$ is an $\mathfrak{R}^N$-valued $(F^{\mathbf{X}}, P)$-martingale. The semimartingale dynamics of the chain $\mathbf{X}$ will be used in later developments in this paper.

For each $t \in \mathcal{T}$, let $r(t)$ be the instantaneous spot interest rate, or the short rate, at time $t$. Then we suppose that the evolution of the short rate $r := \{r(t) \mid t \in \mathcal{T}\}$ over time is governed by a marked point process as

$$r(t) = r(0) + \int_0^t \int_{\mathcal{Z}} z\gamma(du, dz). \tag{2.2}$$

Here $\gamma(\cdot, \cdot)$ is a counting measure corresponding to the marked point process $\{(T_n, Z_n) \mid n = 1, 2, \ldots\}$ with a finite state space $\mathcal{Z} := \{z_1, z_2, \ldots, z_J\}$, where $T_n$ is the $n$th jump time of the short rate and $Z_n$ is the jump size at the $n$th time epoch $T_n$. We suppose that the counting measure $\gamma(\cdot, \cdot)$ and the chain $\mathbf{X}$ do not have common jumps.

Indeed, the counting measure $\gamma(\cdot, \cdot)$ is a special case of a random measure. So, if $\delta_{(T_n, Z_n)}(dt, dz)$ is the delta function at the random point $(T_n, Z_n)$ and $I_E$ is the indicator function of an event $E$, then

$$\gamma(dt, dz) = \sum_{n \geq 1} \delta_{(T_n, Z_n)}(dt, dz) I_{\{T_n < \infty, Z_n \in \mathcal{Z}\}}. \tag{2.3}$$

Note that

$$\int_0^t \int_{\mathcal{Z}} z\gamma(du, dz) = \sum_{n \geq 1} Z_n I_{\{T_n \leq t\}}. \tag{2.4}$$

Further, for any $\mathcal{K} \subseteq \mathcal{Z}$,

$$\gamma(t, \mathcal{K}) := \gamma((0, t] \times \mathcal{K}) = \sum_{n \geq 1} I_{\{Z_n \in \mathcal{K}\}} I_{\{T_n \leq t\}}. \tag{2.5}$$

The probability law of the marked point process is specified by the intensity kernel, (compensator or dual predictable projection), $\lambda_t(dz)$ of $\gamma(dt, dz)$. We suppose that the intensity kernel $\lambda_t(dz)$ has the following form:

$$\lambda_t(dz) := \lambda(t)\Phi_t(\omega, dz). \tag{2.6}$$

Here, $\Phi_t(\omega, dz)$ is a probability transition kernel from $(\Omega \times \mathcal{T}, \mathcal{F} \otimes \mathcal{B}(\mathcal{T}))$ into $(\mathcal{Z}, 2^{\mathcal{Z}})$, where $\mathcal{B}(\mathcal{T})$ is the Borel $\sigma$-field generated by open subsets of $\mathcal{T}$ and $2^{\mathcal{Z}}$ is the power set of $\mathcal{Z}$; $\lambda(t)$ is the stochastic intensity of jump times of the short rate at time $t$. To simplify the notation, we suppress "$\omega$" and write $\Phi_t(dz)$ for $\Phi_t(\omega, dz)$ unless otherwise stated. (We can also consider the case where the jump size distribution of short rate $\Phi_t(dz)$ at time $t$ depends on the credit rating $\mathbf{X}(t)$ and the observed state of the economy $Y(t)$ at time $t$. The results derived in this paper can be easily extended to this case.) Note that the pair $(\lambda(t), \Phi_t(dz))$ is called the local characteristics of $\gamma(dt, dz)$ under $P$ with respect to a filtration to be defined later.

We suppose that this stochastic intensity depends on the level of the short rate $r(t-)$, the corporate credit rating $\mathbf{X}(t-)$ just prior to time $t$, and the observed state of the economy $Y(t)$ at time $t$. For example ,

$$\lambda(t) := \lambda(r(t-), \mathbf{X}(t-), Y(t)) = \langle \boldsymbol{\lambda}(r(t-), Y(t)), \mathbf{X}(t-) \rangle. \tag{2.7}$$

Here, $\boldsymbol{\lambda}(r(t-), Y(t)) := (\lambda(r(t-), \mathbf{e}_1, Y(t)), \lambda(r(t-), \mathbf{e}_2, Y(t)), \ldots, \lambda(r(t-), \mathbf{e}_N, Y(t)))' \in \mathfrak{R}^N$ with $\lambda(r(t-), \mathbf{e}_i, Y(t)) > 0$, for each $i = 1, 2, \ldots, N$; $\lambda(r(t-), \mathbf{e}_i, Y(t))$ is the stochastic intensity of jump times of the short rate when the corporate credit rating is in the $i$th category; the scalar product $\langle \cdot, \cdot \rangle$ selects the component of the vector $\boldsymbol{\lambda}(r(t-), Y(t))$ of stochastic intensities for different rating classes that is in force at time $t$ according to the credit rating $\mathbf{X}(t-)$. When the number of states $N$ of the chain is equal to one, the short-rate model considered here is identical to that in [13].

Let $W := \{W(t) \mid t \in \mathcal{T}\}$ be the standard Brownian motion on $(\Omega, \mathcal{F}, P)$ with respect to its $P$-completed, right-continuous, filtration $F^W := \{\mathcal{F}^W(t) \mid t \in \mathcal{T}\}$. Then we model the evolution of the economic state process $Y := \{Y(t) \mid t \in \mathcal{T}\}$ over time by the following diffusion process:

$$dY(t) = \mu(t, r(t-), Y(t))dt + \sigma(t, r(t-), Y(t))dW(t). \tag{2.8}$$

Note that $Y(t)$ may be interpreted as the logarithm of the GDP at time $t$. In general, we can consider a multidimensional diffusion process to incorporate several economic factors for modeling the short rate. However, to keep the notation and analysis simple, we consider a univariate diffusion process.

## 3. A General Pricing Kernel and Its Properties

In this section, we introduce a general pricing kernel with a view to providing a flexible way to price explicitly the economic, market, and credit risks in the short-rate model presented in the last section. The general pricing kernel is specified by the product of two density processes, one for a measure change for a jump-diffusion process and the other one for a measure change of the Markov chain. A Girsanov transform for the Markov chain is used for

the measure change of the Markov chain. We also analyze some theoretical properties of the general pricing kernel.

Firstly, we specify the information structure of the short-rate model. Recall that $F^X$ is the $P$-completed, right-continuous natural filtration generated by the chain $\mathbf{X}$. Write $F^r := \{\mathcal{F}^r(t) \mid t \in \mathcal{T}\}$ and $F^Y := \{\mathcal{F}^Y(t) \mid t \in \mathcal{T}\}$ for the $P$-completed, right-continuous natural filtrations generated by the short-rate process $r$ and the economic state process $Y$, respectively. For each $t \in \mathcal{T}$, let $\mathcal{G}(t) := \mathcal{F}^X(t) \vee \mathcal{F}^r(t) \vee \mathcal{F}^Y(t)$, the minimal $\sigma$-field generated by $\mathcal{F}^X(t)$, $\mathcal{F}^r(t)$, and $\mathcal{F}^Y(t)$. Write $G := \{\mathcal{G}(t) \mid t \in \mathcal{T}\}$. The enlarged filtration $G$ represents the flow of observable information.

We suppose that the stochastic intensity process $\lambda := \{\lambda(t) \mid t \in \mathcal{T}\}$ is $G$-predictable and satisfies

$$\lambda(t) \in [K_1, K_2], \qquad \forall\, t \in \mathcal{T},\ P\text{-a.s.}, \tag{3.1}$$

for some positive constants $K_1$ and $K_2$ with $K_1 < K_2$.

Suppose $\theta_0 := \{\theta_0(t) \mid t \in \mathcal{T}\}$ and $\theta_1 := \{\theta_1(t) \mid t \in \mathcal{T}\}$ be two real-valued, $G$-predictable stochastic processes on $(\Omega, \mathcal{F}, P)$ such that for all $t \in \mathcal{T}$,

(1) $|\theta_0(t)| < K$, $P$-a.s., for some positive constant $K$;

(2) $|\theta_1(t)| < 1$, $P$-a.s.

Note that $\gamma(\cdot, \cdot)$ is the counting measure having the $(G, P)$-local characteristics given by the pair $(\lambda(t), \Phi_t(dz))$. So the compensated version $\overline{\gamma}(\cdot, \cdot)$ of the counting measure $\gamma(\cdot, \cdot)$ is given by

$$\overline{\gamma}(dt, dz) := \gamma(dt, dz) - \lambda(t)\Phi_t(dz). \tag{3.2}$$

We suppose further that $W$, $\overline{\gamma}$, and $\mathbf{X}$ are orthogonal to each other under $P$.

Consider a $(G, P)$-exponential semimartingale $\Lambda_1 := \{\Lambda_1(t) \mid t \in \mathcal{T}\}$ defined by

$$\Lambda_1(t) := \mathcal{E}\left\{ -\int_0^\cdot \theta_0(u)\,dW(u) - \int_0^\cdot \int_{\mathcal{Z}} \theta_1(u)\overline{\gamma}(du, dz) \right\}(t), \tag{3.3}$$

where $\mathcal{E}\{\cdot\}$ is the stochastic exponential, (see [24], Theorem 13.5 and Remark 13.6 therein).

The following lemma is a slight modification of Lemma 3.2 in [13].

**Lemma 3.1.** *$\Lambda_1$ is a strictly positive supermartingale.*

*Proof.* Since $W$ is an $(F^W, P)$-standard Brownian motion and $W$ is stochastically independent with the chain $\mathbf{X}$ and the short-rate process $r$ under $P$, $W$ is a $(G, P)$-standard Brownian motion. Using Theorem 13.5 in [24], $\Lambda_1$ satisfies

$$\Lambda_1(t) = 1 - \int_0^t \Lambda_1(u)\theta_0(u)\,dW(u) - \int_0^t \int_{\mathcal{Z}} \Lambda_1(u)\theta_1(u)\overline{\gamma}(du, dz). \tag{3.4}$$

Consequently, $\Lambda_1$ is a $(G, P)$-local martingale.

Write $\Delta\gamma(u, \mathscr{Z}) := \gamma(u, \mathscr{Z}) - \gamma(u-, \mathscr{Z})$. Again, by Theorem 13.5 in [24],

$$
\begin{aligned}
\Lambda_1(t) = \exp\left( -\int_0^t \theta_0(u)dW(u) - \int_0^t \int_{\mathscr{Z}} \theta_1(u)\overline{\gamma}(du, dz) - \frac{1}{2}\int_0^t \theta_0^2(u)du \right) \\
\times \prod_{0 \le u \le t} (1 - \theta_1(u)\Delta\gamma(u, \mathscr{Z}))e^{\theta_1(u)\Delta\gamma(u,\mathscr{Z})}.
\end{aligned}
\tag{3.5}
$$

This, together with the assumption on $\theta_1$ and $\gamma(\cdot, \cdot)$, imply that $\Lambda_1(t)$ is strictly positive, $P$-a.s., for all $t \in \mathcal{T}$. Since $\Lambda_1$ is a $(G, P)$-local martingale which is bounded below by zero, it is a strictly positive $(G, P)$-supermartingale. $\square$

The following lemma follows from Lemma 3.3 in Elliott et al. [13]. We give the results without proof.

**Lemma 3.2.** *Let $\mathcal{L}^p(\Omega, \mathcal{F}, P)$ be the space of p-integrable random variables on $(\Omega, \mathcal{F}, P)$. Then, for each $t \in [0, T)$ and $p \in [0, \infty)$,*

$$
\Lambda_1(t) \in \mathcal{L}^p(\Omega, \mathcal{F}, P),
\tag{3.6}
$$

*so $\Lambda_1 := \{\Lambda_1(t) \mid t \in \mathcal{T}\}$ is a positive martingale.*

Suppose that $\mathbf{C} := \{c_{ij}\}_{i,j=1, 2,..., N}$ is a second intensity matrix of the chain $\mathbf{X}$. Then, $c_{ij}$ must satisfy the following conditions:

(1) $c_{ij} \ge 0$, for $i \ne j$;

(2) $\sum_{j=1}^N c_{ji} = 0$, so $c_{ii} \le 0$.

Again, we assume that $c_{ji} > 0$, $i \ne j$, so $c_{ii} < 0$. We suppose further that $|c_{ji}| \le C$, $\forall i, j = 1, 2, \ldots, N$, for some positive constant $C$.

We wish to introduce a new probability measure under which the chain $\mathbf{X}$ has the intensity matrix $\mathbf{C}$ via a Girsanov transform of Markov chains.

Define the following matrix:

$$
\mathbf{D} := \left[ \frac{c_{ij}}{a_{ij}} \right]_{i,j=1, 2,..., N} = [d_{ij}] \text{ say.}
\tag{3.7}
$$

Note that $a_{ij} > 0$, so $\mathbf{D}$ is well defined.

Let $\mathbf{d} := (d_{11}, d_{22}, \ldots, d_{NN})' \in \mathfrak{R}^N$. Then, we define

$$
\mathbf{D}_0 := \mathbf{D} - \mathbf{diag}(\mathbf{d}).
\tag{3.8}
$$

Here, $\mathbf{diag}(\mathbf{y})$ is a diagonal matrix with diagonal elements given by the vector $\mathbf{y}$.

Similarly, $\mathbf{A}_0$ and $\mathbf{C}_0$ are defined, respectively, as

$$
\mathbf{A}_0 := \mathbf{A} - \mathbf{diag}(\mathbf{a}), \qquad \mathbf{C}_0 := \mathbf{C} - \mathbf{diag}(\mathbf{c}),
\tag{3.9}
$$

where $\mathbf{a} := (a_{11}, a_{22}, \ldots, a_{NN})' \in \mathfrak{R}^N$ and $\mathbf{c} := (a_{11}, a_{22}, \ldots, a_{NN})' \in \mathfrak{R}^N$.

Suppose that $\mathbf{N} := \{\mathbf{N}(t) \mid t \in \mathcal{T}\}$ is a vector-valued counting process defined on $(\Omega, \mathcal{F}, \mathcal{P})$, where for each $t \in \mathcal{T}$, $\mathbf{N}(t) := (N_1(t), N_2(t), \ldots, N_N(t))' \in \mathfrak{R}^N$ and $N_j(t)$ represents the number of jumps of the chain $\mathbf{X}$ to state $\mathbf{e}_j$ up to time $t$, for each $j = 1, 2, \ldots, N$. Then,

$$\mathbf{N}(t) = \int_0^t (\mathbf{I} - \mathbf{diag}(\mathbf{X}(u-)))d\mathbf{X}(u), \tag{3.10}$$

where $\mathbf{I}$ is the $(N \times N)$-identity matrix. The integral is defined pathwisely in $\omega \in \Omega$ in the Stieltjes sense.

The following lemma is due to Remark 2.1 in Dufour and Elliott [25].

**Lemma 3.3.** *The process* $\widetilde{\mathbf{N}} := \{\widetilde{\mathbf{N}}(t) \mid t \in \mathcal{T}\}$ *defined by*

$$\widetilde{\mathbf{N}}(t) := \mathbf{N}(t) - \int_0^t \mathbf{A}_0(u)\mathbf{X}(u-)du, \qquad t \in \mathcal{T}, \tag{3.11}$$

*is an* $\mathfrak{R}^N$-*valued* $(F^{\mathbf{X}}, P)$-*martingale.*

Define a scalar-valued process $L := \{L(t) \mid t \in \mathcal{T}\}$ by

$$L(t) := \int_0^t (\mathbf{D}_0\mathbf{X}(u-) - \mathbf{1})'d\widetilde{\mathbf{N}}(u), \tag{3.12}$$

where $\mathbf{1} := (1, 1, \ldots, 1)' \in \mathfrak{R}^N$. Again, the integral is defined pathwisely in $\omega \in \Omega$ in the Stieltjes sense.

Then, we have the following lemma.

**Lemma 3.4.** *For each* $t \in \mathcal{T}$,

$$\Delta L(t) := L(t) - L(t-) > -1, \qquad P\text{-}a.s. \tag{3.13}$$

*Proof.* First, we note that

$$\Delta \widetilde{\mathbf{N}}(t) = \Delta \mathbf{N}(t) := \mathbf{N}(t) - \mathbf{N}(t-) = (\mathbf{I} - \mathbf{diag}(\mathbf{X}(t-)))\Delta \mathbf{X}(t). \tag{3.14}$$

Then,

$$\begin{aligned}
\Delta L(t) &= (\mathbf{D}_0\mathbf{X}(t-) - \mathbf{1})'\Delta \widetilde{\mathbf{N}}(t) \\
&= (\mathbf{D}_0\mathbf{X}(t-) - \mathbf{1})'(\mathbf{I} - \mathbf{diag}(\mathbf{X}(t-)))\Delta \mathbf{X}(t) \\
&= (\mathbf{D}_0\mathbf{X}(t-) - \mathbf{1})'(\mathbf{I} - \mathbf{diag}(\mathbf{X}(t-)))(\mathbf{X}(t) - \mathbf{X}(t-)).
\end{aligned} \tag{3.15}$$

If $\mathbf{X}(t) \neq \mathbf{X}(t-)$, then

$$\Delta L(t) = \sum_{i=1}^{N} (d_{ji} - 1)\langle \mathbf{X}(t), \mathbf{e}_j \rangle \langle \mathbf{X}(t-), \mathbf{e}_i \rangle = d_{j^*i^*} - 1, \qquad P\text{-a.s.,} \tag{3.16}$$

for a unique pair $(i^*, j^*) \in \{1, 2, \ldots, N\}$ with $i^* \neq j^*$.

Since $d_{ji} > 0$, for $i, j = 1, 2, \ldots, N$ with $i \neq j$, we must have

$$\Delta L(t) = d_{j^*i^*} - 1 > -1, \qquad P\text{-a.s.} \tag{3.17}$$

If $\mathbf{X}(t) = \mathbf{X}(t-)$, $\Delta L(t) = 0 > -1$, $P$-a.s. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Consider an $(F^{\mathbf{X}}, P)$-exponential semimartingale $\Lambda_2 := \{\Lambda_2(t) \mid t \in \mathcal{T}\}$ defined by

$$\Lambda_2(t) := \mathcal{E}\{L\}(t). \tag{3.18}$$

Then, by Remark 13.6 in [24],

$$\Lambda_2(t) = 1 + \int_0^t \Lambda_2(u-)dL(u) = 1 + \int_0^t \Lambda_2(u-)(\mathbf{D}_0\mathbf{X}(u-) - \mathbf{1})' d\widetilde{\mathbf{N}}(u). \tag{3.19}$$

The following lemma is due to Dufour and Elliott [25] it follows from the Doléan-Dade stochastic exponential formula, (see [24], Theorem 13.5 therein).

**Lemma 3.5.** $\Lambda_2 := \{\Lambda_2(t) \mid t \in \mathcal{T}\}$ *satisfies*

$$\Lambda_2(t) = \exp\left(-\int_0^t \mathbf{1}'(\mathbf{C}_0 - \mathbf{A}_0)\mathbf{X}(u)du\right) \prod_{0 < u \leq t} (1 + (\mathbf{D}_0\mathbf{X}(u-) - \mathbf{1})'\Delta\mathbf{N}(u)). \tag{3.20}$$

Then, we have the following result.

**Lemma 3.6.** *There exists a $\delta > 0$ such that*

$$\mathcal{E}\{L\}(T) \in L^{2+\delta}(\Omega, \mathcal{F}, P). \tag{3.21}$$

*Proof.* Applying Itô's differentiation rule on $\Lambda_2^{2+\delta}(T)$ (see [24], Theorem 12.13 therein) gives

$$
\begin{aligned}
\Lambda_2^{2+\delta}(T) &= 1 + \int_0^T (2+\delta)\Lambda_2^{1+\delta}(t-)\Lambda_2(t-)(\mathbf{D}_0\mathbf{X}(t-) - \mathbf{1})'(d\mathbf{N}(t) - \mathbf{A}_0\mathbf{X}(t-)dt) \\
&\quad + \sum_{0<t\leq T}\Big[\Lambda^{2+\delta}(t-)\big(1 + (\mathbf{D}_0\mathbf{X}(t-) - \mathbf{1})'\Delta\mathbf{N}(t)\big)^{2+\delta} - \Lambda^{2+\delta}(t-) \\
&\qquad\qquad - (2+\delta)\Lambda^{2+\delta}(t-)(\mathbf{D}_0\mathbf{X}(t-) - \mathbf{1})'\Delta\mathbf{N}(t)\Big] \\
&= 1 - \int_0^T (2+\delta)\Lambda_2^{2+\delta}(t-)\mathbf{1}'(\mathbf{C}_0 - \mathbf{A}_0)\mathbf{X}(t)dt \\
&\quad + \sum_{0<t\leq T}\Lambda^{2+\delta}(t-)\Big[\big(1 + (\mathbf{D}_0\mathbf{X}(t-) - \mathbf{1})'\Delta\mathbf{N}(t)\big)^{2+\delta} - 1\Big] \\
&= \exp\left[(2+\delta)\left(\int_0^T \mathbf{1}'(\mathbf{A}_0 - \mathbf{C}_0)\mathbf{X}(t)dt + \sum_{0<t\leq T}\ln\big(1 + (\mathbf{D}_0\mathbf{X}(t-) - \mathbf{1})'\Delta\mathbf{N}(t)\big)\right)\right].
\end{aligned}
\tag{3.22}
$$

Note that

$$
\sum_{0<t\leq T}\ln\big(1 + (\mathbf{D}_0\mathbf{X}(t-) - \mathbf{1})'\Delta\mathbf{N}(t)\big) \leq \max\{\ln(d_{j^*i^*}), 0\} \leq \max\left\{\ln\left(\frac{C}{A_1}\right),\, 0\right\} := M_1 < \infty.
\tag{3.23}
$$

Further,

$$
\begin{aligned}
\int_0^T \mathbf{1}'(\mathbf{A}_0 - \mathbf{C}_0)\mathbf{X}(t)dt &= \sum_{i=1}^N \int_0^T \mathbf{1}'(\mathbf{A}_0 - \mathbf{C}_0)\mathbf{e}_i\langle\mathbf{X}(t), \mathbf{e}_i\rangle dt \\
&= \sum_{i=1}^N \int_0^T (c_{ii} - a_{ii})\langle\mathbf{X}(t), \mathbf{e}_i\rangle dt \leq (C - A_1)T = M_2 T,
\end{aligned}
\tag{3.24}
$$

where $M_2 := C - A_1 < \infty$.

Consequently,

$$
\Lambda_2^{2+\delta}(T) \leq \exp[(2+\delta)(M_1 + M_2 T)].
\tag{3.25}
$$

Hence, the result follows.  $\square$

Consider a $G$-adapted process $\Lambda := \{\Lambda(t) \mid t \in \mathcal{T}\}$ defined by putting

$$
\Lambda(t) := \Lambda_1(t) \cdot \Lambda_2(t).
\tag{3.26}
$$

Then, we have the following lemma.

**Lemma 3.7.** *The process $\Lambda$ can be written as*

$$\Lambda(t) = \mathcal{E}\left\{L - \int_0^{\cdot} \theta_0(u)dW(u) - \int_0^{\cdot}\int_{\mathcal{Z}} \theta_1(u)\overline{\gamma}(du,dz)\right\}(t). \tag{3.27}$$

*Further, the quadratic covariation process*

$$[\Lambda_1, \ \Lambda_2](t) = 0. \tag{3.28}$$

*Proof.* From Corollary 13.8 in [24],

$$\begin{aligned}\Lambda(t) &:= \Lambda_1(t) \cdot \Lambda_2(t) \\ &= \mathcal{E}\left\{-\int_0^{\cdot} \theta_0(u)dW(u) - \int_0^{\cdot}\int_{\mathcal{Z}} \theta_1(u)\overline{\gamma}(du,dz)\right\}(t)\mathcal{E}\{L\}(t) \\ &= \mathcal{E}\left\{L - \int_0^{\cdot} \theta_0(u)dW(u) - \int_0^{\cdot}\int_{\mathcal{Z}} \theta_1(u)\overline{\gamma}(du,dz) \right. \\ &\quad \left. -\left[L, \int_0^{\cdot} \theta_0(u)dW(u) + \int_0^{\cdot}\int_{\mathcal{Z}} \theta_1(u)\overline{\gamma}(du,dz)\right]\right\}(t).\end{aligned} \tag{3.29}$$

Since the continuous martingale part of $L$ is identical to zero,

$$\left[L, \int_0^{\cdot} \theta_0(u)dW(u)\right](t) = \left\langle L^c, \int_0^{\cdot} \theta_0(u)dW(u)\right\rangle(t) = 0, \tag{3.30}$$

where $[X_1, X_2]$ is the quadratic covariation process of the two processes $X_1$ and $X_2$.

Since $L$ and $\gamma(\cdot,\cdot)$ do not have common jumps and the continuous martingale parts of both $L$ and $\int_0^{\cdot}\int_{\mathcal{Z}} \theta_1(u)\overline{\gamma}(du,dz)$ are identical to zero,

$$\left[L, \int_0^{\cdot}\int_{\mathcal{Z}} \theta_1(u)\overline{\gamma}(du,dz)\right](t) = 0. \tag{3.31}$$

Therefore, the results follow. $\square$

The following theorem is crucial in defining a risk-adjusted probability measure, or a pricing kernel, for bond valuation.

**Theorem 3.8.** *$\Lambda$ is a strictly positive, square integrable $(G, P)$-martingale.*

*Proof.* The strict positivity of $\Lambda$ follows from Lemmas 3.1 and 3.4. Lemmas 3.2 and 3.6 give the square integrability of $\Lambda$. It is not difficult to check that $\Lambda_2$ is an $(F^X, P)$-martingale (see [25]) and $\Lambda_1$ is a $(G, P)$-martingale by Lemma 3.2. Consequently, $\Lambda$ is a $(G, P)$-martingale. $\square$

We now define the risk-adjusted probability measure $P^\dagger$ by putting

$$\left.\frac{dP^\dagger}{dP}\right|_{\mathcal{G}(T)} := \Lambda(T). \tag{3.32}$$

From Lemma 3.7, $\Lambda$ is represented as the stochastic exponential generated by the martingales $L$, $\int_0^{\cdot} \theta_0(u)dW(u)$ and $\int_0^{\cdot} \int_{\mathscr{Z}} \theta_1(u)\bar{\gamma}(du,dz)$, which model the random shocks attributed to transitions of corporate credit ratings, changes in the observed economic factor, and fluctuations of the short rate, respectively. So, if the risk-adjusted probability measure $P^\dagger$ is used to specify a pricing kernel, the pricing kernel can take into account the three sources of risk, namely, the credit, economic, and market risks, in bond valuation.

By Theorem 3.8, $P^\dagger$ is equivalent to $P$ on $\mathcal{G}(T)$. Suppose that there is a money market account $B$ whose balance evolves over time as

$$B(t) = \exp\left(\int_0^t r(u)du\right), \tag{3.33}$$

so $B(t) > 0$, $P$-a.s. The account $B$ is then used as a numéraire, or a unit of account, for bond valuation.

Again, by Theorem 3.8, the random variable

$$\frac{1}{B(T)}\frac{dP^\dagger}{dP} \in \mathcal{L}^2(\Omega, \mathcal{F}, P), \tag{3.34}$$

so the market model for bond valuation considered here is viable.

Consequently, the price of any asset $V \in \mathcal{L}^2(\Omega, \mathcal{G}(T), P)$ at time $t$, denoted by $V(t)$, is evaluated as

$$V(t) = B(t)\mathrm{E}^\dagger\left[\frac{V}{B(T)} \mid \mathcal{G}(t)\right]. \tag{3.35}$$

Here, $\mathrm{E}^\dagger[\,\cdot\,\mid \mathcal{G}(t)]$ is the conditional expectation given $\mathcal{G}(t)$ under $P^\dagger$.

Indeed, Babbs and Selby [26] (see Proposition 3.2 on Page 167 therein) pointed out that to be consistent with general equilibrium, the pricing operator in the pure exchange economy must take the form $\Psi : \mathcal{L}^2(\Omega, \mathcal{G}(T), P) \to \mathfrak{R}^+$, given by

$$\Psi(V) = \mathrm{E}^\dagger\left[\frac{V}{B(T)}\right], \qquad V \in \mathcal{L}^2(\Omega, \mathcal{G}(T), P). \tag{3.36}$$

## 4. A System of Coupled PDEs for Bond Pricing

In this section, we first give the probability laws of the short-rate process, the economic factor, and the Markov chain under the risk-adjusted probability measure $P^\dagger$. Then, we derive a system of coupled partial differential equations governing the price of a pure discount bond under $P^\dagger$.

Firstly, the following theorem is a modification of Proposition 4.1 in [13].

**Theorem 4.1.** *The process $W^\dagger := \{W^\dagger(t) \mid t \in \mathcal{T}\}$ defined by*

$$W^\dagger(t) := W(t) + \int_0^t \theta_0(u)dW(u),  \tag{4.1}$$

*is a $(G, P^\dagger)$-standard Brownian motion.*

*Proof.* The proof resembles that of Proposition 4.1 in [13]. It invokes the use of Theorem 13.19 in [13], the orthogonality assumption of $L$, $\int_0^\cdot \theta_0(u)dW(u)$ and $\int_0^\cdot \int_{\mathcal{Z}} \theta_1(u)\overline{\gamma}(du, dz)$, and Lévy's characterization theorem of Brownian motions. □

The following theorem is an extension of Theorem 4.2 in [13]. The result follows from Theorem T10 in Chapter VIII of [27] and the orthogonal assumption between $L$ and $\gamma(\cdot, \cdot)$.

**Theorem 4.2.** *Let $\alpha := \{\alpha(t) \mid t \in \mathcal{T}\}$ be a process defined by*

$$\alpha(t) := \langle \boldsymbol{\alpha}, \mathbf{X}(t) \rangle,  \tag{4.2}$$

*where $\boldsymbol{\alpha} := (\alpha_1, \alpha_2, \dots, \alpha_N)' \in \mathfrak{R}^N$ and $\alpha_i > 0$, for each $i = 1, 2, \dots, N$.*
*For each $t \in \mathcal{T}$ and $j = 1, 2, \dots, J$, let*

$$\widehat{h}(t, \mathbf{X}(t-)) := \widehat{h}(t, z_j, r(t-), \mathbf{X}(t-)) = (r(t-) + z_j)^+ (\alpha(t-) - r(t-) - z_j)^+.  \tag{4.3}$$

*Suppose that*

$$h(t, \mathbf{X}(t-)) := \frac{\widehat{h}(t, z_j, r(t-), \mathbf{X}(t-))}{\sum_{j=1}^J \widehat{h}(t, z_j, r(t-), \mathbf{X}(t-))\Phi_t(z_j)}.  \tag{4.4}$$

*Define a process $\theta_1^\dagger := \{\theta_1^\dagger(t) \mid t \in \mathcal{T}\}$ by*

$$\theta_1^\dagger(t) := \theta_1^\dagger(t, z_j, \mathbf{X}(t-)) = 1 - \frac{h(t, z_j, r(t-), \mathbf{X}(t-))}{\lambda(t)}.  \tag{4.5}$$

*Then,*

(1) $\sum_{j=1}^J h(t, z_j, r(t-), \mathbf{X}(t-))\Phi_t(z_j) = 1$;

(2) $\gamma(\cdot, \cdot)$ has the $(G, P^\dagger)$-local characteristics $(1, h(t, z_j, r(t-), \mathbf{X}(t-))\Phi_t(z_j))$, so that the intensity of $\gamma(\cdot, \cdot)$ vanishes if $r(t-) + z_j \notin [0, \alpha(t-)]$, for each $j = 1, 2, \dots, J$.

Theorem 4.2 states that the intensity of the jump component of the short-rate process vanishes when $r(t) + z_j \notin [0, \alpha(t-)]$.

The following theorem gives the probability law of the chain $\mathbf{X}$ under the risk-adjusted probability $P^\dagger$. It was due to Lemma 2.3 in [25]. We cite the result without proof.

**Theorem 4.3.** *Under the risk-adjusted measure $P^\dagger$, $\mathbf{X}$ is a Markov chain with intensity matrix $\mathbf{C}$. Consequently, under $P^\dagger$,*

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \mathbf{C}\mathbf{X}(u)\,du + \mathbf{M}^{\mathbf{C}}(t), \tag{4.6}$$

*where $\mathbf{M}^{\mathbf{C}} := \{\mathbf{M}^{\mathbf{C}}(t) \mid t \in \mathcal{T}\}$ is an $\mathfrak{R}^N$-valued $(F^{\mathbf{X}}, P^\dagger)$-martingale.*

We now consider a pure discount bond maturing at a future time $T > t$ with face value equal to one. Let $P(t, T \mid \mathcal{G}(t))$ be a conditional price of the discount bond at time $t$ given $\mathcal{G}(t)$. Then,

$$P(t, T \mid \mathcal{G}(t)) = \mathrm{E}^\dagger\left[\exp\left(-\int_t^T r(u)\,du\right) \mid \mathcal{G}(t)\right]. \tag{4.7}$$

Note that $(r, Y, \mathbf{X})$ is jointly Markov with respect to $G$. Consequently, conditional on $(r(t), Y(t), \mathbf{X}(t)) = (r, y, \mathbf{x})$,

$$
\begin{aligned}
P(t, T \mid \mathcal{G}(t)) &= \mathrm{E}^\dagger\left[\exp\left(-\int_t^T r(u)\,du\right) \mid \mathcal{G}(t)\right] \\
&= \mathrm{E}^\dagger\left[\exp\left(-\int_t^T r(u)\,du\right) \mid r(t) = r, Y(t) = y, \mathbf{X}(t) = \mathbf{x}\right] \\
&= P(t, T, r, y, \mathbf{x}), \qquad P\text{-a.s..}
\end{aligned}
\tag{4.8}
$$

For each $t \in \mathcal{T}$, let

$$
\begin{aligned}
\widetilde{P}(t, T, r, y, \mathbf{x}) &:= \frac{P(t, T, r, y, \mathbf{x})}{B(t)} \\
&= \mathrm{E}^\dagger\left[\exp\left(-\int_0^T r(u)\,du\right) \mid r(t) = r, Y(t) = y, \mathbf{X}(t) = \mathbf{x}\right] \\
&= \mathrm{E}^\dagger\left[\exp\left(-\int_0^T r(u)\,du\right) \mid \mathcal{G}(t)\right].
\end{aligned}
\tag{4.9}
$$

This is the discounted, or normalized, bond price. By definition, the discounted bond price process $\widetilde{P}$ is a $(G, P^\dagger)$-martingale.

**Theorem 4.4.** *Let $P_i := P(t, T, r, y, \mathbf{e}_i)$, for each $i = 1, 2, \ldots, N$. Write*

$$\mathbf{P} := (P_1, P_2, \ldots, P_N)' \in \mathfrak{R}^N, \tag{4.10}$$

*and $\mu(t) := \mu(t, r(t-), Y(t))$ and $\sigma(t) := \sigma(t, r(t-), Y(t))$. Then, $P_i$, $i = 1, 2, \ldots, N$, satisfy the following system of coupled partial differential equations:*

$$\frac{\partial P_i}{\partial t} + (\mu(t) - \theta_0(t)\sigma(t)) \frac{\partial P_i}{\partial y} + \frac{1}{2}\sigma^2(t) \frac{\partial^2 P_i}{\partial y^2} + \langle \mathbf{P}, \mathbf{C}\mathbf{e}_i \rangle - r(t-)P_i$$

$$+ \int_{\mathcal{Z}} (P(t, T, r(t-) + z, Y(t), \mathbf{e}_i) - P(t, T, r(t-), Y(t), \mathbf{e}_i)) \times h(t, \mathbf{e}_i)\Phi_t(dz) = 0, \tag{4.11}$$

*for $i = 1, 2, \ldots, N$.*

*Proof.* Let

$$\overline{\gamma}^\dagger(dt, dz) := \gamma(dt, dz) - h(t, \mathbf{X}(t))\Phi_t(dz)dt. \tag{4.12}$$

Then, applying Itô's differentiation rule to $P(t, T, r(t), Y(t), \mathbf{X}(t))$ gives

$$P(t, T, r(t), Y(t), \mathbf{X}(t))$$

$$= P(0, T, r(0), Y(0), \mathbf{X}(0)) + \int_0^t \frac{\partial P}{\partial u} du + \int_0^t \frac{\partial P}{\partial y} dY(u) + \frac{1}{2} \int_0^t \sigma^2(u) \frac{\partial^2 P}{\partial y^2} du$$

$$+ \sum_{0 < u \leq t} (P(u, T, r(u), Y(u), \mathbf{X}(u)) - P(u, T, r(u-), Y(u), \mathbf{X}(u))) + \int_0^t \langle \mathbf{P}, d\mathbf{X}(u) \rangle$$

$$= P(0, T, r(0), Y(0), \mathbf{X}(0)) + \int_0^t \frac{\partial P}{\partial u} du + \int_0^t (\mu(u) - \theta_0(u)\sigma(u)) \frac{\partial P}{\partial y} du$$

$$+ \int_0^t \sigma(u) \frac{\partial P}{\partial y} dW^\dagger(u) + \frac{1}{2} \int_0^t \sigma^2(u) \frac{\partial^2 P}{\partial y^2} du$$

$$+ \int_0^t \int_{\mathcal{Z}} (P(u, T, r(u-) + z, Y(u), \mathbf{X}(u)) - P(u, T, r(u-), Y(u), \mathbf{X}(u))) \overline{\gamma}^\dagger(du, dz)$$

$$+ \int_0^t \int_{\mathcal{Z}} (P(u, T, r(u-) + z, Y(u), \mathbf{X}(u)) - P(u, T, r(u-), Y(u), \mathbf{X}(u)))$$

$$\times h(u, \mathbf{X}(u))\Phi_u(dz)du + \int_0^t \langle \mathbf{P}, \mathbf{C}\mathbf{X}(u) \rangle du + \int_0^t \left\langle \mathbf{P}, d\mathbf{M}^\mathbf{C}(u) \right\rangle. \tag{4.13}$$

Write $\tilde{P} := \tilde{P}(t, T, r(t), Y(t), \mathbf{X}(t))$ and $\tilde{P}_i := \tilde{P}(t, T, r(t), Y(t), \mathbf{e}_i)$, for each $i = 1, 2, \ldots, N$, and $\tilde{\mathbf{P}} := (\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_N)' \in \mathfrak{R}^N$. Again, applying Itô's differentiation rule to $B^{-1}(t) P(t, T, r(t), Y(t), \mathbf{X}(t))$ gives

$$\tilde{P}(t, T, r(t), Y(t), \mathbf{X}(t))$$

$$= \tilde{P}(0, T, r(0), Y(0), \mathbf{X}(0)) + \int_0^t \frac{\partial \tilde{P}}{\partial u} du + \int_0^t (\mu(u) - \theta_0(u)\sigma(u)) \frac{\partial \tilde{P}}{\partial y} du$$

$$+ \int_0^t \sigma(u) \frac{\partial \tilde{P}}{\partial y} dW^\dagger(u) + \frac{1}{2} \int_0^t \sigma^2(u) \frac{\partial^2 \tilde{P}}{\partial y^2} du$$

$$+ \int_0^t \int_{\mathscr{Z}} \left( \tilde{P}(u, T, r(u-) + z, Y(u), \mathbf{X}(u)) - \tilde{P}(u, T, r(u-), Y(u), \mathbf{X}(u)) \right) \times \overline{\gamma}^\dagger(du, dz)$$

$$+ \int_0^t \int_{\mathscr{Z}} \left( \tilde{P}(u, T, r(u-) + z, Y(u), \mathbf{X}(u)) - \tilde{P}(u, T, r(u-), Y(u), \mathbf{X}(u)) \right)$$

$$\times h(u, \mathbf{X}(u)) \Phi_u(dz) du + \int_0^t \left\langle \tilde{\mathbf{P}}, \mathbf{C}\mathbf{X}(u) \right\rangle du + \int_0^t \left\langle \tilde{\mathbf{P}}, d\mathbf{M}^{\mathbf{C}}(u) \right\rangle - \int_0^t r(u-) \tilde{P} du$$

$$= \tilde{P}(0, T, r(0), Y(0), \mathbf{X}(0))$$

$$+ \int_0^t \left[ \frac{\partial \tilde{P}}{\partial u} + (\mu(u) - \theta_0(u)\sigma(u)) \frac{\partial \tilde{P}}{\partial y} + \frac{1}{2} \sigma^2(u) \frac{\partial^2 \tilde{P}}{\partial y^2} + \left\langle \tilde{\mathbf{P}}, \mathbf{C}\mathbf{X}(u) \right\rangle - r(u-)\tilde{P} \right.$$

$$+ \int_{\mathscr{Z}} \left( \tilde{P}(u, T, r(u-) + z, Y(u), \mathbf{X}(u)) - \tilde{P}(u, T, r(u-), Y(u), \mathbf{X}(u)) \right)$$

$$\left. \times h(u, \mathbf{X}(u)) \Phi_u(dz) \right] du + \int_0^t \sigma(u) \frac{\partial \tilde{P}}{\partial y} dW^\dagger(u)$$

$$+ \int_0^t \int_{\mathscr{Z}} \left( \tilde{P}(u, T, r(u-) + z, Y(u), \mathbf{X}(u)) - \tilde{P}(u, T, r(u-), Y(u), \mathbf{X}(u)) \right) \overline{\gamma}^\dagger(du, dz)$$

$$+ \int_0^t \left\langle \tilde{\mathbf{P}}, d\mathbf{M}^{\mathbf{C}}(u) \right\rangle. \tag{4.14}$$

Note that $\tilde{P}$ is a $(G, P^\dagger)$-martingale, so it is a special semimartingale under $P^\dagger$. By the unique decomposition of a special semimartingale, the finite variation term of the above stochastic integral representation for $\tilde{P}$ must be indistinguishable from the zero process under $P^\dagger$. Consequently, multiplying the integrand of the finite variation term by $B(t)$ gives

$$\frac{\partial P}{\partial t} + (\mu(t) - \theta_0(t)\sigma(t)) \frac{\partial P}{\partial y} + \frac{1}{2} \sigma^2(t) \frac{\partial^2 P}{\partial y^2} + \langle \mathbf{P}, \mathbf{C}\mathbf{X}(t) \rangle - r(t-)P$$

$$+ \int_{\mathscr{Z}} (P(t, T, r(t-) + z, Y(t), \mathbf{X}(t)) - P(t, T, r(t-), Y(t), \mathbf{X}(t))) \tag{4.15}$$

$$\times h(t, \mathbf{X}(t)) \Phi_t(dz) = 0.$$

So, if $\mathbf{X}(t-) = e_i$, $i = 1, 2, \ldots, N$,

$$
\begin{aligned}
&\frac{\partial P_i}{\partial t} + \left(\mu(t) - \theta_0(t)\sigma(t)\right)\frac{\partial P_i}{\partial y} + \frac{1}{2}\sigma^2(t)\frac{\partial^2 P_i}{\partial y^2} + \langle \mathbf{P}, \mathbf{C}\mathbf{e}_i \rangle - r(t-)P_i \\
&\quad + \int_{\mathcal{Z}} \left(P(t, T, r(t-) + z, Y(t), \mathbf{e}_i) - P(t, T, r(t-), Y(t), \mathbf{e}_i)\right) \\
&\quad \times h(t, \mathbf{e}_i)\Phi_t(dz) = 0.
\end{aligned}
\tag{4.16}
$$

$\square$

## 5. Conclusion

A novel short-rate model based on a Markov, regime-switching, marked point process was introduced. This model provides the flexibility in incorporating the impacts of both an observed economic factor and credit ratings in the short-term interest rate. A diffusion process was used to model the evolution of the economic factor over time while credit ratings evolve over time according to a continuous-time, finite-state, Markov chain. A general pricing kernel was introduced to price three different sources of risk, namely, economic, market, and credit risks. We also provided an analysis for some theoretical properties of the pricing kernel. Some properties of the transformed intensity of the jump process were discussed. We also derived a system of coupled partial differential equations governing the evolution of the prices of a pure discount bond with different rating levels over time.

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