

Research Article

Impulse Control of Proportional Reinsurance with Constraints

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We consider an insurance company whose surplus follows a diffusion process with proportional reinsurance and impulse dividend control. Our objective is to maximize expected discounted dividend payouts to shareholders of the company until the time of bankruptcy. To meet some essential requirements of solvency control (e.g., bankruptcy not soon), we impose some constraints on the insurance company's dividend policy. Under two types of constraints, we derive the value functions and optimal control policies of the company.

1. Introduction

Reinsurance is an effective tool for insurance companies to manage and control their exposure to risk, and distributions of dividends are used by firms as a vehicle for distributing some of their profits to their shareholders. The problem of determining an optimal dividend policy can be formulated as a singular/regular stochastic control problem in absence of fixed transaction costs, or an impulse control consisting of lump sum dividends distributed at discrete moments of time with fixed transaction cost. For details, interested readers may refer to Gerber [1], Asmussen and Taksar [2], Paulsen [3], Benkherouf and Bensoussan [4], and Cadenillas et al. [5].

Recently, optimizing dividends payouts with solvency constraints have received much attention. For example, Paulsen [6] and He et al. [7] studied optimal singular dividend problems under barrier constraints with no reinsurance and proportional reinsurance, respectively. Choulli et al. [8] investigated an optimal singular dividend problem under constrained proportional reinsurance. Bai et al. [9] and Ormeci et al. [10] considered optimal impulse dividend problems under different constraints. By these ideas we further discuss

an optimal impulse control of an insurance company with proportional reinsurance policy under some different solvency constraints.

The paper is organized as follows. In Section 2 we establish optimal impulse control problems of the insurance company with proportional reinsurance policy and discrete dividends. In Section 3, we derive the value function and an optimal policy under some constraints of liquid reserves at impulse times. With some constraints of dividends amounts, we obtain the value function and an optimal policy in Section 4. The final section gives concluding remarks.

2. The Model

We fix a complete, filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ on which a real-valued, $(\{\mathcal{F}_t\}, \mathbb{P})$ -standard Brownian motion is defined, where \mathbb{P} is a real-world probability as usual. Consider the following controlled process:

$$X_t = x + \int_0^t \mu u(s) ds + \int_0^t \sigma u(s) dW_s - \sum_{n=1}^{\infty} I_{\{\tau_n < t\}} \xi_n, \quad (2.1)$$

where $\mu > 0$, $\sigma > 0$, $u(t) \in [0, 1]$, and $\{\tau_i; i = 1, 2, \dots\}$ is an increasing sequence of stopping times and $\{\xi_i; i = 1, 2, \dots\}$ is a sequence of random variables, associated with amounts of the dividends paid to shareholders of an insurance company.

Definition 2.1. A pair

$$\pi \equiv \{u; \mathcal{S}\} \equiv \{u; \tau_1, \tau_2, \dots, \tau_n, \dots; \xi_1, \xi_2, \dots, \xi_n, \dots\} \quad (2.2)$$

is an admissible policy of an insurance company with initial capital x if it satisfies the following conditions:

- (1) for each $i = 1, 2, \dots$ and each $t \geq 0$, $\{\tau_i \leq t\} \in \mathcal{F}_t$ and $\xi_i \in \mathcal{F}_{\tau_i}$;
- (2) the induced dividend process, say Q , defined by

$$Q_t := \sum_{n=1}^{\infty} I_{\{\tau_n < t\}} \xi_n \quad (2.3)$$

is $\{\mathcal{F}_t\}$ adapted, increasing, and càdlàg;

- (3) $0 < \xi_i \leq X_{\tau_i-}$;
- (4) the stochastic differential equation for $X := \{X_t \mid t \geq 0\}$ admits a unique strong solution;
- (5) $u(t) \in [0, 1]$;
- (6) $P(\lim_{n \rightarrow \infty} \tau_n \leq T) = 0$, for all $T \geq 0$.

We write $\Pi(x)$ for the space of these admissible policies.

For each $\pi \in \Pi(x)$, we write $\{X_t^\pi \mid t \geq 0\}$ for the surplus process of the insurance company associated with π . Here the superscript π is added to emphasize the dependence of the surplus process X^π on the strategy π . The ruin time corresponding to π is defined as

$$\tau^\pi := \inf\{t \geq 0 : X_t^\pi \leq 0\} \quad (2.4)$$

To simplify the notation, we suppress the superscript π and write $\tau := \tau^\pi$.

The goal of the insurance company is to select an optimal strategy $\pi \in \Pi(x)$ so as to maximize the expected present value of dividends before bankruptcy.

Let K , ($K > 0$), be the fixed transaction cost attributed to the advisory and consulting fees and k , ($0 < k < 1$), the proportional transaction cost due to taxes on dividends. Then the optimization problem of the insurance company is to select $\pi \in \Pi(x)$ so as to maximize the following performance function:

$$J(x, \pi) := E \left[\sum_{n=1}^{\infty} e^{-\lambda \tau_n} (-K + k \xi_n) I_{\{\tau_n \leq \tau\}} \right], \quad (2.5)$$

where λ is the impatient factor and $\lambda > 0$, that is, to determine the value function

$$V(x) := \sup \{ J(x, \pi); \pi \in \Pi(x) \}, \quad (2.6)$$

and the optimal strategy π^* such that $V(x) = J(x, \pi^*)$.

The value function $V(x)$ is also called an optimal return function.

Without imposing any constraints, Cadenillas et al. [5] investigated the model under the performance function (2.5) and showed that the (s, S) dividend policy is optimal, that is, when the surplus reaches a barrier level S , it is reduced to s via a dividend payment, and the process continues. To meet some requirements of solvency control (e.g., bankruptcy not soon), we impose some constraints on the insurance company's dividend policy. Under solvency constraints $X_{\tau_i^-} \geq \bar{\zeta}$, $X_{\tau_i} \geq \underline{\zeta}$, and $0 \leq \xi_i \leq d$, the optimal control problems are presented in Sections 3 and 4, respectively

In what follows, we still use $V(x)$ to denote the value function different cases of constraints.

3. Case I $X_{\tau_i^-} \geq \bar{\zeta}$, $X_{\tau_i} \geq \underline{\zeta}$

Without reinsurance, the constraints $X_{\tau_i^-} \geq \bar{\zeta}$ and $X_{\tau_i} \geq \underline{\zeta}$ were considered by Bai et al. [9]. To prove our main results, we first recall some results (Propositions 3.1–3.3) without constraints, see Cadenillas et al. [5].

Proposition 3.1. *The optimal return function V is a continuous, nondecreasing function in x satisfying $V(0) = 0$.*

To simplify our notation, we define

$$\begin{aligned} \gamma &= \frac{\lambda}{(\mu^2/2\sigma^2) + \lambda}, & x_0 &= \frac{(1-\gamma)\sigma^2}{\mu}, \\ \theta_+ &= \frac{-\mu + \sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2}, & \theta_- &= \frac{-\mu - \sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2}, \\ a_1 &= \frac{\gamma x_0^{\gamma-1} - \theta_- x_0^\gamma}{\theta_+ - \theta_-}, & a_2 &= \frac{\theta_+ x_0^\gamma - \gamma x_0^{\gamma-1}}{\theta_+ - \theta_-}. \end{aligned} \quad (3.1)$$

Proposition 3.2. *The function $v(x)$, subject to the linear growth condition $v(x) \leq k(x + \mu/c)$, is continuously differentiable on $(0, \infty)$ and is twice continuously differentiable on $(0, x_1^{\tilde{C}}) \cup (x_1^{\tilde{C}}, \infty)$, where*

$$v(x) = \begin{cases} \tilde{C}x^\gamma, & 0 \leq x < x_0, \\ \tilde{C}(a_1 e^{\theta_+(x-x_0)} + a_2 e^{\theta_-(x-x_0)}), & x_0 \leq x \leq x_1^{\tilde{C}}, \\ k(x - \tilde{x}^{\tilde{C}}) + v(\tilde{x}^{\tilde{C}}) - K, & x \geq x_1^{\tilde{C}}. \end{cases} \quad (3.2)$$

Here \tilde{C} , $\tilde{x}^{\tilde{C}}$, and $x_1^{\tilde{C}}$ satisfy the following conditions:

$$v'(\tilde{x}^{\tilde{C}}) = v'(x_1^{\tilde{C}}) = k, \quad v(x_1^{\tilde{C}}) - v(\tilde{x}^{\tilde{C}}) = k(x_1^{\tilde{C}} - \tilde{x}^{\tilde{C}}) - K. \quad (3.3)$$

Proposition 3.3. *The control*

$$\pi^* = (u^*, \mathcal{T}^*, \xi^*) = (u^*; \tau_1^*, \tau_2^*, \dots, \tau_n^*, \dots; \xi_1^*, \xi_2^*, \dots, \xi_n^*, \dots) \quad (3.4)$$

defined by

$$u^*(t) = u^*(X_t^*) = \begin{cases} \frac{\mu}{\sigma^2(1-\gamma)} X_t^*, & 0 \leq X_t^* \leq x_0, \\ 1, & X_t^* \geq x_0, \end{cases} \quad (3.5)$$

$$\tau_1^* = \inf\{t \geq 0 : X_t^* = x_1^{\tilde{C}}\},$$

$$\xi_1^* = x_1^{\tilde{C}} - \tilde{x}^{\tilde{C}},$$

and for every $n \geq 2$

$$\tau_n^* = \inf\{t \geq \tau_{n-1}^* : X_t^* = x_1^{\tilde{C}}\}, \quad (3.6)$$

$$\xi_n^* = x_1^{\tilde{C}} - \tilde{x}^{\tilde{C}},$$

where X^* is the solution of the stochastic differential equation

$$X_t^* = x + \int_0^t \mu u^*(X_s^*) ds + \int_0^t \sigma u^*(X_s^*) dW_s - (x_1^{\tilde{C}} - \tilde{x}^{\tilde{C}}) \sum_{n=1}^{\infty} I_{\{\tau_n^* \leq t\}}, \quad (3.7)$$

is the QVI control associated with the function $v(x)$ defined by (3.2). This control is optimal, and the function $v(x)$ coincides with the value function. That is,

$$V(x) = v(x) = J(x, \pi^*) = J(x; u^*, \mathcal{T}^*, \xi^*). \quad (3.8)$$

For a function $\psi(x)$, we define the operator by

$$\mathcal{L}^u \psi(x) = \frac{1}{2} \sigma^2 u^2 \psi''(x) + \mu u \psi'(x) - \lambda \psi(x). \quad (3.9)$$

Define, for $\underline{\zeta} > \tilde{x}^{\bar{C}}$,

$$v_{\underline{\zeta}}(x) := \begin{cases} Cx^\gamma, & 0 \leq x < x_0, \\ C(a_1 e^{\theta_+(x-x_0)} + a_2 e^{\theta_-(x-x_0)}), & x_0 \leq x \leq \bar{\zeta}_1, \\ k(x - \underline{\zeta}) + v_{\underline{\zeta}}(\underline{\zeta}) - K, & x > \bar{\zeta}_1, \end{cases} \quad (3.10)$$

where C and $\bar{\zeta}_1$ satisfy the following conditions:

$$v'_{\underline{\zeta}}(\bar{\zeta}_1) = k, \quad v_{\underline{\zeta}}(\bar{\zeta}_1) - v_{\underline{\zeta}}(\underline{\zeta}) = k(\bar{\zeta}_1 - \underline{\zeta}) - K. \quad (3.11)$$

We can easily prove that $v'_{\underline{\zeta}}(x)$ is convex on $(0, \bar{\zeta}_1)$ and $\bar{\zeta}_1 > x_1^{\bar{C}}$.

Lemma 3.4. For $\underline{\zeta} > \tilde{x}^{\bar{C}}$,

- (a) $\max_{u \in [0,1]} \mathcal{L}^u v_{\underline{\zeta}}(x) = 0$ for $0 \leq x \leq \bar{\zeta}_1$ and $\max_{u \in [0,1]} \mathcal{L}^u v_{\underline{\zeta}}(x) < 0$ for $x > \bar{\zeta}_1$,
- (b) for $y \geq x \geq \underline{\zeta}$, one has

$$v_{\underline{\zeta}}(y) - v_{\underline{\zeta}}(x) \geq k(y - x) - K, \quad (3.12)$$

and the equality holds when $x = \underline{\zeta}$ and $y \geq \bar{\zeta}_1$.

Proof. (a) For $x < x_0$,

$$\begin{aligned} u^*(x) &= \text{Arg max}_{u \in [0,1]} \left[\frac{1}{2} \sigma^2 u^2 \gamma (\gamma - 1) + \mu u \gamma x - \lambda x^2 \right] = -\frac{\mu x}{\sigma^2 (\gamma - 1)}, \\ \max_{u \in [0,1]} \mathcal{L}^u v_{\underline{\zeta}}(x) &= Cx^{\gamma-2} \max_{u \in [0,1]} \left[\frac{1}{2} \sigma^2 u^2 \gamma (\gamma - 1) + \mu u \gamma x - \lambda x^2 \right] \\ &= Cx^{\gamma-2} \left[\frac{1}{2} \sigma^2 u^{*2} \gamma (\gamma - 1) + \mu u^* \gamma x - \lambda x^2 \right] \\ &= 0. \end{aligned} \quad (3.13)$$

Let

$$x^* = \text{Arg min}_{\underline{\zeta}} v'_{\underline{\zeta}}(x). \quad (3.14)$$

That is, x^* satisfies

$$a_1\theta_+^2 e^{\theta_+(x^*-x_0)} + a_2\theta_-^2 e^{\theta_-(x^*-x_0)} = 0. \quad (3.15)$$

For $x_0 < x < x^*$,

$$\bar{u}(x) = \text{Arg} \max_{u \in [-\infty, \infty]} \mathcal{L}^u v_{\underline{\zeta}}(x) = -\frac{\mu}{\sigma^2} \frac{a_1\theta_+ e^{\theta_+(x-x_0)} + a_2\theta_- e^{\theta_-(x-x_0)}}{a_1\theta_+^2 e^{\theta_+(x-x_0)} + a_2\theta_-^2 e^{\theta_-(x-x_0)}}. \quad (3.16)$$

We can easily show that $\bar{u}(x)$ is an increasing function of x on (x_0, x^*) and $\bar{u}(x) \geq \bar{u}(x_0) = 1$.

For $x^* < x < \bar{\zeta}_1$, the function $v'_{\underline{\zeta}}(x)$ is increasing and $v''_{\underline{\zeta}}(x) > 0$. Thus for $x_0 < x < \bar{\zeta}_1$,

$$\begin{aligned} u^*(x) &= \text{Arg} \max_{u \in [0,1]} \mathcal{L}^u v_{\underline{\zeta}}(x) = 1, \\ \max_{u \in [0,1]} \mathcal{L}^u v_{\underline{\zeta}}(x) &= \frac{1}{2} \sigma^2 C \left[a_1\theta_+^2 e^{\theta_+(x-x_0)} + a_2\theta_-^2 e^{\theta_-(x-x_0)} \right] \\ &\quad + \mu C \left[a_1\theta_+ e^{\theta_+(x-x_0)} + a_2\theta_- e^{\theta_-(x-x_0)} \right] \\ &\quad - \lambda C \left[a_1 e^{\theta_+(x-x_0)} + a_2 e^{\theta_-(x-x_0)} \right] \\ &= 0. \end{aligned} \quad (3.17)$$

From the above steps of the proof, we notice that $(1/2)\sigma^2 v''_{\underline{\zeta}}(\bar{\zeta}_1^-) + \mu k - \lambda v_{\underline{\zeta}}(\bar{\zeta}_1) = 0$. Thus, for $x > \bar{\zeta}_1$,

$$\begin{aligned} \max_{u \in [0,1]} \mathcal{L}^u v_{\underline{\zeta}}(x) &= \max_{u \in [0,1]} \left\{ \mu u k - \lambda \left[k(x - \underline{\zeta}) + v_{\underline{\zeta}}(\underline{\zeta}) - K \right] \right\} \\ &= \mu k - \lambda \left[k(x - \underline{\zeta}) + v_{\underline{\zeta}}(\underline{\zeta}) - K \right] \\ &\leq \mu k - \lambda v_{\underline{\zeta}}(\bar{\zeta}_1) \\ &= -\frac{1}{2} \sigma^2 v''_{\underline{\zeta}}(\bar{\zeta}_1^-) \\ &< 0. \end{aligned} \quad (3.18)$$

(b) By the observation, $v'_{\underline{\zeta}}(x) \leq k$ on $(\underline{\zeta}, \infty)$. Consequently,

$$v_{\underline{\zeta}}(y) - v_{\underline{\zeta}}(x) - k(y-x) = \int_x^y (v'_{\underline{\zeta}}(z) - k) dz \quad (3.19)$$

is smallest for $x = \underline{\zeta}$ and $y \geq \bar{\zeta}_1$ with constrained condition $y \geq x \geq \underline{\zeta}$, and then it equals $-K$. This completes the proof. \square

To consider the case where $X_{\tau_i^-} \geq \bar{\zeta}$ and $X_{\tau_i} \geq \underline{\zeta}$, we first consider the case where $X_{\tau_i} \geq \underline{\zeta}$.

Proposition 3.5. *Assume the dividend policy has to satisfy $X_{\tau_i} \geq \underline{\zeta}$ for some positive $\underline{\zeta}$, that is, the surplus is not allowed to be less than $\underline{\zeta}$ immediately after the dividend payout. Then,*

- (a) if $\underline{\zeta} \leq \tilde{x}^{\bar{C}}$, the optimal policy and the value function are as in Proposition 3.3;
- (b) if $\underline{\zeta} > \tilde{x}^{\bar{C}}$, the optimal policy and the value function are

$$u^*(t) = u^*(X_t^*) = \begin{cases} \frac{\mu}{\sigma^2(1-\gamma)} X_t^*, & 0 \leq X_t^* < x_0, \\ 1, & x \geq x_0, \end{cases} \quad (3.20)$$

$$\tau_1^* = \inf\{t \geq 0 : X_t^* = \bar{\zeta}_1\},$$

$$\xi_1^* = \bar{\zeta}_1 - \underline{\zeta},$$

and for every $n \geq 2$

$$\tau_n^* = \inf\{t \geq \tau_{n-1}^* : X_t^* = \bar{\zeta}_1\}, \quad (3.21)$$

$$\xi_n^* = \bar{\zeta}_1 - \underline{\zeta},$$

and $V(x) = v_{\underline{\zeta}}(x)$.

Proof. Part (a) is obvious since the optimal policy is feasible under the constraint.

The idea of the proof of Part (b) is similar to that of Corollary 2.2 of Alvarez and Lempa [11].

Since $v_{\underline{\zeta}}(x)$ is not twice continuously differentiable at $\bar{\zeta}_1$, we cannot use Ito's differentiation rule directly. However, we can show that there exists a sequence $\{v_{\underline{\zeta},n}\}_{n=1}^{\infty}$ of mappings $v_{\underline{\zeta},n} \in \mathcal{C}^2(\mathbb{R}_+)$ such that as $n \rightarrow \infty$

- (1) $v_{\underline{\zeta},n} \rightarrow v_{\underline{\zeta}}$ uniformly on compact subsets of \mathbb{R}_+ ;
- (2) $\mathcal{L}^u(v_{\underline{\zeta},n}) \rightarrow \mathcal{L}^u(v_{\underline{\zeta}})$ uniformly on compact subsets of $\mathbb{R}_+ \setminus \mathcal{N}$, where \mathcal{N} is a subset of \mathbb{R}_+ which of measure zero;
- (3) $\{\mathcal{L}^u(v_{\underline{\zeta},n})\}_{n=1}^{\infty}$ is locally bounded on \mathbb{R}_+ .

Applying Ito's differentiation rule to the mapping $(t, x) \mapsto e^{-\lambda t} v_{\underline{\zeta},n}(x)$, conditioning on \mathcal{F}_{τ_j} , and reordering terms yield

$$e^{-\lambda \tau_j} v_{\underline{\zeta},n}(X_{\tau_j}) = E_{\mathcal{F}_{\tau_j}} \left[e^{-\lambda \tau_{j+1}} v_{\underline{\zeta},n}(X_{\tau_{j+1}^-}) \right] - E_{\mathcal{F}_{\tau_j}} \left[\int_{\tau_j}^{\tau_{j+1}^-} e^{-\lambda s} \mathcal{L}^u v_{\underline{\zeta},n}(X_s) ds \right], \quad (3.22)$$

where $X_{\tau_j} \geq \underline{\zeta}$.

Letting $n \rightarrow \infty$, applying Fatou's theorem, and invoking the use of the variational inequality $\mathcal{L}^a v_{\underline{\zeta}}(x) \leq 0$ then result in the inequality

$$e^{-\lambda \tau_j} v_{\underline{\zeta}}(X_{\tau_j}) \geq E_{\mathcal{F}_{\tau_j}} \left[e^{-\lambda \tau_{j+1}} v_{\underline{\zeta}}(X_{\tau_{j+1}-}) \right]. \quad (3.23)$$

Taking expectation in both sides, we have

$$E \left[e^{-\lambda \tau_j} v_{\underline{\zeta}}(X_{\tau_j}) \right] \geq E \left[e^{-\lambda \tau_{j+1}} v_{\underline{\zeta}}(X_{\tau_{j+1}-}) \right]. \quad (3.24)$$

Letting $\tau_0 = 0$, summing over j , and applying the nonnegativity of the mapping $v_{\underline{\zeta}}(x)$ give

$$v_{\underline{\zeta}}(x) \geq \sum_{j=1}^n E \left[e^{-\lambda \tau_j} \left(v_{\underline{\zeta}}(X_{\tau_j-}) - v_{\underline{\zeta}}(X_{\tau_j}) \right) I_{\{\tau_j < \tau\}} \right]. \quad (3.25)$$

Since $X_{\tau_j} = X_{\tau_j-} - \xi_j$, $v_{\underline{\zeta}}(y) - v_{\underline{\zeta}}(x) \geq k(y - x) - K$ with $y \geq x \geq \underline{\zeta}$, we find that

$$v_{\underline{\zeta}}(x) \geq E \left[\sum_{j=1}^n e^{-\lambda \tau_j} (k \xi_j - K) I_{\{\tau_j < \tau\}} \right]. \quad (3.26)$$

Letting $n \rightarrow \infty$ and invoking the use of the dominated convergence then imply that

$$v_{\underline{\zeta}}(x) \geq E \left[\sum_{j=1}^{\infty} e^{-\lambda \tau_j} (k \xi_j - K) I_{\{\tau_j < \tau\}} \right]. \quad (3.27)$$

Using the strategy of Part (b) gives the equality

$$v_{\underline{\zeta}}(x) = E \left[\sum_{j=1}^{\infty} e^{-\lambda \tau_j^*} (k \xi_j^* - K) I_{\{\tau_j^* < \tau\}} \right]. \quad (3.28)$$

□

Theorem 3.6. Assume the dividend policy has to satisfy $X_{\tau_i-} \geq \bar{\zeta}$ and $X_{\tau_i} \geq \underline{\zeta}$ for some positive $\underline{\zeta}$ and $\bar{\zeta}$. Then,

- (a) if $\underline{\zeta} \leq \tilde{x}^{\bar{C}}$ and $\bar{\zeta} \leq x_1^{\bar{C}}$, the optimal policy and value function are as in Proposition 3.3;
- (b) if $\underline{\zeta} \leq \tilde{x}^{\bar{C}}$ and $\bar{\zeta} > x_1^{\bar{C}}$, the optimal policy and value function are

$$u^*(t) = u^*(X_t^*) = \begin{cases} \frac{\mu}{\sigma^2(1-\gamma)} X_t^*, & 0 \leq X_t^* < x_0, \\ 1, & X_t^* \geq x_0, \end{cases} \quad (3.29)$$

$$\tau_1^* = \inf \{ t \geq 0 : X_t^* = \bar{\zeta} \},$$

$$\xi_1^* = \bar{\zeta} - \tilde{x}^{\bar{C}},$$

and for every $n \geq 2$

$$\begin{aligned} \tau_n^* &= \inf\{t \geq \tau_{n-1}^* : X_t^* = \bar{\zeta}\}, \\ \xi_n^* &= \bar{\zeta} - \tilde{x}^{\bar{C}}, \\ V(x) = v_{\tilde{x}^{\bar{C}}, \bar{\zeta}}(x) &:= \begin{cases} Cx^\gamma, & 0 \leq x < x_0, \\ C(a_1 e^{\theta_+(x-x_0)} + a_2 e^{\theta_-(x-x_0)}), & x_0 \leq x \leq \bar{\zeta}, \\ k(x - \tilde{x}^{\bar{C}}) + v_{\tilde{x}^{\bar{C}}, \bar{\zeta}}(\tilde{x}^{\bar{C}}) - K, & x > \bar{\zeta}, \end{cases} \end{aligned} \quad (3.30)$$

and C satisfies the following conditions:

$$v_{\tilde{x}^{\bar{C}}, \bar{\zeta}}(\bar{\zeta}) - v_{\tilde{x}^{\bar{C}}, \bar{\zeta}}(\tilde{x}^{\bar{C}}) = k(\bar{\zeta} - \tilde{x}^{\bar{C}}) - K; \quad (3.31)$$

(c) if $\underline{\zeta} > \tilde{x}^{\bar{C}}$ and $\bar{\zeta} \leq \bar{\zeta}_1$, the optimal policy and value function are as in Proposition 3.5;

(d) if $\underline{\zeta} > \tilde{x}^{\bar{C}}$ and $\bar{\zeta} > \bar{\zeta}_1$, the optimal policy and value function are

$$\begin{aligned} u^*(t) = u^*(X_t^*) &= \begin{cases} \frac{\mu}{\sigma^2(1-\gamma)} X_t^*, & 0 \leq X_t^* < x_0, \\ 1, & X_t^* \geq x_0, \end{cases} \\ \tau_1^* &= \inf\{t \geq 0 : X_t^* = \bar{\zeta}\}, \\ \xi_1^* &= \bar{\zeta} - \underline{\zeta}, \end{aligned} \quad (3.32)$$

and for every $n \geq 2$

$$\begin{aligned} \tau_n^* &= \inf\{t \geq \tau_{n-1}^* : X_t^* = \bar{\zeta}\}, \\ \xi_n^* &= \bar{\zeta} - \underline{\zeta}, \\ V(x) = v_{\underline{\zeta}, \bar{\zeta}}(x) &:= \begin{cases} Cx^\gamma, & 0 \leq x < x_0, \\ C(a_1 e^{\theta_+(x-x_0)} + a_2 e^{\theta_-(x-x_0)}), & x_0 \leq x \leq \bar{\zeta}, \\ k(x - \underline{\zeta}) + v_{\underline{\zeta}, \bar{\zeta}}(\underline{\zeta}) - K, & x > \bar{\zeta}, \end{cases} \end{aligned} \quad (3.33)$$

and C satisfies the following condition:

$$v_{\underline{\zeta}, \bar{\zeta}}(\bar{\zeta}) - v_{\underline{\zeta}, \bar{\zeta}}(\underline{\zeta}) = k(\bar{\zeta} - \underline{\zeta}) - K. \quad (3.34)$$

Proof. Parts (a) and (c) are obvious since the optimal policy is feasible under the constraint. The proofs of parts (b) and (d) are similar to Theorem 3.2 in [9], so we state the result here without giving the proof. \square

4. Case II $0 \leq \xi_i \leq d$

Without reinsurance, the constraint $0 \leq \xi_i \leq d$ was considered by Ormeci et al. [10].

If $d \leq (K/k)$, by definition (2.5), we know that the optimal policy is $\tau_1^* = \infty$. So we only consider the case where $d > K/k$.

If $\xi_n^* = x_1^{\tilde{C}} - \tilde{x}^{\tilde{C}} \leq d$, the control band policy that is optimal for the unconstrained problem is also optimal for the constrained problem. If $\xi_n^* = x_1^{\tilde{C}} - \tilde{x}^{\tilde{C}} > d$, we will prove that a control band policy is also an optimal policy for the constrained problem. To prove this result, we use the Lagrangian relaxation, that is, to introduce a Lagrange multiplier $\delta \geq 0$. For each scalar $\delta \geq 0$ and policy π , we define the Lagrangian function

$$\begin{aligned} J(x, \pi, \delta) &:= E \left[\sum_{n=1}^{\infty} e^{-\lambda \tau_n} [-K + k\xi_n + \delta(d - \xi_n)] I_{\{\tau_n \leq \tau\}} \right] \\ &= E \left[\sum_{n=1}^{\infty} e^{-\lambda \tau_n} [-(K - \delta d) + (k - \delta)\xi_n] I_{\{\tau_n \leq \tau\}} \right]. \end{aligned} \quad (4.1)$$

The resulting unconstrained problem is equivalent to the original problem with parameters $K - \delta d$ and $k - \delta$ and gives an upper bound on the objective function of the original constrained problem. That is,

$$J(x, \pi) \leq J(x, \pi, \delta), \quad V(x) \leq \sup_{0 \leq \delta \leq K/d} J(x, \pi, \delta), \quad \text{for } 0 \leq \delta \leq K/d. \quad (4.2)$$

By conditions $0 \leq \delta \leq K/d$, $d > K/k$, we deduce that $\delta < k$. In the following, we find a control band policy that achieves this bound thereby proving its optimality.

For $\delta \in [0, K/d]$, if there exists an $\tilde{x} \in (0, x_0]$ satisfying

$$\begin{aligned} C\gamma\tilde{x}^{\gamma-1} &= k - \delta, \\ C\left(a_1\theta_+e^{\theta_+(\tilde{x}+d-x_0)} + a_2\theta_-e^{\theta_-(\tilde{x}+d-x_0)}\right) &= k - \delta, \end{aligned} \quad (4.3)$$

then \tilde{x} satisfies

$$\gamma\tilde{x}^{\gamma-1} = a_1\theta_+e^{\theta_+(\tilde{x}+d-x_0)} + a_2\theta_-e^{\theta_-(\tilde{x}+d-x_0)}, \quad (4.4)$$

$$C = \frac{k - \delta}{\gamma\tilde{x}^{\gamma-1}}. \quad (4.5)$$

Define

$$\begin{aligned} f(\delta) &:= (k - \delta)d - (\nu(\tilde{x} + d) - \nu(\tilde{x})) \\ &= (k - \delta)d - C \left(a_1 e^{\theta_+(\tilde{x}+d-x_0)} + a_2 e^{\theta_-(\tilde{x}+d-x_0)} - \gamma \tilde{x}^{\gamma-1} \right), \end{aligned} \quad (4.6)$$

where C and $\nu(x)$ are given by (4.5) and (3.2), respectively.

Obviously, we have $f(K/d) > 0$. It can be shown that $f(0) < I(\tilde{C}) = K$, where $I(\tilde{C})$ can be seen in Equation (5.25) of Cadenillas et al. [5]. For $f(\delta)$ being a decreasing function of δ , the equation

$$f(\delta) = K - \delta d \quad (4.7)$$

has a unique solution $\delta^* \in (0, K/d)$.

If there does not exist an $\tilde{x} \in (0, x_0]$ satisfying (4.3), then there must exist an $\tilde{x} \in (x_0, \infty)$ such that

$$\begin{aligned} C \left(a_1 \theta_+ e^{\theta_+(\tilde{x}-x_0)} + a_2 \theta_- e^{\theta_-(\tilde{x}-x_0)} \right) &= k - \delta, \\ C \left(a_1 \theta_+ e^{\theta_+(\tilde{x}+d-x_0)} + a_2 \theta_- e^{\theta_-(\tilde{x}+d-x_0)} \right) &= k - \delta, \end{aligned} \quad (4.8)$$

which results in \tilde{x} satisfying

$$\begin{aligned} a_1 \theta_+ e^{\theta_+(\tilde{x}-x_0)} + a_2 \theta_- e^{\theta_-(\tilde{x}-x_0)} &= a_1 \theta_+ e^{\theta_+(\tilde{x}+d-x_0)} + a_2 \theta_- e^{\theta_-(\tilde{x}+d-x_0)}, \\ C &= \frac{k - \delta}{a_1 \theta_+ e^{\theta_+(\tilde{x}-x_0)} + a_2 \theta_- e^{\theta_-(\tilde{x}-x_0)}}. \end{aligned} \quad (4.9)$$

Define

$$\begin{aligned} g(\delta) &:= (k - \delta)d - (\nu(\tilde{x} + d) - \nu(\tilde{x})) \\ &= (k - \delta)d - C \left[a_1 e^{\theta_+(\tilde{x}+d-x_0)} + a_2 e^{\theta_-(\tilde{x}+d-x_0)} - a_1 e^{\theta_+(\tilde{x}-x_0)} - a_2 e^{\theta_-(\tilde{x}-x_0)} \right], \end{aligned} \quad (4.10)$$

where C and $\nu(x)$ are given by (4.11) and (3.2), respectively.

By a similar analysis, we also can show that the equation

$$g(\delta) = K - \delta d \quad (4.11)$$

has a unique solution $\delta^* \in (0, K/d)$.

Theorem 4.1. Assume the dividend policy has to satisfy $0 \leq \xi_n \leq d$. Then,

- (a) if $x_1^{\tilde{C}} - \tilde{x}^{\tilde{C}} \leq d$, the optimal policy and value function are as in Proposition 3.3;
 (b) if $x_1^{\tilde{C}} - \tilde{x}^{\tilde{C}} > d$, the optimal policy and value function are

$$u^*(t) = u^*(X_t^*) = \begin{cases} \frac{\mu}{\sigma^2(1-\gamma)} X_t^*, & 0 \leq X_t^* < x_0, \\ 1, & X_t^* \geq x_0, \end{cases} \quad (4.12)$$

$$\tau_1^* = \inf\{t \geq 0 : X_t^* = \tilde{x} + d\},$$

$$\xi_1^* = d,$$

and for every $n \geq 2$

$$\tau_n^* = \inf\{t \geq \tau_{n-1}^* : X_t^* = \tilde{x} + d\},$$

$$\xi_n^* = d,$$

$$V(x) = v_d(x) := \begin{cases} Cx^\gamma, & 0 \leq x < x_0, \\ C(a_1 e^{\theta_+(x-x_0)} + a_2 e^{\theta_-(x-x_0)}), & x_0 \leq x \leq \tilde{x} + d, \\ (k - \delta^*)(x - \tilde{x}) + v_d(\tilde{x}) - (K - \delta^*d), & x > \tilde{x} + d, \end{cases} \quad (4.13)$$

and C and \tilde{x} satisfy the following conditions:

$$v_d'(\tilde{x}) = k - \delta^*, \quad v_d'(\tilde{x} + d) = k - \delta^*, \quad v_d(\tilde{x} + d) - v_d(\tilde{x}) = (k - \delta^*)d - (K - \delta^*d). \quad (4.14)$$

Proof. We only prove Part (b). By the above analysis and a similar proof of Cadenillas et al. [5],

$$\begin{aligned} \sup_{0 \leq \xi_i < \infty} J(x, \pi, \delta^*) &= E \left[\sum_{n=1}^{\infty} e^{-\lambda \tau_n^*} [-K + k\xi_n^* + \delta^*(d - \xi_n^*)] I_{\{\tau_n^* \leq \tau_{n+1}^* = d\}} \right] \\ &= E \left[\sum_{n=1}^{\infty} e^{-\lambda \tau_n^*} [-K + k\xi_n^*] I_{\{\tau_n^* \leq \tau_{n+1}^* = d\}} \right] \\ &= J(x, \pi^*). \end{aligned} \quad (4.15)$$

However,

$$J(x, \pi^*) \leq V(x) \leq \sup_{0 \leq \xi \leq d} J(x, \pi, \delta^*) \leq \sup_{0 \leq \xi_i < \infty} J(x, \pi, \delta^*). \quad (4.16)$$

By (4.15) and (4.16), we obtain

$$J(x, \pi^*) = V(x), \quad (4.17)$$

which results in the claim of the theorem. \square

5. Conclusion

We have discussed some important issues about the combined optimal reinsurance and dividend problem in the presence of both fixed and proportional transaction costs. We supposed that the goal of the insurance company is to maximize the expected present value of dividends and formulated the problem into an optimal impulse control problem. Under some cases of constraints, we provided a detailed mathematical analysis for the solution of the problem and derived that the optimal dividend strategy is still an (s, S) -(band)-policy.

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