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Research Article

General Decay Stability for Stochastic Functional Differential Equations with Infinite Delay

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So far there are not many results on the stability for stochastic functional differential equations with infinite delay. The main aim of this paper is to establish some new criteria on the stability with general decay rate for stochastic functional differential equations with infinite delay. To illustrate the applications of our theories clearly, this paper also examines a scalar infinite delay stochastic functional differential equations with polynomial coefficients.

1. Introduction

Stability is one of the central problems for both deterministic and stochastic dynamic systems. Due to introduction of stochastic factors, stochastic stability mainly includes almost sure stability and the moment stability. In a series of papers (see [1–5]), Mao et al. examined the moment exponential stability and almost sure exponential stability for various stochastic systems.

In many cases we may find that the Lyapunov exponent equals zero, namely, the equation is not exponentially stable, but the solution does tend to zero asymptotically. By this phenomenon, Mao [6] considered polynomial stability of stochastic system, which shows that solution tends to zero polynomially. Then in [7], he extended these two classes of stability into the general decay stability.

In general, time delay and system uncertainty are commonly encountered and are often the source of instability (see [8]). Many studies focused on stochastic systems with delay. Especially, infinite delay systems have received the increasing attention in the recent years since they play important roles in many applied fields (cf. [7, 9–13]). Under the Lipschitz condition and the linear growth condition, Wei and Wang [14] built the existence-and-uniqueness theorem of global solutions to stochastic functional differential equations with infinite delay. There is also some other literature to consider stochastic functional differential equations with infinite delay and we here only mention [15–17].

However, to the best knowledge of the authors, there are not many results on the stability with general decay rate for stochastic functional equations with infinite delay. It is therefore interesting to consider the stability of infinite delay stochastic systems. The main aim of this paper is to establish some new criteria for pth moment stability and almost surely asymptotic stability with general decay rate of the global solution to stochastic functional differential equations with infinite delay

$$\mathrm{d}x(t) = f(t, x(t), x_t)\mathrm{d}t + g(t, x(t), x_t)\mathrm{d}w(t), \tag{1.1}$$

where $f = (f_1, \ldots, f_d)^{\mathrm{T}} : \mathbb{R}_+ \times \mathbb{R}^d \times C^b((-\infty, 0]; \mathbb{R}^d) \to \mathbb{R}^d$, and $g = [g_{ij}]_{d\times r} : \mathbb{R}_+ \times \mathbb{R}^d \times C^b((-\infty, 0]; \mathbb{R}^d) \to \mathbb{R}^{d\times r}$ are Borel measurable functionals, and $w(t)$ is an r-dimensional Brownian motion. Without the linear growth condition, we will show that (1.1) has the following properties.

(i) This equation almost surely admits a global solution on $[0, \infty)$.

(ii) There exists a pair of positive constants p and q such that this global solution has properties

$$\begin{aligned} &\limsup_{t\to\infty} \frac{\ln \mathrm{E}|x(t,\xi)|^p}{\ln \psi(t)} \leqslant -q, \\ &\limsup_{t\to\infty} \frac{\ln |x(t,\xi)|}{\ln \psi(t)} \leqslant -\frac{q}{p}, \quad \text{a.s.,} \end{aligned} \tag{1.2}$$

where $\psi(t)$ is a general decay function defined in the next section, namely, this solution is pth moment and almost surely asymptotically stable with general decay rate.

In the next section, we introduce some necessary notation and definitions. Section 3 gives the main result of this paper by establishing a new criteria for pth moment stability and almost surely asymptotic stability with general decay rate for the global solution of (1.1). To make our results more applicable, Section 4 gives the further result. To illustrate the application of our result, Section 5 considers a scalar stochastic functional differential equation with infinite delay in detail.

2. Preliminaries

Throughout this paper, unless otherwise specified, we use the following notation. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a complete probability space with the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions, that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. $w(t)$ is an r-dimensional Brownian motion defined on this probability space.

Let $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{R}_{++} = (0, +\infty)$, and $\mathbb{R}_- = (-\infty, 0]$. Let $|x|$ be the Euclidean norm of vector $x \in \mathbb{R}^n$. If A is a vector or matrix, its transpose is denoted by A^{T}. For a matrix A, its trace norm is denoted by $|A| = \sqrt{\operatorname{trace}(A^{\mathrm{T}}A)}$. Denote by $C_b = C^b(\mathbb{R}_-; \mathbb{R}^d)$ the family of all bounded continuous functions φ from $\mathbb{R}_-$ to $\mathbb{R}^d$ with the norm $\|\varphi\| = \sup_{-\infty<\theta\leqslant 0}|\varphi(\theta)|$, which forms a Banach space. In this paper, *const* always represents some positive constants whose precise value is not important. If $x(t)$ is an $\mathbb{R}^d$-valued stochastic process on $\mathbb{R}$, for any $t \geqslant 0$, define $x_t = x_t(\theta) = \{x(t+\theta) : \theta \in \mathbb{R}_-\}$. $C^2(\mathbb{R}^d, \mathbb{R})$ denotes the family of continuously twice differentiable $\mathbb{R}$-valued functions defined on $\mathbb{R}^d$. For any $V(x) \in C^2(\mathbb{R}^d, \mathbb{R}_+)$, define an operator $\mathcal{L}V : \mathbb{R}_+ \times \mathbb{R}^d \times C_b \to \mathbb{R}$ by

$$\mathcal{L}V(t,x,\varphi) = V_x(x) f(t,x,\varphi) + \frac{1}{2}\operatorname{trace}\left[g^{\mathrm{T}}(t,x,\varphi)V_{xx}(x)g(t,x,\varphi)\right], \tag{2.1}$$

where

$$V_x(x) = \left(\frac{\partial V(x)}{\partial x_1}, \frac{\partial V(x)}{\partial x_2}, \ldots, \frac{\partial V(x)}{\partial x_d}\right), \qquad V_{xx}(x) = \left[\frac{\partial^2 V(x)}{\partial x_i \partial x_j}\right]_{d\times d}. \tag{2.2}$$

If $x(t)$ is a solution of (1.1), for any $V(x) \in C^2(\mathbb{R}^d, \mathbb{R})$, applying the Itô formula yields

$$\mathrm{d}V(x(t)) = LV(x(t))\mathrm{d}t + V_x(x(t))g(t,x(t),x_t)\mathrm{d}w(t), \tag{2.3}$$

where $LV(x(t)) = \mathcal{L}V(t,x(t),x_t)$.

Let us introduce the following ψ-type function, which will be used as the decay function.

Definition 2.1. The function $\psi : \mathbb{R} \to (0,\infty)$ is said to be the ψ-type function if it satisfies the following conditions:

(i) it is continuous and nondecreasing in $\mathbb{R}$ and differentiable in $\mathbb{R}_+$,

(ii) $\psi(0) = 1$ and $\psi(\infty) = \infty$,

(iii) $\phi := \sup_{t\geqslant 0}\psi_1(t) < \infty$, where $\psi_1(t) = \psi\prime(t)/\psi(t)$,

(iv) for any $\theta \leqslant 0$ and $t \geqslant 0$, $\psi(t) \leqslant \psi(-\theta)\psi(t+\theta)$.

It is is easy to find that functions $\psi(t) = e^{\gamma t}$ and $\psi(t) = (1+t^+)^{\overline{\gamma}}$ for any $\gamma, \overline{\gamma} > 0$ are ψ-type functions.

For any $p, q \geqslant 0$ and $\varphi \in C_b$, define

$$\mathcal{T}_{p,q}(\varphi) = \int_{-\infty}^{0} \psi^q(\theta)|\varphi(\theta)|^p \mathrm{d}\theta \tag{2.4}$$

and $C(p,q) = \{\varphi \in C_b : \mathcal{T}_{p,q}(\varphi) < \infty\}$. Denote by M_0 the family of all probability measures on $\mathbb{R}_-$. For any $\mu \in M_0$ and $\varepsilon \geqslant 0$, define

$$M_\varepsilon = \left\{\mu \in M_0 : \mu_\varepsilon := \int_{-\infty}^{0} \psi^\varepsilon(-\theta)\mathrm{d}\mu(\theta) < \infty\right\}. \tag{2.5}$$

We also impose the following standard assumption on coefficients f and g.

Assumption 2.2. Let f and g satisfy the Local Lipschitz condition. That is, for every integer $n \geqslant 1$, there is $k_n > 0$ such that

$$\left|f(t,x,\varphi) - f(t,\overline{x},\overline{\varphi})\right| \vee \left|g(t,x,\varphi) - g(t,\overline{x},\overline{\varphi})\right| \leqslant k_n\left(|x-\overline{x}| + \|\varphi - \overline{\varphi}\|\right), \tag{2.6}$$

for all $t \geqslant 0$ and those $x, \overline{x} \in \mathbb{R}^n$, $\varphi, \overline{\varphi} \in C_b$ with $|x| \vee |\overline{x}| \vee \|\varphi\| \vee \|\overline{\varphi}\| \leqslant n$.

Let us present the continuous semimartingale convergence theory (cf. [18]).

Lemma 2.3. *Let $M(t)$ be a real-value local martingale with $M(0) = 0$ a.s. Let ζ be a nonnegative $\mathcal{F}_0$-measurable random variable. If $X(t)$ is a nonnegative continuous $\mathcal{F}_t$-adapted process and satisfies*

$$X(t) \leq \zeta + M(t) \quad \text{for } t \geq 0, \tag{2.7}$$

then $\mathbb{E}X(t) \leq \zeta$ and $X(t)$ is almost surely bounded, namely, $\lim_{t\to\infty} X(t) < \infty$, a.s.

3. Main Results

In this section, we establish the stability result with general decay rate for (1.1). This result includes the global existence and uniqueness of the solution, the pth moment stability, and almost surely asymptotic stability with general decay rate.

In order for a stochastic differential equation to have a unique global solution for any given initial value, the coefficients of this equation are generally required to satisfy the linear growth condition and the local Lipschitz condition (see [18, 19]) or a given non-Lipschitz condition and the linear growth condition (cf. [20, 21]). These show that the linear growth condition plays an important role to suppress the potential explosion of solutions and guarantee existence of global solutions. References [16, 22] extended these two classes conditions to infinite delay cases. However, many well-known infinite delay systems such that the Lotka-Volterra (see [13]) do not satisfy the linear growth condition. It is therefore necessary to examine the global existence of the solution for (1.1).

It is well known for stochastic differential equations that the linear growth condition for global solutions may be replaced by the use of the Lyapunov functions [23, 24]. By this idea, this paper establishes the existence-and-uniqueness theorem for (1.1).

For $i = 1, 2, \ldots, k$, let $\zeta_i, \alpha_i \in \mathbb{R}_+$ and probability measures $\mu_i \in M_\varepsilon$. Define $\Gamma_\varepsilon : \mathbb{R}^n \times C_b \to \mathbb{R}$ as

$$\Gamma_\varepsilon(x,\varphi) = \sum_{i=0}^{k} \zeta_i \left(\int_{-\infty}^{0} \left|\varphi(\theta)\right|^{\alpha_i} \mathrm{d}\mu_i(\theta) - \mu_{i\varepsilon}|x|^{\alpha_i} \right), \tag{3.1}$$

where $\mu_{i\varepsilon}$ is defined by (2.5). Then the following theorem follows.

Theorem 3.1. *Assume that there exist positive constants a, p, ε, ζ_i, α_i and probability measures $\mu_i \in M_\varepsilon$, where $i = 1, 2, \ldots, k$, such that for any $x \in \mathbb{R}^d$ and $\varphi \in C_b$, the function $V(x) = |x|^p$ satisfies*

$$\mathcal{L}V(t,x,\varphi) \leqslant \Gamma_\varepsilon(x,\varphi) - a|x|^p. \tag{3.2}$$

Under Assumption 2.2, there exists a constant $q > 0$ such that for any $\xi \in C(\widehat{\alpha}, q)$, where $\widehat{\alpha} = \min_{0\leqslant i\leqslant k}\{\alpha_i\}$, (1.1) *almost surely admits a unique global solution $x(t)$ on $[0,\infty)$ and this solution has the properties* (1.2)*.*

Proof. For sufficiently small $q \in (0, \varepsilon)$, fix the initial data $\xi \in C(\widehat{\alpha}, q)$. We divide this proof into the two steps.

Step 1 (existence and uniqueness of the global solution). Under Assumption 2.2, (1.1) has a unique maximal local solution $x(t)$ on $[0, \rho_e)$ (see [21]), where ρ_e is the explosion time. If we can show $\rho_e = \infty$, a.s., then $x(t)$ is actually a global solution. Let n_0 be a positive integer such that $\sup_{\theta\leqslant 0}|\xi(\theta)| < n_0$. For each integer $n \geqslant n_0$, define the stopping time

$$\sigma_n = \inf\{t \in [0, \rho_e) : |x(t)|^p \geqslant n\}. \tag{3.3}$$

Obviously, σ_n is increasing and $\sigma_n \to \sigma_\infty \leqslant \rho_e$ as $n \to \infty$. Thus, to prove $\rho_e = \infty$ a.s., it is sufficient to show that $\sigma_\infty = \infty$ a.s., which is equivalent to the statement that for any $t > 0$, $\mathbb{P}(\sigma_n \leqslant t) \to 0$ as $n \to \infty$.

For any $t \geqslant 0$, define $t_n = t \wedge \sigma_n$. Applying the Itô formula to $\psi^q(t)V(x(t))$ yields

$$\begin{aligned}
n\mathbb{P}(\sigma_n \leqslant t) &= \mathrm{E}\left[I_{\{\sigma_n\leqslant t\}}V(x(t_n))\right] \\
&\leqslant \mathrm{E}V(x(t_n)) \\
&\leqslant \mathrm{E}\left[\psi^q(t_n)V(x(t_n))\right] \\
&= \text{const} + \mathrm{E}\int_0^{t_n} L\left[\psi^q(s)V(x(s))\right]\mathrm{d}s \\
&= \text{const} + \mathrm{E}\int_0^{t_n} \psi^q(s)\left[LV(x(s)) + q\psi_1(s)V(x(s))\right]\mathrm{d}s \\
&\leqslant \text{const} + \mathrm{E}\int_0^{t_n} \psi^q(s)\left[LV(x(s)) + q\phi V(x(s))\right]\mathrm{d}s \\
&\leqslant \text{const} + \mathrm{E}\int_0^{t_n} \psi^q(s)\left[\Gamma_\varepsilon(x(s), x_s) - a|x(s)|^p + q\phi V(x(s))\right]\mathrm{d}s.
\end{aligned} \tag{3.4}$$

Note that by (2.5), $\mu_{i\varepsilon} \geq \mu_{iq}$ for $q \leq \varepsilon$. By the Fubini theorem and a substitution technique, we have

$$\begin{aligned}
&\int_0^{t_n} \psi^q(s)\Gamma_\varepsilon(x(s), x_s)\mathrm{d}s \\
&\quad = \sum_{i=0}^{k}\zeta_i\left[\int_{-\infty}^{0}\mathrm{d}\mu_i(\theta)\int_0^{t_n}\psi^q(s)|x(s+\theta)|^{\alpha_i}\mathrm{d}s - \mu_{i\varepsilon}\int_0^{t_n}\psi^q(s)|x(s)|^{\alpha_i}\mathrm{d}s\right]
\end{aligned}$$

$$\begin{aligned}
&\leqslant \sum_{i=0}^{k}\zeta_i\left[\int_{-\infty}^{0}\mathrm{d}\mu_i(\theta)\int_{\theta}^{t_n+\theta}\psi^q(s-\theta)|x(s)|^{\alpha_i}\mathrm{d}s-\mu_{iq}\int_{0}^{t_n}\psi^q(s)|x(s)|^{\alpha_i}\mathrm{d}s\right]\\
&\leqslant \sum_{i=0}^{k}\zeta_i\left[\int_{-\infty}^{0}\psi^q(-\theta)\mathrm{d}\mu_i(\theta)\int_{\theta}^{t_n+\theta}\psi^q(s)|x(s)|^{\alpha_i}\mathrm{d}s-\mu_{iq}\int_{0}^{t_n}\psi^q(s)|x(s)|^{\alpha_i}\mathrm{d}s\right]\\
&\leqslant \sum_{i=0}^{k}\zeta_i\left[\mu_{iq}\int_{-\infty}^{t_n}\psi^q(s)|x(s)|^{\alpha_i}\mathrm{d}s-\mu_{iq}\int_{0}^{t_n}\psi^q(s)|x(s)|^{\alpha_i}\mathrm{d}s\right]\\
&=\sum_{i=0}^{k}\zeta_i\mu_{iq}\int_{-\infty}^{0}\psi^q(\theta)|\xi(\theta)|^{\alpha_i}\mathrm{d}\theta.
\end{aligned} \tag{3.5}$$

Noting that $\xi\in C(\widehat{\alpha},q)$, we have $\xi\in C(\alpha_i,q)$, which implies that for all $i=1,\ldots,k$,

$$\int_{-\infty}^{0}\psi^q(\theta)|\xi(\theta)|^{\alpha_i}\mathrm{d}\theta<\infty. \tag{3.6}$$

Hence, there exists

$$\int_{0}^{t_n}\psi^q(s)\Gamma_\varepsilon(x(s),x_s)\mathrm{d}s<\infty. \tag{3.7}$$

By (3.4) and (3.7), we have

$$n\mathbb{P}(\sigma_n\leqslant t)\leqslant \mathrm{const}+\mathrm{E}\int_{0}^{t_n}\psi^q(s)\left[q\phi V(x(s))-a|x(s)|^p\right]\mathrm{d}s. \tag{3.8}$$

Choosing q sufficiently small such that $q\phi\leqslant a$, by (3.8) we have $n\mathbb{P}(\sigma_n\leqslant t)\leqslant$ const, which implies that $\mathbb{P}(\sigma_n\leqslant t)\to 0$ as $n\to\infty$.

Step 2 (Proof of (1.2)). Define

$$h(t)=\psi^q(t)V(x(t)). \tag{3.9}$$

By the Itô formula and (3.2),

$$\begin{aligned}
h(t)&=h(0)+\int_{0}^{t}\psi^q(s)\left[LV(x(s))+q\psi_1(s)V(x(s))\right]\mathrm{d}s+M(t)\\
&\leqslant h(0)+\int_{0}^{t}\psi^q(s)\left[\Gamma_\varepsilon(x(s),x_s)-a|x|^p+q\phi V(x(s))\right]\mathrm{d}s+M(t),
\end{aligned} \tag{3.10}$$

where

$$M(t) = \int_0^t \psi^q(s) V_x(x(s)) g(s, x(s), x_s) \mathrm{d}w(s) \tag{3.11}$$

is a continuous local martingale with $M(0) = 0$. Similar to (3.7), there exists

$$\int_0^t \psi^q(s) \Gamma_\varepsilon(x(s), x_s) \mathrm{d}s < \infty. \tag{3.12}$$

By (3.10), (3.12), noting that $q\phi \leqslant a$,

$$\begin{aligned} h(t) &\leqslant \text{const} + \int_0^t \psi^q(s)\left(q\phi - a\right)|x(s)|^p \mathrm{d}s + M(t) \\ &\leqslant \text{const} + M(t). \end{aligned} \tag{3.13}$$

By Lemma 2.3, we have

$$\limsup_{t\to\infty} \mathrm{E}h(t) < \infty, \qquad \limsup_{t\to\infty} h(t) < \infty, \quad \text{a.s.}, \tag{3.14}$$

which implies the required assertions. □

4. Further Result

In Theorem 3.1, it is not convenient to check condition (3.2) since it is not related to coefficients f and g explicitly. To make our theory more applicable, let us impose the following assumption on coefficients f and g.

Assumption 4.1. There exist positive constants σ, $\widetilde{\sigma}$, α, β, ε, and nonnegative constants $\overline{\sigma}$, λ, $\overline{\lambda}$, $\widetilde{\lambda}$, σ_i, $\overline{\sigma}_i$, λ_j, $\overline{\lambda}_j$, α_i, β_j, such that for any $x \in \mathbb{R}^n$, $\varphi \in C_b$,

$$\begin{aligned} x^{\mathrm{T}} f(t, x, \varphi) \leqslant{}& -\sigma|x|^{\alpha+2} + \overline{\sigma}\int_{-\infty}^0 \left|\varphi(\theta)\right|^{\alpha+2} \mathrm{d}\mu(\theta) - \widetilde{\sigma}|x|^2 \\ &+ \sum_{i=0}^k \left(\sigma_i |x|^{\alpha_i+2} + \overline{\sigma}_i \int_{-\infty}^0 \left|\varphi(\theta)\right|^{\alpha_i+2} \mathrm{d}\mu_i(\theta) \right), \end{aligned} \tag{4.1}$$

$$\begin{aligned} \left|g(t, x, \varphi)\right| \leqslant{}& \lambda|x|^{\beta+1} + \overline{\lambda}\int_{-\infty}^0 \left|\varphi(\theta)\right|^{\beta+1} \mathrm{d}\nu(\theta) + \widetilde{\lambda}|x| \\ &+ \sum_{j=0}^l \left(\lambda_j |x|^{\beta_j+1} + \overline{\lambda}_j \int_{-\infty}^0 \left|\varphi(\theta)\right|^{\beta_j+1} \mathrm{d}\nu_j(\theta) \right), \end{aligned} \tag{4.2}$$

where $0 \leqslant \alpha_0 < \alpha_1 < \cdots < \alpha_k < \alpha$, $0 \leqslant \beta_0 < \beta_1 < \cdots < \beta_l < \beta$, $2\beta \leqslant \alpha$, and $\mu, \mu_i, \nu, \nu_j \in M_\varepsilon$.

We also need the following lemma.

Lemma 4.2. *Let $\alpha, p > 0$. Assume that $\alpha_0, \alpha_1, \dots, \alpha_k$, $c_0, c_1, \dots, c_k$ are nonnegative constants such that $0 \leqslant \alpha_0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_k \leqslant \alpha$, $b > c =: \sum_{i=0}^{k} c_k$ and $a > c\rho$, where*

$$\rho = \begin{cases} 1, & \text{if } \alpha_0 = 0, \\ 0, & \text{if } \alpha_0 = \alpha, \\ (\alpha - \alpha_0)\left(\dfrac{\alpha_0^{\alpha_0}}{\alpha^{\alpha}}\right)^{1/(\alpha-\alpha_0)}, & \text{if } \alpha_0 \in (0, \alpha), \end{cases} \tag{4.3}$$

then, there is $\overline{a} \in (0, a)$ such that for all $t \geqslant 0$,

$$at^p + bt^{\alpha+p} - \sum_{i=0}^{k} c_i t^{\alpha_i + p} \geqslant \overline{a} t^p. \tag{4.4}$$

Proof. Noting that $a > c\rho$, choose the constant $\widetilde{a}$ such that

$$c\rho < \widetilde{a} < a. \tag{4.5}$$

If we can show that for any $t \in [0, \infty)$, $F(t) =: \widetilde{a} + bt^{\alpha} - \sum_{i=0}^{k} c_i t^{\alpha_i} \geqslant 0$, then the inequality

$$a + bt^{\alpha} - \sum_{i=0}^{k} c_i t^{\alpha_i} \geqslant a - \widetilde{a} \tag{4.6}$$

holds. Let $\overline{a} = a - \widetilde{a}$. This is equivalent to prove that

$$at^p + bt^{\alpha+p} - \sum_{i=0}^{k} c_i t^{\alpha_i + p} \geqslant \overline{a} t^p. \tag{4.7}$$

For all $t \in (1, +\infty)$, there exists $F(t) \geqslant \widetilde{a} + bt^{\alpha} - ct^{\alpha}$. By $\widetilde{a} > c\rho \geqslant 0$ and $b > c$, we have $F(t) \geqslant \widetilde{a} + bt^{\alpha} - ct^{\alpha} > 0$.

For all $t \in [0, 1]$, there exists $F(t) \geqslant F_*(t) =: \widetilde{a} + bt^{\alpha} - ct^{\alpha_0}$. To prove $F(t) \geqslant 0$, we consider three cases of α_0, respectively.

Case 1. $\alpha_0 = 0$. By $\alpha_0 = 0$, we have $F_*(t) = \widetilde{a} + bt^{\alpha} - c$ and $\widetilde{a} \in (c, a)$. Then there exists $F(t) \geqslant F_*(t) > 0$.

Case 2. $\alpha_0 = \alpha$. By $\alpha_0 = \alpha$, we have $F_*(t) = \widetilde{a} + bt^{\alpha} - ct^{\alpha}$ and $\widetilde{a} \in (0, a)$. Noting $b > c$, we obtain $F(t) \geqslant F_*(t) > 0$.

Case 3. $\alpha_0 \in (0, \alpha)$. Without the loss of generality, we assume that $c > 0$. Obviously, on $(0, +\infty)$ the derivative function $F'_*(t) = b\alpha t^{\alpha-1} - c\alpha_0 t^{\alpha_0 - 1}$ has a unique null point $t_0 =: (\alpha_0 c/\alpha b)^{1/(\alpha-\alpha_0)} < 1$. We can compute that

$$\begin{aligned} F_*(t_0) &= \tilde{a} + b\left(\frac{\alpha_0 c}{\alpha b}\right)^{\alpha/(\alpha-\alpha_0)} - c\left(\frac{\alpha_0 c}{\alpha b}\right)^{\alpha_0/(\alpha-\alpha_0)} \\ &= \tilde{a} - c\left(\frac{c}{b}\right)^{\alpha_0/(\alpha-\alpha_0)} (\alpha - \alpha_0)\left(\frac{\alpha_0^{\alpha_0}}{\alpha^\alpha}\right)^{1/(\alpha-\alpha_0)}. \end{aligned} \tag{4.8}$$

Since $0 < \alpha_0 < \alpha$ and $b > c$, we know that

$$0 < \left(\frac{c}{b}\right)^{\alpha_0/(\alpha-\alpha_0)} < 1. \tag{4.9}$$

By (4.8) and (4.9), we obtain that $F_*(t_0) > \tilde{a} - c\rho > 0$. Then we have that for any $t \in [0, 1]$, $F(t) \geqslant F_*(t) \geqslant F_*(t_0) > 0$. The proof is completed. □

For the purpose of simplicity, we introduce the following notations:

$$\begin{gathered} \sigma. = \sum_{i=0}^{k} \sigma_i, \qquad \overline{\sigma}. = \sum_{i=0}^{k} \overline{\sigma}_i, \qquad \lambda. = \sum_{j=0}^{l} \lambda_j, \qquad \overline{\lambda}. = \sum_{j=0}^{l} \overline{\lambda}_j, \\ Q = \sigma - \sigma. - \overline{\sigma}. - \overline{\sigma}, \qquad S = \lambda + \overline{\lambda} + \tilde{\lambda} + \lambda. + \overline{\lambda}. \end{gathered} \tag{4.10}$$

Then the following theorem follows.

Theorem 4.3. *Let Assumptions 2.2 and 4.1 hold. Assume that*

$$2Q > S\left(S - \tilde{\lambda}\right), \tag{4.11}$$

$$2\tilde{\sigma} - 2\rho(\sigma. + \overline{\sigma}.) > S\left[\tilde{\lambda} + \rho\left(S - \tilde{\lambda}\right)\right], \tag{4.12}$$

where ρ is defined by Lemma 4.2 except that α_0 is replaced by $\alpha_0 \wedge 2\beta$. For any

$$p \in (2, p_1 \wedge p_2), \tag{4.13}$$

where

$$p_1 = 1 + \frac{2Q}{S\left(S - \tilde{\lambda}\right)}, \qquad p_2 = 1 + \frac{2\tilde{\sigma} - 2\rho(\sigma. + \overline{\sigma}.)}{S\left[\tilde{\lambda} + \rho\left(S - \tilde{\lambda}\right)\right]}, \tag{4.14}$$

there exists a positive constant q such that for any initial data $\xi \in C((\alpha_0 \wedge 2\beta_0) + p, q)$, (1.1) admits a unique global solution $x(t)$ on $[0, \infty)$ and this solution has the properties (1.2).

Proof. Define $V(x) = |x|^p$ for $p > 2$. Applying (2.1) gives

$$\begin{aligned}
\mathcal{L}V(t, x, \varphi) &= p|x|^{p-2}x^{\mathrm{T}} f + \frac{p}{2}(p-2)|x|^{p-4}\left|g^{\mathrm{T}} x\right|^2 + \frac{p}{2}|x|^{p-2}\left|g\right|^2 \\
&\leqslant p|x|^{p-2}x^{\mathrm{T}} f + \frac{p}{2}(p-1)|x|^{p-2}\left|g\right|^2 \\
&=: I_1 + I_2.
\end{aligned} \tag{4.15}$$

By (4.1) and the Young inequality,

$$\begin{aligned}
I_1 &= p|x|^{p-2}x^{\mathrm{T}} f \\
&\leqslant p|x|^{p-2}\Bigg[-\sigma|x|^{\alpha+2} + \overline{\sigma}\int_{-\infty}^{0}\left|\varphi(\theta)\right|^{\alpha+2}\mathrm{d}\mu(\theta) - \widetilde{\sigma}|x|^2 \\
&\qquad + \sum_{i=0}^{k}\left(\sigma_i|x|^{\alpha_i+2} + \overline{\sigma}_i\int_{-\infty}^{0}\left|\varphi(\theta)\right|^{\alpha_i+2}\mathrm{d}\mu_i(\theta)\right)\Bigg] \\
&\leqslant -p\left(\sigma - \overline{\sigma}\frac{p-2}{\alpha+p}\right)|x|^{\alpha+p} - p\widetilde{\sigma}|x|^p + p\overline{\sigma}\frac{\alpha+2}{\alpha+p}\int_{-\infty}^{0}\left|\varphi(\theta)\right|^{\alpha+p}\mathrm{d}\mu(\theta) \\
&\quad + p\sum_{i=0}^{k}\left(\sigma_i + \overline{\sigma}_i\frac{p-2}{\alpha_i+p}\right)|x|^{\alpha_i+p} + p\sum_{i=0}^{k}\overline{\sigma}_i\frac{\alpha_i+2}{\alpha_i+p}\int_{-\infty}^{0}\left|\varphi(\theta)\right|^{\alpha_i+p}\mathrm{d}\mu_i(\theta).
\end{aligned} \tag{4.16}$$

Recall the following elementary inequality: for any $\lambda_j \geqslant 0$ and $x_j \in \mathbb{R}$, $j = 0, 1, \ldots, n$, applying the Hölder inequality yields

$$\left(\sum_{j=0}^{n}\lambda_j x_j\right)^2 \leqslant \sum_{j=0}^{n}\lambda_j\sum_{j=0}^{n}\lambda_j x_j^2. \tag{4.17}$$

By (4.2) and (4.17), applying the Young inequality and the Hölder inequality, we have

$$\begin{aligned}
I_2 &= \frac{p}{2}(p-1)|x|^{p-2}\left|g\right|^2 \\
&\leqslant \frac{p(p-1)}{2}|x|^{p-2}\Bigg[\lambda|x|^{\beta+1} + \overline{\lambda}\int_{-\infty}^{0}\left|\varphi(\theta)\right|^{\beta+1}\mathrm{d}\nu(\theta) + \widetilde{\lambda}|x| \\
&\qquad + \sum_{j=0}^{l}\left(\lambda_j|x|^{\beta_j+1} + \overline{\lambda}_j\int_{-\infty}^{0}\left|\varphi(\theta)\right|^{\beta_j+1}\mathrm{d}\nu_j(\theta)\right)\Bigg]^2
\end{aligned}$$

$$\leqslant \frac{Sp(p-1)}{2}\left[\left(\lambda+\bar{\lambda}\frac{p-2}{2\beta+p}\right)|x|^{2\beta+p}+\bar{\lambda}\frac{2\beta+2}{2\beta+p}\int_{-\infty}^{0}\left|\varphi(\theta)\right|^{2\beta+p}d\nu(\theta)+\tilde{\lambda}|x|^{p}\right.$$
$$\left.+\sum_{j=0}^{l}\left(\lambda_j+\bar{\lambda}_j\frac{p-2}{2\beta_j+p}\right)|x|^{2\beta_j+p}+\sum_{j=0}^{l}\bar{\lambda}_j\frac{2\beta_j+2}{2\beta_j+p}\int_{-\infty}^{0}\left|\varphi(\theta)\right|^{2\beta_j+p}d\nu_j(\theta)\right]. \quad (4.18)$$

Substituting (4.16) and (4.18) into (4.15) yields

$$\mathcal{L}V(t,x,\varphi)\leqslant\Gamma_{\varepsilon}(x,\varphi)-\frac{p}{2}H(x), \quad (4.19)$$

where

$$\begin{aligned}\Gamma_{\varepsilon}(x,\varphi)=&\sum_{i=0}^{k}p\bar{\sigma}_i\frac{\alpha_i+2}{\alpha_i+p}\left(\int_{-\infty}^{0}\left|\varphi(\theta)\right|^{\alpha_i+p}d\mu_i(\theta)-\mu_{i\varepsilon}|x|^{\alpha_i+p}\right)\\&+p\bar{\sigma}\frac{\alpha+2}{\alpha+p}\left(\int_{-\infty}^{0}\left|\varphi(\theta)\right|^{\alpha+p}d\mu(\theta)-\mu_{\varepsilon}|x|^{\alpha+p}\right)\\&+\sum_{j=0}^{l}\frac{Sp(p-1)}{2}\bar{\lambda}_j\frac{2\beta_j+2}{2\beta_j+p}\left(\int_{-\infty}^{0}\left|\varphi(\theta)\right|^{2\beta_j+p}d\nu_j(\theta)-\nu_{j\varepsilon}|x|^{2\beta_j+p}\right)\\&+\frac{Sp(p-1)}{2}\bar{\lambda}\frac{2\beta+2}{2\beta+p}\left(\int_{-\infty}^{0}\left|\varphi(\theta)\right|^{2\beta+p}d\nu(\theta)-\nu_{\varepsilon}|x|^{2\beta+p}\right),\end{aligned} \quad (4.20)$$

whose expression is similar to (3.1) and

$$H(x)=a|x|^{p}+b(\varepsilon)|x|^{\alpha+p}-\tilde{c}(\varepsilon)|x|^{2\beta+p}-\sum_{i=0}^{k}c_i(\varepsilon)|x|^{\alpha_i+p}-\sum_{j=0}^{l}\tilde{c}_j(\varepsilon)|x|^{2\beta_j+p}, \quad (4.21)$$

in which

$$\begin{aligned}a&=2\tilde{\sigma}-S(p-1)\tilde{\lambda},\\b(\varepsilon)&=2\sigma-2\bar{\sigma}\frac{p-2}{\alpha+p}-2\bar{\sigma}\frac{\alpha+2}{\alpha+p}\mu_{\varepsilon},\\\tilde{c}(\varepsilon)&=S(p-1)\left(\lambda+\bar{\lambda}\frac{p-2}{2\beta+p}+\bar{\lambda}\frac{2\beta+2}{2\beta+p}\mu_{\varepsilon}\right),\\c_i(\varepsilon)&=2\sigma_i+2\bar{\sigma}_i\frac{p-2}{\alpha_i+p}+2\bar{\sigma}_i\frac{\alpha_i+2}{\alpha_i+p}\mu_{i\varepsilon},\\\tilde{c}_j(\varepsilon)&=S(p-1)\left(\lambda_j+\bar{\lambda}_j\frac{p-2}{2\beta_j+p}+\bar{\lambda}_j\frac{2\beta_j+2}{2\beta_j+p}\nu_{j\varepsilon}\right).\end{aligned} \quad (4.22)$$

Let $c(\varepsilon) = \widetilde{c}(\varepsilon) + \sum_{i=0}^{k} c_i(\varepsilon) + \sum_{j=0}^{l} \widetilde{c}_j(\varepsilon)$. Note that $b(0) = 2(\sigma - \overline{\sigma})$, $\widetilde{c}(0) = S(p-1)(\lambda + \overline{\lambda})$, $c_i(0) = 2(\sigma_i + \overline{\sigma}_i)$, $\widetilde{c}_j(0) = S(p-1)(\lambda_j + \overline{\lambda}_j)$, $c(0) = S(p-1)(S - \widetilde{\lambda}) + 2(\sigma_{.} + \overline{\sigma}_{.})$. By (4.13), we obtain that $\widetilde{c}(0) > 0$, $c(0) > 0$, and $\widetilde{c}_j(0) > 0$ for all $0 \leqslant j \leqslant l$. By (4.11) and (4.13), we have $b(0) > c(0)$. By (4.12) and (4.13), we obtain $a(0) > \rho c(0)$. Choose sufficiently small ε such that

$$a > \rho c(\varepsilon), \qquad b(\varepsilon) > c(\varepsilon). \tag{4.23}$$

By (4.23) and Lemma 4.2, there exists a constant $\overline{a} \in (0, a)$ such that

$$\overline{a}|x|^p \leqslant a|x|^p + b(\varepsilon)|x|^{\alpha+p} - \widetilde{c}(\varepsilon)|x|^{2\beta+p} - \sum_{i=0}^{k} c_i(\varepsilon)|x|^{\alpha_i+p} - \sum_{j=0}^{l} \widetilde{c}_j(\varepsilon)|x|^{2\beta_j+p}. \tag{4.24}$$

By (4.19), (4.21), and (4.24), we therefore have

$$\mathcal{L}V(t, x, \varphi) \leqslant \Gamma_{\varepsilon}(x, \varphi) - \frac{p}{2}\overline{a}|x|^p, \tag{4.25}$$

which implies that condition (3.2) is satisfied. By (4.20), (4.25), and the fact that $0 \leqslant \alpha_0 < \alpha_1 < \cdots < \alpha_k < \alpha$ and $0 \leqslant \beta_0 < \beta_1 < \cdots < \beta_l < \beta$, applying Theorem 3.1 yields that there exists $q > 0$, such that for any $\xi \in C((\alpha_0 \wedge 2\beta_0) + p, q)$, the desired assertions hold. The proof is completed. $\square$

5. A Scalar Case

To illustrate the application of our result, this section considers a scalar stochastic functional differential equations

$$\begin{aligned} \mathrm{d}x(t) = & \left[\sum_{r=1}^{n} x^r(t)u_r(t) + \sum_{0\leqslant r<r+s\leqslant n} x^r(t)\int_{-\infty}^{0} x^s(t+\theta)u_{rs}(t,\theta)\mathrm{d}\theta\right]\mathrm{d}t \\ & + \left[\sum_{k=1}^{m} x^k(t)v_k(t) + \sum_{0\leqslant k<k+l\leqslant m} x^k(t)\int_{-\infty}^{0} x^l(t+\theta)v_{kl}(t,\theta)\mathrm{d}\theta\right]\mathrm{d}w(t), \end{aligned} \tag{5.1}$$

where for $r = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, m$, $u_r(t), v_k(t) \in C(\mathbb{R}_+)$, for $0 \leqslant r < r+s \leqslant n$ and $0 \leqslant k < k+l \leqslant m, u_{rs}(t,\theta), v_{kl}(t,\theta) \in C(\mathbb{R}_+ \times \mathbb{R}_-), n \geqslant 3$ is an odd number, $m \geqslant 2$, and $2m \leqslant n+1$. In this section, $\sum_{0\leq r<r+s\leq n} := \sum_{r=0}^{n}\sum_{s=0}^{n}$ with $r+s \leq n$ and $\sum_{0\leq k<r+l\leq m}$ has similar

explanation. Assume

$$\begin{aligned} u_1(t) &\leqslant -a_1 < 0, \\ u_n(t) &\leqslant -a_n < 0, \\ |u_r(t)| &\leqslant a_r, \quad \text{where } 2 \leqslant r \leqslant n-1, \\ |u_{rs}(t,\theta)| &\leqslant a_{rs} 2\varepsilon (1-\theta)^{-1-2\varepsilon}, \quad \text{where } 0 \leqslant r < r+s \leqslant n, \end{aligned} \tag{5.2}$$

$$\begin{aligned} |v_k(t)| &\leqslant b_k, \quad \text{where } 1 \leqslant k \leqslant m, \\ |v_{kl}(t,\theta)| &\leqslant b_{kl} 2\varepsilon (1-\theta)^{-1-2\varepsilon}, \quad \text{where } 0 \leqslant k < k+l \leqslant m, \end{aligned} \tag{5.3}$$

in which a_r, a_{rs}, b_k, b_{kl} are nonnegative constants and $\varepsilon > 0$. Define

$$\begin{aligned} f(t,x,\varphi) &= \sum_{r=1}^{n} x^r(t) u_r(t) + \sum_{0 \leqslant r < r+s \leqslant n} x^r(t) \int_{-\infty}^{0} \varphi^s(\theta) u_{rs}(t,\theta) \mathrm{d}\theta, \\ g(t,x,\varphi) &= \sum_{k=1}^{m} x^k(t) v_k(t) + \sum_{0 \leqslant k < k+l \leqslant m} x^k(t) \int_{-\infty}^{0} \varphi^l(\theta) v_{kl}(t,\theta) \mathrm{d}\theta. \end{aligned} \tag{5.4}$$

It is obvious that $f(t,x,\varphi)$ and $g(t,x,\varphi)$ satisfy the local Lipschtiz condition. By (5.4), (5.1) can be rewritten as (1.1).

Choose the Ψ-type function $\varphi(t) = 1 + t^+$. Let $\mathrm{d}\mu(\theta) = 2\varepsilon(1-\theta)^{-1-2\varepsilon}\mathrm{d}\theta$. It is obvious that $\int_{-\infty}^{0} \mathrm{d}\mu(\theta) = 1$ and

$$\int_{-\infty}^{0} \varphi^{\varepsilon}(-\theta)\mathrm{d}\mu(\theta) = \int_{-\infty}^{0} \left[1 + (-\theta)^+\right]^{\varepsilon} 2\varepsilon(1-\theta)^{-1-2\varepsilon}\mathrm{d}\theta = \int_{-\infty}^{0} 2\varepsilon(1-\theta)^{-1-\varepsilon}\mathrm{d}\theta = 2 < \infty, \tag{5.5}$$

which shows that $\mu \in M_{\varepsilon}$.

By (5.2) and the Young inequality, we have that

$$\begin{aligned} x^{\mathrm{T}} f(t,x,\varphi) &\leqslant -a_1|x|^2 + \sum_{i=1}^{n-2} a_{i+1}|x|^{i+2} - a_n|x|^{n+1} + \sum_{0 \leqslant r < r+s \leqslant n} a_{rs}|x|^{r+1} \int_{-\infty}^{0} \left|\varphi(\theta)\right|^s \mathrm{d}\theta \\ &\leqslant -a_1|x|^2 + \sum_{i=1}^{n-2} a_{i+1}|x|^{i+2} - a_n|x|^{n+1} \\ &\quad + \sum_{0 \leqslant r < r+s \leqslant n} a_{rs} \left(\frac{r+1}{r+s+1}|x|^{r+s+1} + \frac{s}{r+s+1} \int_{-\infty}^{0} \left|\varphi(\theta)\right|^{r+s+1} \mathrm{d}\mu(\theta) \right) \end{aligned}$$

$$
\begin{aligned}
&= -\left(a_n - \sum_{0\leqslant r<r+s=n} \frac{(r+1)a_{rs}}{n+1}\right)|x|^{n-1+2} + \sum_{0\leqslant r<r+s=n} \frac{sa_{rs}}{n+1}\int_{-\infty}^{0} \left|\varphi(\theta)\right|^{n-1+2} \mathrm{d}\mu(\theta) \\
&\quad - a_1|x|^2 + \frac{1}{2}a_{01}|x|^2 + \sum_{i=1}^{n-2}\left(a_{i+1} + \sum_{0\leqslant r<r+s=i+1} \frac{(r+1)a_{rs}}{i+2}\right)|x|^{i+2} \\
&\quad + \sum_{i=0}^{n-2} \sum_{0\leqslant r<r+s=i+1} \frac{sa_{rs}}{i+2}\int_{-\infty}^{0} \left|\varphi(\theta)\right|^{i+2} \mathrm{d}\mu(\theta) \\
&=: -\sigma|x|^{\alpha+2} + \overline{\sigma}\int_{-\infty}^{0} \left|\varphi(\theta)\right|^{\alpha+2} \mathrm{d}\mu(\theta) - \widetilde{\sigma}|x|^2 \\
&\quad + \sum_{i=0}^{n-2}\left(\sigma_i|x|^{\alpha_i+2} + \overline{\sigma}_i\int_{-\infty}^{0} \left|\varphi(\theta)\right|^{\alpha_i+2} \mathrm{d}\mu(\theta)\right),
\end{aligned}
\tag{5.6}
$$

which shows that condition (4.1) holds with

$$
\begin{gathered}
\sigma = a_n - \sum_{0\leqslant r<r+s=n} \frac{(r+1)a_{rs}}{n+1}, \qquad \overline{\sigma} = \sum_{0\leqslant r<r+s=n} \frac{sa_{rs}}{n+1}, \qquad \widetilde{\sigma} = a_1, \\
\sigma_i = a_{i+1} + \sum_{1\leqslant r<r+s=i+1} \frac{(r+1)a_{rs}}{i+2}, \qquad \overline{\sigma}_i = \sum_{1\leqslant r<r+s=i+1} \frac{sa_{rs}}{i+2}, \\
\alpha = n-1, \qquad \alpha_i = i.
\end{gathered}
\tag{5.7}
$$

By (5.3) and the Young inequality, we get that

$$
\begin{aligned}
\left|g(t,x,\varphi)\right| &\leqslant b_1|x| + \sum_{j=1}^{m-1} b_j|x|^j + b_m|x|^m + \sum_{0\leqslant k<k+l\leqslant m} b_{kl}|x|^k \int_{-\infty}^{0} \left|\varphi(\theta)\right|^l \mathrm{d}\theta \\
&\leqslant b_1|x| + \sum_{j=1}^{m-1} b_j|x|^j + b_m|x|^m + \sum_{0\leqslant k<k+l\leqslant m} b_{kl}\left(\frac{k}{k+l}|x|^{k+l} + \frac{l}{k+l}\int_{-\infty}^{0} \left|\varphi(\theta)\right|^{k+l} \mathrm{d}\mu(\theta)\right) \\
&= \left(b_m + \sum_{0\leqslant k<k+l=m} \frac{kb_{kl}}{m}\right)|x|^{m-1+1} + \sum_{0\leqslant k<k+l=m} \frac{lb_{kl}}{m}\int_{-\infty}^{0} \left|\varphi(\theta)\right|^{m-1+1} \mathrm{d}\mu(\theta) + b_1|x| \\
&\quad + \sum_{j=1}^{m-2}\left(b_{j+1} + \sum_{0\leqslant k<k+l=j+1} \frac{kb_{kl}}{j+1}\right)|x|^{j+1} + \sum_{j=0}^{m-2} \sum_{0\leqslant k<k+l=j+1} \frac{lb_{kl}}{j+1}\int_{-\infty}^{0} \left|\varphi(\theta)\right|^{j+1} \mathrm{d}\mu(\theta) \\
&=: \lambda|x|^{\beta+1} + \overline{\lambda}\int_{-\infty}^{0} \left|\varphi(\theta)\right|^{\beta+1} \mathrm{d}\mu(\theta) - \widetilde{\lambda}|x| \\
&\quad + \sum_{j=0}^{m-2}\left(\lambda_j|x|^{\beta_j+1} + \overline{\lambda}_j\int_{-\infty}^{0} \left|\varphi(\theta)\right|^{\beta_j+1} \mathrm{d}\mu(\theta)\right),
\end{aligned}
\tag{5.8}
$$

which shows that condition (4.2) holds with

$$\lambda = b_m + \sum_{0\leqslant k<k+l=m} \frac{kb_{kl}}{m}, \qquad \overline{\lambda} = \sum_{0\leqslant k<k+l=m} \frac{lb_{kl}}{m}, \qquad \widetilde{\lambda} = b_1,$$
$$\overline{\lambda}_j = \sum_{0\leqslant k<k+l=j+1} \frac{lb_{kl}}{j+1}, \qquad \lambda_j = \begin{cases} 0, & \text{if } j=0, \\ b_{j+1} + \displaystyle\sum_{0\leqslant k<k+l=j+1} \frac{kb_{kl}}{j+1}, & \text{if } 1\leqslant j\leqslant m-2, \end{cases} \tag{5.9}$$
$$\beta = m-1, \qquad \beta_j = j.$$

By $2m \leqslant n+1$, we have $2(m-1) \leqslant n-1$, which implies $2\beta \leqslant \alpha$. It is easy to see that $\overline{\sigma}$, $\widetilde{\sigma}$, λ, $\overline{\lambda}$, and $\widetilde{\lambda}$ are positive, and σ_i, $\overline{\sigma}_i$, λ_j, $\overline{\lambda}_j$, are nonnegative, where $0 \leqslant i \leqslant n-2, 0 \leqslant j \leqslant m-2$.

By the parameters in Theorem 4.3, we can compute

$$\rho = 1, \qquad \sigma. + \overline{\sigma}. = \sum_{i=2}^{n-1} a_i + \sum_{0\leqslant r<r+s\leqslant n-1} a_{rs},$$
$$S = b. + \sum_{0\leqslant k<k+l\leqslant m} b_{kl}, \qquad Q = a_n - \left(\sum_{i=2}^{n-1} a_i + \sum_{0\leqslant r<r+s\leqslant n} a_{rs}\right), \tag{5.10}$$
$$S - \lambda_0 = \sum_{j=2}^{m} b_j + \sum_{0\leqslant k<k+l\leqslant m} b_{kl}.$$

In Assumption 4.1, the parameter σ is positive, so it is required that

$$a_n > \sum_{0\leqslant r<r+s=n} \frac{(r+1)a_{rs}}{n+1}. \tag{5.11}$$

Let

$$W_1 = \sum_{i=2}^{n-1} a_i + \sum_{0\leqslant r<r+s\leqslant n-1} a_{rs},$$
$$W_2 = b. + \sum_{0\leqslant k<k+l\leqslant m} b_{kl}, \tag{5.12}$$
$$W_3 = \sum_{i=2}^{n-1} a_i + \sum_{0\leqslant r<r+s\leqslant n} a_{rs}.$$

To apply Theorem 4.3, it is necessary to test that (4.11)–(4.13) are satisfied. This requires that

$$a_1 > W_1 + \frac{1}{2}W_2^2, \tag{5.13}$$

$$a_n > W_3 + \frac{1}{2}W_2(W_2 - b_1). \tag{5.14}$$

Obviously, (5.11) can be obtained from (5.14). By (4.14),

$$p_1 = 1 + \frac{2(a_n - W_3)}{W_2(W_2 - b_1)}, \qquad p_2 = 1 + \frac{2(a_1 - W_1)}{W_2^2}. \tag{5.15}$$

Thus, we have the following corollary from Theorem 4.3.

Corollary 5.1. *Let conditions* (5.2), (5.3), (5.13), *and* (5.14) *be satisfied, where* W_1, W_2, *and* W_3 *are given in* (5.12). *For any* $p \in (2, p_1 \wedge p_2)$, *where* p_1 *and* p_2 *are given in* (5.15), *there exist* $q > 0$, *for any* $\xi \in C(p, q)$, (5.1) *has a unique global solution* $x(t) = x(t, \xi)$, *and this solution has properties*

$$\begin{aligned} &\limsup_{t\to\infty} \frac{\ln \mathrm{E}|x(t,\xi)|^p}{\ln(1+t^+)} \leqslant -q, \\ &\limsup_{t\to\infty} \frac{\ln |x(t,\xi)|}{\ln(1+t^+)} \leqslant -\frac{q}{p}, \quad a.s. \end{aligned} \tag{5.16}$$

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