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*Research Article*

# First Passage Time Moments of Jump-Diffusions with Markovian Switching

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Using an integral equation associated with generalized backward Kolmogorov's equation for the transition probability density function, recurrence relations are derived for the moments of the time of first exit of jump-diffusions with Markovian switching. The results are used to find the expectation of first passage time of some financial models.

## 1. Introduction

Owing to the increasing demands on regime-switching diffusions in financial engineering and wireless communications, much attention has been drawn to switching jump diffusion processes. For example, one of the early efforts of using such hybrid models for financial applications can be traced back to [1, 2], in which both the appreciation rate and the volatility of a stock depend on a continuous Markov chain. The introduction of such models makes it possible to describe stochastic volatility in a relatively simpler manner. A stock market may be considered to have two "modes" or "regimes", up and down, resulting from the state of the underlying economy, the general mood of investors in the market, and so on. The rationale is that in the different modes or regimes, the volatility and return rates are very different. For instance, in a stock market, the regimes can be roughly divided into two states, bull market and bear market. The market sentiment and reaction to the two states are in stark contrast. Normally, a bare market is more volatile than that of a bull market. Another need is to capture the features of insurance policies that are subject to the economic or political environment changes. It is thus sensible and necessary to take such regime changes into consideration. Also due to the needs from modeling points of view, formulation of dividend optimization problem often results in complex models in order to take into consideration various scenarios arising in finance and insurance practice. For example, in lieu of the diffusion models or jump-diffusion models, one may consider such models with regime switching modulated by a continuous time Markov chain, in which the Markov chain is used to delineate the random environment. Adopting a Markov regime-switching model is an easy way to capture all the cyclical features of the drift and volatility of asset return depending on market environment. For a long time, lots of empirical evidences have supported the existence of regime-switching property in financial markets. For example, Lewellen found that expected returns on equities change over time in [3]. Schwert concluded that the volatilities of stock returns also vary substantially over time in [2].

Taking the factors above into consideration, this paper is concerned with jump-diffusion involving Markovian switching regimes. In the models, there are finite set of regimes or configurations and a switching process that dictates which regime to take at any given instance. At each time $t$, once the configuration is determined by the switching process, dynamics of the system follows a jump-diffusion process. It evolves until the next jump takes place. Then, the postjump location is determined and the process sojourns in the new location following the evolution of another jump-diffusion motions.

First passage time distribution for stochastic processes are key quantities in many fields of sciences, such as mathematical physics, neurology, and also in mathematical finance. For instance, in the latter case, it is required for the pricing of some path-dependent options and for estimating the risk of default in the structural approach. Tuckwell derived the recurrence relationship for the moments of the time of first exit of a temporally homogeneous Makov process from a set in the phase space (see [4]). Numerical algorithm for computing the probability of the first exit time from a bounded domain for multidimensional diffusions was given in [5]. Kou and Wang obtained explicit solutions of the Laplace transforms, of both the distribution of the first passage times and the joint distribution of a double exponential jump diffusion process (see [6]). However, to the best of our knowledge, there are few literatures concerning the first passage time of jump-diffusions with Markovian switching. Based on these, in this paper, following a similar method as in [4], we show that the moment of the first passage time can be represented as the unique weak solution of the backward Kolmogorov equation associated to the process subject to appropriate boundaries conditions. The results are used to find the expectation of first passage time of some financial models.

## 2. Derivation of the Recurrence Relations for the Moments

Consider the following jump-diffusions with Markovian switching:

$$dX(t) = \sigma(X(t), Z(t))dB(t) + b(X(t), Z(t))dt + \int_{R^d\setminus\{0\}} c(X(t-), Z(t-), u)N(dt, du),$$

$$P\{Z(t+\Delta t) = l \mid Z(t) = k\} = \begin{cases} q_{kl}\Delta + o(\Delta) & \text{if } k \neq l, \\ 1 + q_{kl}\Delta + o(\Delta) & \text{if } k = l. \end{cases} \tag{2.1}$$

We denote the transition probability family of the jump-diffusion with Markovian switching $(X(t), Z(t))$ by $\{P(t,(x,k),A) : t \geq 0,\ (x,k) \in R\times M,\ A \in \mathcal{B}(R\times M)\}$. Assume that $P(t,(x,k),A)$ has density $p(t,(x,k),\cdot)$, then it satisfies the Kolmogorov backward equation.

*Remark 2.1.* When the switching process $Z(t)$ is missing, it reduces to the Ito's formula for jump-diffusion processes studied by Tuckwell in [4]; when the Poisson jump is missing, it reduces to diffusions with regime switching investigated in [7]; when both the switching process and the Poisson jumps are missing, it reduces to the Ito's formula for the usual diffusion processes and this model has been studied by many authors, see [5, 6] and references therein.

In order to assure the existence of solutions of (2.1), we need to make the following assumption.

*Assumption A.* Assume that $c(x,k,u)$ is $(\mathcal{B} \times M) \times \mathcal{B}(R \setminus \{0\})$ measurable function and that $\sigma(x,k)$ and $b(x,k)$ are continuously differentiable in $x$ and they satisfy the Lipschitz condition and the linear growth condition as follows: for some constant $H > 0$,

$$\begin{gathered} \left|\sigma(x,k) - \sigma(y,k)\right|^2 + \left|b(x,k) - b(y,k)\right|^2 \leq H\left|x - y\right|^2, \\ |\sigma(x,k)|^2 + |b(x,k)|^2 \leq H\left(1 + |x|^2\right), \end{gathered} \tag{2.2}$$

for all $x, y \in R$ and $k \in M$.

Under Assumption A, [8] has proved that (2.1) determines a unique right continuously strong Markov process $(X(t), Z(t))$ with left-hand limits and shows that $(X(t), Z(t))$ has the following generator:

$$\mathcal{A} f(x,k) = L(k) f(x,k) + \Omega(k) f(x,k) + Q f(x,k), \quad f(x,k) \in C_0^2(R \times M), \tag{2.3}$$

where

$$\begin{gathered} L(k) f(x,k) = \frac{1}{2}\sigma^2(x,k)\frac{d^2}{dx^2} f(x,k) + b(x,k)\frac{d}{dx} f(x,k), \\ \Omega(k) f(x,k) = \int_{R\setminus\{0\}} [f(x + c(x,k,u),k) - f(x,k)]\Pi(du), \\ Q f(x,k) = \sum_{l \in N} q_{kl}\left(f(x,l) - f(x,k)\right). \end{gathered} \tag{2.4}$$

$\Pi(\cdot)$ is the measure defined on $\mathcal{B}(R)$ such that

$$E[\nu(t,B)] = t\Pi(B), \quad B \in \mathcal{B}(R), \tag{2.5}$$

and the jump intensity

$$\Lambda = \int_R \Pi(du) \tag{2.6}$$

is assumed to be finite. $\nu(t,\cdot)$ is a temporally homogeneous Poisson random measure.

We now further assume that conditions are satisfied for the transition probability of $(X(t), Z(t))$ to have a density, $p((y, j), t_2 \mid (x, k), t_1)$, with $t_1 < t_2$. Denote

$$p((y, j), t_2 \mid (x, k), t_1) \triangleq p(t_2 - t_1, (x, k), (y, j)). \tag{2.7}$$

Using a similar method as in [9], we can show that the density satisfies the backward equation, that is, for fixed $(y, j) \in R \times N$ and $t_2$,

$$-\frac{\partial p}{\partial t_1} = L(k)p + \Omega(k)p + Qp \triangleq \mathcal{A}p. \tag{2.8}$$

**Lemma 2.2** (see [8]). *There exists a unique strong Markov process* $(X(t), Z(t))$ *satisfying* (2.1) *with* $D([0, \infty), R \times M)$ *as its path space.*

We now prove the following result.

**Theorem 2.3.** *Let* $(X(t), Z(t))$ *be the temporally homogeneous process whose transition density satisfy the Kolmogorov backward equation* (2.8). *Let* $A$ *be an open set in the phase space* $R \times M$ *and define, for* $X(0) = (x, k) \in A$,

$$\tau(x, k) = \inf\{t : X(t) \neq A\}, \tag{2.9}$$

*which is the time of first exit from* $A$. *One sets*

$$t_n(x, k) = E\left[\tau(x, k)^n\right], \quad n = 0, 1, 2, \ldots \tag{2.10}$$

*if these quantities exist. One, then, sets the following:*

(i) *the probability* $t_0(x, k)$ *that* $X(t)$ *ever leaves the set* $A$ *satisfies the equation*

$$\begin{aligned} &-\Lambda t_0(x, k) + \frac{1}{2}\sigma(x, k)^2 \frac{d^2 t_0(x, k)}{dx^2} + b(x, k)\frac{dt_0(x, k)}{dx} \\ &\quad + \int_R t_0(x + c(x, k, u), k)\Pi(du) + \sum_{l \in N} q_{kl}(t_0(x, l) - t_0(x, k)) = 0, \end{aligned} \tag{2.11}$$

*with boundary conditions* $t_0(x, k) = 1$, $(x, k) \notin A$,

(ii) *if the solution of* (2.11) *is* $t_0(x, k) = 1$, *for all* $(x, k) \in A$, *then the moments of* $\tau(x, k)$ *satisfy the recurrence relations*

$$\begin{aligned} &-\Lambda t_n(x, k) + \frac{1}{2}\sigma(x, k)^2 \frac{d^2 t_n(x, k)}{dx^2} + b(x, k)\frac{dt_n(x, k)}{dx} \\ &\quad + \int_R t_n(x + c(x, k, u), k)\Pi(du) + \sum_{l \in N} q_{kl}(t_n(x, l) - t_n(x, k)) = -n t_{n-1}(x, k), \\ &\qquad n = 1, 2, \ldots, \end{aligned} \tag{2.12}$$

*with boundary conditions* $t_n(x, k) = 0$ *for* $(x, k) \notin A$.

*Proof.* For fixed $(y,j) \in R \times M$, $p((y,j),t_2|(x,k),t_1)$ satisfy the backward Kolmogorov equation

$$-\frac{\partial p}{\partial t_1} = L(k)p + \Omega(k)p + Qp. \tag{2.13}$$

Consider the functional

$$U(t_1,t_2) = \int_{t_1}^{t_2} \Phi(X(t'),Z(t'),t')dt', \tag{2.14}$$

where $\Phi(\cdot,\cdot,\cdot)$ is defined on $R \times M \times [0,\infty)$ and takes nonnegative values. The Laplace transform for $U(t_1,t_2)$ is

$$\begin{aligned}
&E\left[e^{-\lambda U(t_1,t_2)} \mid (X(t_1),Z(t_1)) = (x,k)\right] \\
&= \sum_{j=1}^{N}\int_{-\infty}^{\infty} E\left[e^{-\lambda U(t_1,t_2)} \mid (X(t_2),Z(t_2)) = (y,j),(X(t_1),Z(t_1)) = (x,k)\right] \\
&\quad \cdot P((X(t_2),Z(t_2)) \in dy \times j \mid (X(t_1),Z(t_1)) = (x,k)) \\
&= \sum_{j=1}^{N}\int_{-\infty}^{+\infty} E\left[e^{-\lambda U(t_1,t_2)} \mid (X(t_2),Z(t_2)) = (y,j),(X(t_1),Z(t_1)) = (x,k)\right] p((y,j),t_2 \mid (x,k),t_1)dy \\
&= \sum_{j=1}^{N}\int_{-\infty}^{+\infty} r((x,k),t_1 \mid (y,j),t_2;\lambda)dy,
\end{aligned} \tag{2.15}$$

where

$$\begin{aligned}
&r((x,k),t_1 \mid (y,j),t_2;\lambda) \\
&\quad \triangleq E\left[e^{-\lambda U(t_1,t_2)} \mid (X(t_2),Z(t_2)) = (y,j),(X(t_1),Z(t_1)) = (x,k)\right] p((y,j),t_2 \mid (x,k),t_1).
\end{aligned} \tag{2.16}$$

According to [1], we have that $r$ satisfies the integral equation

$$\begin{aligned}
r((x,k),t_1 \mid (y,j),t_2;\lambda) = p((y,j),t_2 \mid (x,k),t_1)& \\
-\lambda \sum_{k=1}^{N}\int_{t_1}^{t_2}\int dt'dx' p((x',k'),t' \mid (x,k),t_1)\Phi((x',k'),t')& \\
\cdot r((x,k),t_1 \mid (x',k'),t';\lambda).&
\end{aligned} \tag{2.17}$$

Applying the operator $\mathscr{A} + \partial/\partial t_1$ to (2.17) and combining with (2.15), we obtain

$$-\frac{\partial r}{\partial t_1}((x,k),t_1 \mid (y,j),t_2;\lambda) = [\mathscr{A} - \lambda\Phi((x,k),t_1)]r((x,k),t_1 \mid (y,j),t_2;\lambda). \tag{2.18}$$

The Laplace transform of $U(t_1,t_2)$

$$\widehat{\Psi}((x,k),t_1,t_2;\lambda) = \sum_{j=1}^{N}\int_0^\infty r((x,k),t_1 \mid (y,j),t_2;\lambda)dy \tag{2.19}$$

satisfies

$$\mathscr{A} + \frac{\partial}{\partial t_1}\widehat{\Psi}((x,k),t_1,t_2;\lambda) = \lambda\Phi(x,t_1)\widehat{\Psi}((x,k),t_1,t_2;\lambda). \tag{2.20}$$

Let $A$ be an open set in $R \times M$, and define the function $\Phi((X(t),Z(t)),t)$ by

$$\Phi((X(t),Z(t)),t) = \begin{cases} 1 & \text{if } (X(t),Z(t)) \notin A, \\ 0 & \text{if } (X(t),Z(t)) \in A, \end{cases} \tag{2.21}$$

so that the corresponding Laplace transform $\widehat{\Psi}_A$ satisfies

$$\left(\mathscr{A} + \frac{\partial}{\partial t_1}\widehat{\Psi}_A((x,k),t_1,t_2;\lambda)\right) = 0, \quad (x,k) \in A. \tag{2.22}$$

Given $(X(0),Z(0)) = (x,k) \in A$, the probability that $(X(t),Z(t))$ is in $A$ in the time interval $[t_1,t_2]$ is

$$P_A((x,k),t_1,t_2) = \lim_{\lambda\to\infty}\widehat{\Psi}_A((x,k),t_1,t_2;\lambda), \tag{2.23}$$

where

$$\begin{aligned} P_A((x,k),t_1,t_2) &\triangleq P_x((X(t),Z(t)) \in A,\ t \in [t_1,t_2]), \\ \widehat{\Psi}_A((x,k),t_1,t_2;\lambda) &= E_x e^{-\lambda\Phi((X(t),Z(t)),t)} \\ &= e^{-\lambda\cdot 1}P_x((X(t),Z(t)) \notin A,\ t \in [t_1,t_2]) \\ &\quad + e^{-\lambda\cdot 0}P_x((X(t),Z(t)) \in A,\ t \in [t_1,t_2]). \end{aligned} \tag{2.24}$$

Let $\lambda \to \infty$, then we have

$$\begin{aligned}\left(\mathcal{A}+\frac{\partial}{\partial t_1}\right)P_A((x,k),t_1,t_2) &= \left(\mathcal{A}+\frac{\partial}{\partial t_1}\right)\lim_{\lambda\to\infty}\widehat{\Psi}_A((x,k),t_1,t_2;\lambda)\\ &= \lim_{\lambda\to\infty}\left(\mathcal{A}+\frac{\partial}{\partial t_1}\right)\widehat{\Psi}_A((x,k),t_1,t_2;\lambda)\\ &= 0.\end{aligned} \tag{2.25}$$

So, we obtain the equation

$$\left(\mathcal{A}+\frac{\partial}{\partial t_1}\right)P_A((x,k),t_1,t_2) = 0, \quad (x,k)\in A, \tag{2.26}$$

with initial conditions

$$P_A((x,k),t_1,t_2) = \begin{cases} 1 & \text{if } (x,k)\in A,\\ 0 & \text{if } (x,k)\notin A \end{cases} \tag{2.27}$$

and boundary conditions

$$P_A((x,k),t_1,t_2) = 0, \quad (x,k)\notin A. \tag{2.28}$$

On the one hand, since the Markov switching process $(X(t),Z(t))$ is homogeneous, we set $t_2 - t_1 = t$ and put $P_A = P_A((x,k),t)$

$$-\frac{\partial P_A}{\partial t} = \mathcal{A}P_A((x,k),t). \tag{2.29}$$

On the other hand,

$$P_A((x,k),t) = P(\tau_A(x,k) > t) = 1 - P(\tau_A(x,k)\le t) = 1 - F_A((x,k),t), \tag{2.30}$$

where $F_A((x,k),t)$ is the distribution function of $\tau_A(x,k)$ and this is determined by

$$\frac{\partial F_A}{\partial t} = \mathcal{A}F_A((x,k),t), \quad (x,k)\in A, \tag{2.31}$$

with initial condition

$$F_A((x,k),0) = \begin{cases} 0 & \text{if } x\in A,\\ 1 & \text{if } x\notin A \end{cases} \tag{2.32}$$

and the boundary conditions $F_A((x,k),t) = 1$ if $(x,k) \notin A$. From (2.31), the Laplace-Stieltjes transform of $F_A$ defined by

$$\widetilde{F}_A((x,k),s) = \int_0^\infty e^{-st} F_A((x,k),dt) \tag{2.33}$$

satisfies

$$s\widetilde{F}_A((x,k),s) = \mathcal{A}\widetilde{F}_A((x,k),s), \quad (x,k) \in A, \tag{2.34}$$

with boundary conditions $\widetilde{F}_A(x,s) = 0$ for $x \notin A$.

Since

$$\widetilde{F}_A((x,k),s) = E_{(x,k)} e^{-s\tau_A(x,k)} = \sum_{n=0}^{\infty} \frac{(-s)^n}{n!} t_n(x,k), \tag{2.35}$$

where $t_n(x,k) = E_{(x,k)}[\tau_A^n]$, $n = 0,1,2,\ldots$.

Substituting (2.35) into (2.34), we have the recurrence relation

$$\mathcal{A}t_n(x,k) = -nt_{n-1}(x,k), \quad (x,k) \in A, \ n = 1,2,\ldots. \tag{2.36}$$

In particular, the probability that $(X(t), Z(t))$ ever leaves $A$, satisfying

$$\mathcal{A}t_0(x,k) = 0, \quad (x,k) \in A, \tag{2.37}$$

with boundary conditions $t_0(x,k) = 1$ for $(x,k) \notin A$. □

*Remark 2.4.* (1) When there is no Markov switching, it is the results of [4].

(2) When there is no Poisson jump, it gives the first passage time of regime-switching diffusion, which is the model studied in [7, 10].

(3) When both Poisson's jump and Markov's switching are disappeared, it is the result of [1].

## 3. Application to Finance

For the regime-switching diffusion without Poisson's jumps, there are a number of works on financial models. For example, bounds to the first passage time density and distribution function of alternating Brownian motion are given in [10]. The problem of hedging an European call option for a modulated diffusion model is studied in [11], and a generalization of the Black-Scholes formula for the corresponding option price is shown there. The financial market models based on telegraph processes and alternative jump diffusions are treated in [12]. In this section, we add an extra Poisson jump term to the continuous diffusion models and consider the first passage time moment of the stock price composed of a geometric Brownian motion and Poisson jump.

Consider a stock whose price satisfies the following stochastic differential-integral equation and initial condition

$$dS(t) = S(t)(Z(t)dt + \sigma dB(t)) + \int_{R\setminus\{0\}} c(S(t-), Z(t-), u)N(dt, du),$$
$$S(0) = S_0. \tag{3.1}$$

In (3.1), $S_0$ and $\sigma$ are constants, $B(t)$ is a standard Brownian motion, $Z(t)$ is a jump Markov process with two states $a,\ b$, and $a > 0,\ b < 0$. The generator of $Z(t)$ is given by

$$B = \begin{pmatrix} -p & p \\ -q & q \end{pmatrix}. \tag{3.2}$$

The constant $a$ gives the rate of increase, $b$ the rate of decrease, $1/p$ is the expected time the drift $Z(t)$ stays in state $a$, and $1/q$ is the expected time the drift $Z(t)$ stays in state $b$. In (3.1), the stock price is composed of a geometric Brownian motion and Poisson jump. In the geometric Brownian motion, the drift rate is a two-state jump Markov process. One state of the drift rate is positive. In this state, the stock price should increase. The other state of the drift rate is negative. In this state, the stock price should decrease. Hence, the drift rate changes back and forth between positive and negative values. Discontinuous jump process is modeled by a Poisson distribution. These discontinuous price jumps are usually a result of outages, transmission constraints, and so forth.

The generator of (3.1) for $k = a, b$ is

$$L(k)f(x,k) = \frac{1}{2}\sigma^2 x^2 f''(x) + x \cdot k \cdot f'(x),$$
$$\Omega(k)f(x,k) = \int_{R\setminus\{0\}} \left[f(x + c(x,k,u),k) - f(x,k)\right]\Pi(du),$$
$$Qf(x,a) = p\big(f(x,b) - f(x,a)\big),$$
$$Qf(x,b) = -q\big(f(x,a) - f(x,b)\big). \tag{3.3}$$

When there is no Poisson jump, it reduces to the model in [13].

**Proposition 3.1.** *The moments of first passage time for $(x,a) \in A$ satisfy*

$$-\Lambda t_n(x,a) + \frac{1}{2}\sigma^2 x^2 \frac{d^2 t_n(x,a)}{dx^2} + ax\frac{dt_n(x,a)}{dx}$$
$$+ \int_{R\setminus\{0\}} t_n(x + c(x,a,u))\Pi(du) + p(t_n(x,b) - t_n(x,a) = -nt_{n-1}(x,a)), \tag{3.4}$$

*with boundary conditions* $t_n(x,a) = 0$ *for* $(x,a) \notin A$ *and for* $(x,b) \in A$ *satisfy*

$$\begin{aligned} &-\Lambda t_n(x,b) + \frac{1}{2}\sigma^2 x^2 \frac{d^2 t_n(x,b)}{dx^2} + ax\frac{dt_n(x,b)}{dx} \\ &\quad + \int_{R\setminus\{0\}} t_n(x + c(x,b,u))\Pi(du) - q(t_n(x,a) - t_n(x,b) = -nt_{n-1}(x,b)), \end{aligned} \tag{3.5}$$

*with boundary conditions* $t_n(x,b) = 0$ *for* $(x,b) \notin A$.

*Example 3.2.* Consider the following jump-diffusion processes with Markov's switching:

$$dS(t) = S(t)(Z(t)dt + \sigma dB(t)) + \int_{R\setminus\{0\}} \beta(Z(t-))uN(dt,du), \tag{3.6}$$

where constant $\sigma > 0$, $B(t)$ is a one-dimensional Brownian motion; $N(dt,du)$ is a stationary point process and independent of $B(t)$ such that $\widetilde{N}(dt,du) = N(dt,du) - \Pi(du)dt$ is the compensated Poisson random measure on $[0,\infty) \times R$. $\beta(1)$ and $\beta(2)$ are any given real numbers, and $Z(t)$ is a two-state random jump process on $M = \{1,-1\}$ with generator given by

$$\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}. \tag{3.7}$$

When $\Lambda = \sigma^2 = 1$ and $\beta(1) = \beta(-1) = 1$, we have the following recursive relation for the moments of first passage time of (3.6):

$$-t_n(x,1) + \frac{1}{2}x^2 t_n''(x,1) + xt_3'(x,1) + \int t_n(x+u)\pi(du) + (t_n(x,-1) - t_n(x,1)) = -nt_{n-1}(x,1) \tag{3.8}$$

with boundary conditions $t_n(x,1) = 0$, for $(x,1) \notin A$

$$\begin{aligned} &-t_n(x,-1) + \frac{1}{2}x^2 t_n''(x,-1) + xt_3'(x,-1) + \int t_n(x+u)\pi(du) - 2(t_n(x,1) - t_n(x,-1)) \\ &\quad = -nt_{n-1}(x,-1), \end{aligned} \tag{3.9}$$

with boundary conditions $t_n(x,-1) = 0$ for $(x,-1) \notin A$.

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