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Research Article

Large Deviations for Stochastic Differential Equations on S^d Associated with the Critical Sobolev Brownian Vector Fields

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We obtain a large deviation principle for the stochastic differential equations on the sphere S^d associated with the critical Sobolev Brownian vector fields.

1. Introduction

The purpose of our paper is to prove a large deviation principle on the asymptotic behavior of the stochastic differential equations on the sphere S^d associated with a critical Sobolev Brownian vector field which was constructed by Fang and Zhang [1].

Recall that Schilder theorem states that if B is the real Brownian motion and $C_0[0,1]$ is the space of real continuous functions defined on $[0,1]$, null at 0, which endowed with the uniform norm, then for any open set $G \subset C_0[0,1]$ and closed set $F \subset C_0[0,1]$,

$$\begin{aligned} &\liminf_{\varepsilon \to 0} \varepsilon^2 \log P(\varepsilon B \in G) \geq -\inf_{f \in G} I_0(f), \\ &\limsup_{\varepsilon \to 0} \varepsilon^2 \log P(\varepsilon B \in F) \leq -\inf_{f \in F} I_0(f), \end{aligned} \tag{1.1}$$

with

$$I_0(f) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^1 \left|\dot{f}\right|^2 ds, & f \text{ absolutely continuous}, \\ \infty, & \text{otherwise}. \end{cases} \tag{1.2}$$

This result was then generalized by Freidlin and Wentzell in their famous paper [2] by considering the Itö equation

$$dx_t^{\varepsilon} = \varepsilon\sigma\left(x_t^{\varepsilon}\right)dW(t) + b\left(x_t^{\varepsilon}\right)dt, \quad x_0^{\varepsilon} = x. \tag{1.3}$$

They proved a large deviation principle for the above equation under usual Lipschitz conditions.

Recently, Ren and Zhang in [3] proved a large deviation principle for flows associated with differential equations with non-Lipschitz coefficients by using the weak convergence approach which is systematically developed in [4], and as an application, they established a Schilder Theorem for Brownian motion on the group of diffeomorphisms of the circle.

In this paper, we consider the large deviation principle of the critical Sobolev isotropic Brownian flows on the sphere S^d which is defined by the following SDE:

$$dx_t = \sum_{\ell=1}^{\infty}\left\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}A_{\ell,k}^1(x_t)\circ dB_{\ell,k}^1(t) + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}A_{\ell,k}^2(x_t)\circ dB_{\ell,k}^2(t)\right\}, \quad x_0 = x, \tag{1.4}$$

where $A_{\ell,k}^i$ are eigenvector fields of Laplace operator Δ on the sphere S^d with respect to the metric $H^{(d+2)/2}$. $D_{\ell,1} = \dim \mathcal{G}_\ell$, $D_{\ell,2} = \dim \mathfrak{D}_\ell$, $\mathcal{G}_\ell$, and $\mathfrak{D}_\ell$ are the eigenspaces of eigenvalues $-c_{\ell,d}$ and $-c_{\ell,\delta}$, respectively.

The authors in [1] consider the stochastic differential equations on S^d

$$dx_t^n = \sum_{\ell=1}^{2^n}\left\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}A_{\ell,k}^1\left(x_t^n\right)\circ dB_{\ell,k}^1(t) + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}A_{\ell,k}^2\left(x_t^n\right)\circ dB_{\ell,k}^2(t)\right\}, \quad x_0^n = x. \tag{1.5}$$

Let $\theta_n(t) = d(x_t^n, x_t^{n+1})$, and there exists a real-valued Brownian motion $W_n(t)$ such that

$$d\theta_n(t) = -\sigma_n(t)dW_n(t) - B_n(t)dt, \tag{1.6}$$

therefore, the coefficients of SDEs which defined the Brownian motion on S^d with respect to the metric $H^{d+2/2}$ are non-Lipschitz (see Lemma 4.2 or page 582–585 [1] and Theorem 2.3 in [1]).

Because of the complex structure of this equation, it seems hard to prove the large deviation principle for the small perturbation of the equation by using its recursive approximating system as Ren and Zhang did in [3]. We will adopt a different approach which is similar to those of Fang and Zhang [1] and Liang [5]. We first work with the solution $x^{n,\varepsilon}$ of (5.1) (below) driven by finitely many Brownian motions, and this equation has smooth coefficients, so the large deviation principle for this equation is well known. Next, we show that $x^{n,\varepsilon} \to x^{\varepsilon}$ is exponentially fast, which together with the special relation of rate functions guaranties that the large deviation estimate of $x^{n,\varepsilon}$ can be transferred to x^{ε}, where x^{ε} is the solution of the small perturbed system (3.1).

The rest of the paper is organized as follows. In Section 2, we recall the critical Sobolev isotropic Brownian flows on the sphere S^d. In Section 3, we introduce the main result. Section 4 is devoted to the study of the rate function. The large deviation principle is proved in Section 5.

2. Framework

Let Δ be the Laplace operator on S^d, acting on vector fields. The spectrum of Δ is given by spectrum $(\Delta) = \{-c_{\ell,d};\ \ell \geq 1\} \cup \{-c_{\ell,\delta};\ \ell \geq 1\}$, where $c_{\ell,d} = \ell(\ell+d-1)$, $c_{\ell,\delta} = (\ell+1)(\ell+d-2)$. Let $\mathcal{G}_\ell$ be the eigenspace associated to $c_{\ell,d}$ and $\mathfrak{D}_\ell$ the eigenspace associated to $c_{\ell,\delta}$. Their dimensions will be denoted by $D_{\ell,1} = \dim \mathcal{G}_\ell$, $D_{\ell,2} = \dim \mathfrak{D}_\ell$. It is known (see [6]) that

$$D_{\ell,1} = O\left(\ell^{d-1}\right), \quad D_{\ell,2} = O\left(\ell^{d-1}\right) \quad \text{as } \ell \longrightarrow +\infty. \tag{2.1}$$

Denote by $\{A^i_{\ell,k};\ k = 1, \ldots, D_{\ell,i},\ \ell \geq 1\}$ for $i = 1, 2$ the orthonormal basis of $\mathcal{G}_\ell$ and $\mathfrak{D}_\ell$ in L^2; that is,

$$\int_{S^d} \left\langle A^i_{\ell,k}(x), A^j_{\alpha,\beta}(x) \right\rangle dx = \delta_{ij}\delta_{\ell\alpha}\delta_{k\beta}, \tag{2.2}$$

where δ_{ij} is the Kronecker symbol and dx is the normalized Riemannian measure on S^d, which is the unique one invariant by actions of $g \in \mathrm{SO}(d+1)$. By Weyl theorem, the vector fields $\{A^i_{\ell,k}\}$ are smooth. For more detailed properties of the eigenvector fields, we refer the reader to Appendix A in [1].

Let $s > 0$ and $H^s(S^d)$ be the Sobolev space of vector fields on S^d, which is the completion of smooth vector fields with respect to the norm

$$\|V\|^2_{H^s} = \int_{S^d} \langle (-\Delta + 1)^s V, V \rangle dx. \tag{2.3}$$

Then, $\{A^1_{\ell,k}/(1 + c_{\ell,d})^{s/2},\ A^2_{\ell,\beta}/(1 + c_{\ell,\delta})^{s/2};\ \ell \geq 1,\ 1 \leq k \leq D_{\ell,1},\ 1 \leq \beta \leq D_{\ell,2}\}$ is an orthonormal basis of H^s. If we consider

$$a_\ell = \frac{a}{(\ell-1)^{1+\alpha}}, \quad b_\ell = \frac{b}{(\ell-1)^{1+\alpha}}, \quad \alpha > 0,\ a,\ b > 0,\ \ell \geq 2, \tag{2.4}$$

then

$$\sqrt{\frac{a_\ell}{D_{\ell,1}}} = O\left(\frac{1}{\ell^{(\alpha+d)/2}}\right), \qquad \sqrt{\frac{b_\ell}{D_{\ell,2}}} = O\left(\frac{1}{\ell^{(\alpha+d)/2}}\right). \tag{2.5}$$

Let $\{B^i_{\ell,k}(t);\ \ell \geq 1,\ 1 \leq k \leq D_{\ell,i}\}$ for $i = 1, 2$ be two family of independent standard Brownian motions defined on a probability space $(\Omega, \mathcal{F}, P)$. Consider the series

$$W_t(\omega) = \sum_{\ell \geq 1} \left\{ \sqrt{\frac{da_\ell}{D_{\ell,1}}} \sum_{k=1}^{D_{\ell,1}} B^1_{\ell,k}(t) A^1_{\ell,k} + \sqrt{\frac{db_\ell}{D_{\ell,2}}} \sum_{k=1}^{D_{\ell,2}} B^2_{\ell,k}(t) A^2_{\ell,k} \right\}, \tag{2.6}$$

which converges in L^2, uniformly with respect to t in any compact subset of $[0, +\infty[$. According to (2.5), $(W_t)_{t \geq 0}$ is a *cylindrical* Brownian motion in the Sobolev space $H^{(\alpha+d)/2}$. Moreover,

W_t takes values in the space $H^s(S^d)$ for any $0 < s < \alpha/2$. By Sobolev embedding theorem, in order to ensure that W_t takes values in the space of C^2 vector fields, α must be *larger* than $d+2$. In this later case, the classical Kunita's framework [7] can be applied to integrate the vector field W_t so that we obtain a flow of diffeomorphisms. For the case of small α, the notion of statistical solutions was introduced in [6], and the phenomenon of phase transition appears. It was also shown in [6] that the statistical solutions give rise to a flow of maps if $\alpha > 2$ and the solution is not a flow of maps if $0 < \alpha < 2$ The critical case $\alpha = 2$ was studied in [1]. Instead of introducing $(W_t)_{t\geq 0}$ as in (2.6), the authors in [1] consider first the stochastic differential equations on S^d

$$dx_t^n = \sum_{\ell=1}^{2^n} \left\{ \sqrt{\frac{da_\ell}{D_{\ell,1}}} \sum_{k=1}^{D_{\ell,1}} A_{\ell,k}^1(x_t^n) \circ dB_{\ell,k}^1(t) + \sqrt{\frac{db_\ell}{D_{\ell,2}}} \sum_{k=1}^{D_{\ell,2}} A_{\ell,k}^2(x_t^n) \circ dB_{\ell,k}^2(t) \right\}, \quad x_0^n = x. \tag{2.7}$$

Using the specific properties of eigenvector fields, it was proved that $x_t^n(x)$ converges uniformly in $(t, x) \in [0, T] \times S^d$ to a solution of the sde (2.8) below. We quote the following result from [1].

Theorem A (see [1]). *Let $\alpha = 2$ in definition* (2.4). *Then, the stochastic differential equation on S^d*

$$dx_t = \sum_{\ell=1}^{\infty} \left\{ \sqrt{\frac{da_\ell}{D_{\ell,1}}} \sum_{k=1}^{D_{\ell,1}} A_{\ell,k}^1(x_t) \circ dB_{\ell,k}^1(t) + \sqrt{\frac{db_\ell}{D_{\ell,2}}} \sum_{k=1}^{D_{\ell,2}} A_{\ell,k}^2(x_t) \circ dB_{\ell,k}^2(t) \right\}, \quad x_0 = x \tag{2.8}$$

admits a unique strong solution $(x_t(x))_{t\geq 0}$, which gives rise to a flow of homeomorphisms.

In the case of the circle S^1, this property of flows of homeomorphisms was discovered in [8] then studied in [9, 10].

3. Statement of the Result

Consider the small perturbation of (2.8)

$$dx_t^\varepsilon = \varepsilon \sum_{\ell=1}^{\infty} \left\{ \sqrt{\frac{da_\ell}{D_{\ell,1}}} \sum_{k=1}^{D_{\ell,1}} A_{\ell,k}^1(x_t^\varepsilon) \circ dB_{\ell,k}^1(t) + \sqrt{\frac{db_\ell}{D_{\ell,2}}} \sum_{k=1}^{D_{\ell,2}} A_{\ell,k}^2(x_t^\varepsilon) \circ dB_{\ell,k}^2(t) \right\}, \quad x_0^\varepsilon = x \tag{3.1}$$

Equation (3.1) has a unique strong solution $(x_t^\varepsilon(x))_{t\geq 0}$ according to Theorem A, denoted by x_t^ε.

We consider the abstract Wiener space $(\Omega, \mathscr{H}, \mathcal{F}, P)$ associated with Wiener processes $W(s) = \{B_{\ell,k}^i(t);\ \ell \geq 1,\ 1 \leq k \leq D_{\ell,i},\ i = 1, 2\}$. P is the Wiener measure and

$$\mathscr{H} = \left\{ h = \left\{ h_{\ell,k}^i(t) \right\},\ \ell \geq 1,\ 1 \leq k \leq D_{\ell,i},\ i = 1, 2,\ \|h\|_{\mathscr{H}}^2 < \infty \right\} \tag{3.2}$$

is the Cameron-Martin space associated with W, where

$$\|h\|_{\mathscr{H}}^2 = \sum_{\ell\geq 1}\left\{\sum_{k=1}^{D_{\ell,1}}\int_0^T \left|\dot{h}_{\ell,k}^1(t)\right|^2 dt + \sum_{k=1}^{D_{\ell,2}}\int_0^T \left|\dot{h}_{\ell,k}^2(t)\right|^2 dt\right\}. \tag{3.3}$$

The purpose of this paper is to prove a large deviation principle for the family $\{x^\varepsilon,\ \varepsilon > 0\}$ in the space $C_x([0,T],S^d)$ and the collection of continuous functions f from $[0,T]$ into S^d with $f(0) = x$. To state the result, let us introduce the rate function. For any $h \in \mathscr{H}$, let $\{S^h(t),\ t \in [0,T]\}$ be the solution of

$$dS^h(t) = \sum_{\ell=1}^{\infty}\left\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}} A_{\ell,k}^1\left(S^h(t)\right)\dot{h}_{\ell,k}^1(t)dt + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}} A_{\ell,k}^2\left(S^h(t)\right)\dot{h}_{\ell,k}^2(t)dt\right\}, \quad S^h(0) = x. \tag{3.4}$$

And for any $f \in C_x([0,T],S^d)$, let

$$I(f) = \inf\left\{\frac{1}{2}\|h\|_{\mathscr{H}}^2 : f = S^h,\ h \in \mathscr{H}\right\}. \tag{3.5}$$

We recall the definition of the good rate function.

Definition 3.1. A function I mapping a metric space E into $[0,\infty]$ is called a good rate function if for each $a < \infty$, the level set $\{f \in E : I(f) \leq a\}$ is compact.

Our main result reads as follows.

Theorem 3.2. *Let x_t^ε be the solution of* (3.1) *on $C_x([0,T],S^d)$, then $\{x_t^\varepsilon,\ \varepsilon > 0\}$ satisfies a large deviation principle with a good rate function $I(f)$, $f \in C_x([0,T],S^d)$; that is,*

(i) *for any closed subset $C \subset C_x([0,T],S^d)$,*

$$\limsup_{\varepsilon\to 0}\varepsilon^2 \log P(x^\varepsilon \in C) \leq -\inf_{f\in C} I(f), \tag{3.6}$$

(ii) *for any open set $G \subset C_x([0,T],S^d)$,*

$$\liminf_{\varepsilon\to 0}\varepsilon^2 \log P(x^\varepsilon \in G) \geq -\inf_{f\in G} I(f). \tag{3.7}$$

4. Skeleton Equation and the Rate Function

Theorem 4.1. *For any $h \in \mathscr{H}$,* (3.4) *has a unique solution, denoted by $S^h(t)$.*

In order to prove Theorem 4.1, we now introduce the following estimates which is Theorem 2.3 in [1].

Lemma 4.2. *Let*

$$\sigma_n(\theta) = -\frac{\sqrt{U_n(\theta)}}{\sin\theta},$$
$$B_n(\theta) = \frac{V_n(\theta)}{\sin\theta} + \frac{1}{2}\frac{\cos\theta}{\sin^3\theta}U_n(\theta). \tag{4.1}$$

U_n, V_n is defined respectively, by (2.14) and (2.13) in [1]. Then, there exist some constants $N > 0$, $c > 0$ such that for any $n > N$,

$$\sigma_n^2(\theta) \le C\theta^2 \log\frac{2\pi}{\theta} + 2^{-n},$$
$$-B_n(\theta) \le C\theta \log\frac{2\pi}{\theta} + 2^{-n}. \tag{4.2}$$

Proof of Theorem 4.1. Let $S^{n,h}$ be the solution of the following system:

$$dS^{n,h}(t) = \sum_{\ell=1}^{2^n}\left\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}A^1_{\ell,k}\left(S^{n,h}(t)\right)\dot{h}^1_{\ell,k}(t)dt + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}A^2_{\ell,k}\left(S^{n,h}(t)\right)\dot{h}^2_{\ell,k}(t)dt\right\},$$
$$S^h(0) = x. \tag{4.3}$$

Since $A^i_{\ell,k}$ are smooth, the solution of (4.3) exists.

For $x, y \in S^d$, consider the Riemannian distance $d(x,y)$ defined by

$$\cos d(x,y) = \langle x, y\rangle, \tag{4.4}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in R^{d+1}. Let $|\cdot|$ denote the Euclidean distance. We have the relation

$$|x-y| \le d(x,y) \le \frac{\pi}{2}|x-y|. \tag{4.5}$$

Our aim is to show that $S^{n,h}$ converges to a solution of (3.4). By the chain rule,

$$d\left\langle S^{n,h}(t), S^{n+1,h}(t)\right\rangle = \left\langle dS^{n,h}(t), S^{n+1,h}(t)\right\rangle + \left\langle S^{n,h}(t), dS^{n+1,h}(t)\right\rangle$$
$$= \sum_{\ell=1}^{2^n}\left[\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\left\langle S^{n+1,h}(t), A^1_{\ell,k}\left(S^{n,h}(t)\right)\right\rangle\dot{h}^1_{\ell,k}(t)\right.$$
$$\left.+\sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\left\langle S^{n+1,h}(t), A^2_{\ell,k}\left(S^{n,h}(t)\right)\right\rangle\dot{h}^2_{\ell,k}(t)\right]dt$$

$$+\sum_{\ell=1}^{2^{n+1}}\left[\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\left\langle S^{n,h}(t), A^1_{\ell,k}\left(S^{n+1,h}(t)\right)\right\rangle \dot{h}^1_{\ell,k}(t) + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\left\langle S^{n,h}(t), A^2_{\ell,k}\left(S^{n+1,h}(t)\right)\right\rangle \dot{h}^2_{\ell,k}(t)\right]dt. \tag{4.6}$$

Let $\theta_t^n = d(S^{n,h}(t), S^{n+1,h}(t))$, then

$$d\theta_t^n = -\frac{1}{\sin\theta_t^n}d\left\langle S^{n,h}(t), S^{n+1,h}(t)\right\rangle. \tag{4.7}$$

Let

$$\begin{aligned}
&I_1(t)\\
&= -\int_0^t \frac{1}{\sin\theta_s^n}\\
&\quad\times\sum_{\ell=1}^{2^n}\left[\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\left(\left\langle S^{n+1,h}(s), A^1_{\ell,k}\left(S^{n,h}(s)\right)\right\rangle + \left\langle S^{n,h}(s), A^1_{\ell,k}\left(S^{n+1,h}(s)\right)\right\rangle\right)\dot{h}^1_{\ell,k}(s)\right.\\
&\qquad\left.+\sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\left(\left\langle S^{n+1,h}(s), A^2_{\ell,k}\left(S^{n,h}(s)\right)\right\rangle + \left\langle S^{n,h}(s), A^2_{\ell,k}\left(S^{n+1,h}(s)\right)\right\rangle\right)\dot{h}^2_{\ell,k}(s)\right]ds,\\
&I_2(t)\\
&= -\int_0^t \frac{1}{\sin\theta_s^n}\sum_{\ell=2^n+1}^{2^{n+1}}\left[\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\left\langle S^{n,h}(s), A^1_{\ell,k}\left(S^{n+1,h}(s)\right)\right\rangle \dot{h}^1_{\ell,k}(s)\right.\\
&\qquad\left.+\sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\left\langle S^{n,h}(s), A^2_{\ell,k}\left(S^{n+1,h}(s)\right)\right\rangle \dot{h}^2_{\ell,k}(s)\right]ds.
\end{aligned} \tag{4.8}$$

We have

$$\begin{aligned}
I_1^2(t) \le \Bigg(&\int_0^t \frac{1}{\sin\theta_s^n}\\
&\times\left(\sum_{\ell=1}^{2^n}\left[\frac{da_\ell}{D_{\ell,1}}\sum_{k=1}^{D_{\ell,1}}\left(\left\langle S^{n+1,h}(s), A^1_{\ell,k}\left(S^{n,h}(s)\right)\right\rangle + \left\langle S^{n,h}(s), A^1_{\ell,k}\left(S^{n+1,h}(s)\right)\right\rangle\right)^2\right.\right.\\
&\qquad\left.\left.+\frac{db_\ell}{D_{\ell,2}}\sum_{k=1}^{D_{\ell,2}}\left(\left\langle S^{n+1,h}(s), A^2_{\ell,k}\left(S^{n,h}(s)\right)\right\rangle + \left\langle S^{n,h}(s), A^2_{\ell,k}\left(S^{n+1,h}(s)\right)\right\rangle\right)^2\right]\right)^{1/2}\\
&\times\left(\sum_{\ell=1}^{2^n}\left[\sum_{k=1}^{D_{\ell,1}}\left|\dot{h}^1_{\ell,k}(s)\right|^2 + \sum_{k=1}^{D_{\ell,2}}\left|\dot{h}^2_{\ell,k}(s)\right|^2\right]\right)^{1/2} ds\Bigg)^2.
\end{aligned} \tag{4.9}$$

Using Proposition A.4 in [1] and Lemma 4.2, we see that

$$
\begin{aligned}
I_1^2(t) \le &\left(\int_0^t \left(2\sum_{\ell=1}^{2^n}\left\{a_\ell\left[1-\cos\theta_s^n\gamma_\ell(\cos\theta_s^n)+\sin^2\theta_s^n\gamma_\ell'(\cos\theta_s^n)\right]+b_\ell\left[1-\gamma_\ell(\cos\theta_s^n)\right]\right\}\right)^{1/2}\right.\\
&\left.\times\left(\sum_{\ell=1}^{2^n}\left[\sum_{k=1}^{D_{\ell,1}}\left|\dot h_{\ell,k}^1(s)\right|^2+\sum_{k=1}^{D_{\ell,2}}\left|\dot h_{\ell,k}^2(s)\right|^2\right]\right)^{1/2} ds\right)^2\\
\le &\, 2\int_0^t\left(\sum_{\ell=1}^{2^n}\left\{a_\ell\left[1-\cos\theta_s^n\gamma_\ell(\cos\theta_s^n)+\gamma_\ell'(\cos\theta_s^n)\right]+b_\ell\left[1-\gamma_\ell(\cos\theta_s^n)\right]\right\}\right)ds\\
&\times\int_0^t\sum_{\ell=1}^{2^n}\left[\sum_{k=1}^{D_{\ell,1}}\left|\dot h_{\ell,k}^1(s)\right|^2+\sum_{k=1}^{D_{\ell,2}}\left|\dot h_{\ell,k}^2(s)\right|^2\right]ds\\
\le &\, 2\|h\|_{\mathcal{H}}^2\int_0^t\left(\sum_{\ell=1}^{2^n}\left\{a_\ell\left[1-\cos\theta_s^n\gamma_\ell(\cos\theta_s^n)+\gamma_\ell'(\cos\theta_s^n)\right]+b_\ell\left[1-\gamma_\ell(\cos\theta_s^n)\right]\right\}\right)ds.
\end{aligned}
\tag{4.10}
$$

Similarly, we have

$$
\begin{aligned}
I_2^2(t) \le &\left(\int_0^t \frac{1}{\sin\theta_s^n}\left(\sum_{\ell=2^n+1}^{2^{n+1}}\left[\frac{da_\ell}{D_{\ell,1}}\sum_{k=1}^{D_{\ell,1}}\left\langle S^{n,h}(s), A_{\ell,k}^1\left(S^{n+1,h}(s)\right)\right\rangle^2\right.\right.\right.\\
&\left.\left.+\frac{db_\ell}{D_{\ell,2}}\sum_{k=1}^{D_{\ell,2}}\left\langle S^{n,h}(s), A_{\ell,k}^2\left(S^{n+1,h}(s)\right)\right\rangle^2\right]\right)^{1/2}\\
&\left.\times\left(\sum_{\ell=2^n+1}^{2^{n+1}}\left[\sum_{k=1}^{D_{\ell,1}}\left|\dot h_{\ell,k}^1(s)\right|^2+\sum_{k=1}^{D_{\ell,2}}\left|\dot h_{\ell,k}^2(s)\right|^2\right]\right)^{1/2} ds\right)^2\\
\le &\, 2\|h\|_{\mathcal{H}}^2\int_0^t\sum_{\ell=2^n}^{2^{n+1}}(a_\ell+b_\ell)ds.
\end{aligned}
\tag{4.11}
$$

Therefore,

$$
\begin{aligned}
\left|\theta_t^n\right|^2 &\le 2I_1^2(t)+2I_2^2(t)\\
&\le 4\|h\|_{\mathcal{H}}^2\left[\int_0^t\sum_{\ell=1}^{2^n}\left\{a_\ell\left[1-\cos\theta_s^n\gamma_\ell(\cos\theta_s^n)+\gamma_\ell'(\cos\theta_s^n)\right]+b_\ell\left[1-\gamma_\ell(\cos\theta_s^n)\right]\right\}ds\right.\\
&\qquad\left.+\int_0^t\sum_{\ell=2^n+1}^{2^{n+1}}(a_\ell+b_\ell)ds\right]
\end{aligned}
$$

$$\begin{aligned}&\leq 4\int_0^T \sigma_n^2(\theta_s^n)ds\\&\leq 4\int_0^t\left((\theta_s^n)^2\log\frac{2\pi}{\theta_s^n}+2^{-n}\right)ds.\end{aligned} \tag{4.12}$$

Using the similar arguments as that in [1], the above inequality implies that there exist constants C_1, C_2 such that

$$\left|\theta_t^n\right|^2 \leq C_1 2^{-ne^{-C_2 t}}, \tag{4.13}$$

and C_1, C_2 are independent of n, t. Hence,

$$\left|S^{n,h}(t)-S^{n+1,h}(t)\right| \leq \left|\theta_t^n\right| \leq C_1 2^{-ne^{-C_2 t}}. \tag{4.14}$$

Thus, $S^{n,h}(t)$ uniformly converges to some function S^h in $C_x([0,T],S^d)$.

Next, we show that $\{S^h(t),\ t\geq 0\}$ satisfies (3.4).

It suffices to show that for any $u\in S^d$,

$$\begin{aligned}d\left\langle u,S^h(t)\right\rangle = \sum_{\ell=1}^{\infty}\Bigg\{&\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\left\langle u,A_{\ell,k}^1\left(S^h(t)\right)\right\rangle \dot{h}_{\ell,k}^1(t)dt\\&+\sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\left\langle u,A_{\ell,k}^2\left(S^h(t)\right)\right\rangle \dot{h}_{\ell,k}^2(t)dt\Bigg\}.\end{aligned} \tag{4.15}$$

Set $\eta_t=\langle u,S^h(t)\rangle$, $\eta_t^n=\langle u,S^{n,h}(t)\rangle$, $\theta_t^n=d(u,S^{n,h}(t))$, and $\theta_t=d(u,S^h(t))$.

Fix $N_0>0$, and by Proposition A.4 in [1] and Lemma 4.2, we have

$$\begin{aligned}&\left|\sum_{\ell=N_0}^{2^n}\int_0^t\left\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\left\langle u,A_{\ell,k}^1\left(S^{n,h}(s)\right)\right\rangle\dot{h}_{\ell,k}^1(s)ds+\sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\left\langle u,A_{\ell,k}^2\left(S^{n,h}(s)\right)\right\rangle\dot{h}_{\ell,k}^2(s)ds\right\}\right|^2\\&\quad\leq\int_0^t\sum_{\ell=N_0}^{2^n}\left(\frac{da_\ell}{D_{\ell,1}}\sum_{k=1}^{D_{\ell,1}}\left\langle u,A_{\ell,k}^1\left(S^{n,h}(s)\right)\right\rangle^2+\frac{db_\ell}{D_{\ell,2}}\sum_{k=1}^{D_{\ell,2}}\left\langle u,A_{\ell,k}^2\left(S^{n,h}(s)\right)\right\rangle^2\right)ds\\&\qquad\times\int_0^t\sum_{\ell=N_0}^{2^n}\left(\sum_{k=1}^{D_{\ell,1}}\left|\dot{h}_{\ell,k}^1(s)\right|^2+\sum_{k=1}^{D_{\ell,2}}\left|\dot{h}_{\ell,k}^2(s)\right|^2ds\right)\\&\quad=\int_0^t\sum_{\ell=N_0}^{2^n}(a_\ell+b_\ell)\sin^2\theta_s^n ds\times\int_0^t\sum_{\ell=N_0}^{2^n}\left(\sum_{k=1}^{D_{\ell,1}}\left|\dot{h}_{\ell,k}^1(s)\right|^2+\sum_{k=1}^{D_{\ell,2}}\left|\dot{h}_{\ell,k}^2(s)\right|^2\right)ds\\&\quad\leq t\|h\|_{\mathscr{H}}^2\sum_{N_0}^{2^n}(a_\ell+b_\ell)\leq t\|h\|_{\mathscr{H}}^2\sum_{N_0}^{\infty}(a_\ell+b_\ell).\end{aligned} \tag{4.16}$$

Thus, for any $\varepsilon > 0$, there exists $N_0 > 0$, for $2^n > N_0$,

$$\left|\sum_{\ell=N_0}^{2^n}\int_0^t\left\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\left\langle u, A^1_{\ell,k}\left(S^{n,h}(s)\right)\right\rangle \dot{h}^1_{\ell,k}(s)ds+\sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\left\langle u, A^2_{\ell,k}\left(S^{n,h}(s)\right)\right\rangle \dot{h}^2_{\ell,k}(s)ds\right\}\right| < \frac{\varepsilon}{2}. \tag{4.17}$$

By similar reasons, we also have

$$\begin{aligned}&\left|\sum_{\ell=N_0}^{\infty}\int_0^t\left\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\left\langle u, A^1_{\ell,k}\left(S^{h}(s)\right)\right\rangle \dot{h}^1_{\ell,k}(s)ds+\sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\left\langle u, A^2_{\ell,k}\left(S^{h}(s)\right)\right\rangle \dot{h}^2_{\ell,k}(s)\right\}ds\right|\\ &\quad\le t\|h\|^2_{\mathscr{H}}\sum_{N_0}^{\infty}(a_\ell+b_\ell) < \frac{\varepsilon}{2}.\end{aligned} \tag{4.18}$$

On the other hand, because $S^{n,h} \Rightarrow S^h$ in $C_x([0,T], S^d)$, for any $\varepsilon > 0$, one can find $N_1 > 0$ such that for $n > N_1$,

$$\begin{aligned}&\left|\sum_{\ell=1}^{N_0-1}\int_0^t\left\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\left\langle u, A^1_{\ell,k}\left(S^{n,h}(s)\right)-A^1_{\ell,k}\left(S^{h}(s)\right)\right\rangle \dot{h}^1_{\ell,k}(s)ds\right.\right.\\ &\qquad\left.\left.+\sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\left\langle u, A^2_{\ell,k}\left(S^{n,h}(s)\right)-A^2_{\ell,k}\left(S^{h}(s)\right)\right\rangle \dot{h}^2_{\ell,k}(s)ds\right\}\right| < \frac{\varepsilon}{2}.\end{aligned} \tag{4.19}$$

Therefore, for any $\varepsilon > 0$, one can find $N_2 > 0$ such that for $n > N_2$,

$$\begin{aligned}&\left|\sum_{\ell=1}^{N_0-1}\int_0^t\left\{\frac{da_\ell}{D_{\ell,1}}\sum_{k=1}^{D_{\ell,1}}\left\langle u, A^1_{\ell,k}\left(S^{n,h}(s)\right)-A^1_{\ell,k}\left(S^{h}(s)\right)\right\rangle \dot{h}^1_{\ell,k}(s)ds\right.\right.\\ &\qquad\left.\left.+\sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\left\langle u, A^2_{\ell,k}\left(S^{n,h}(s)\right)-A^2_{\ell,k}\left(S^{h}(s)\right)\right\rangle \dot{h}^2_{\ell,k}(s)ds\right\}\right|\\ &\quad+\left|\sum_{\ell=N_0}^{2^n}\int_0^t\left\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\left\langle u, A^1_{\ell,k}\left(S^{n,h}(s)\right)\right\rangle \dot{h}^1_{\ell,k}(s)ds\right.\right.\\ &\qquad\left.\left.+\sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\left\langle u, A^2_{\ell,k}\left(S^{n,h}(s)\right)\right\rangle \dot{h}^2_{\ell,k}(s)ds\right\}\right|\\ &\quad+\left|\sum_{\ell=N_0}^{\infty}\int_0^t\left\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\left\langle u, A^1_{\ell,k}\left(S^{h}(s)\right)\right\rangle \dot{h}^1_{\ell,k}(s)ds\right.\right.\\ &\qquad\left.\left.+\sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\left\langle u, A^2_{\ell,k}\left(S^{h}(s)\right)\right\rangle \dot{h}^2_{\ell,k}(s)ds\right\}\right|\\ &\quad< \varepsilon.\end{aligned} \tag{4.20}$$

Since ε is arbitrary, we obtain that

$$
\begin{aligned}
d\left\langle u, S^h(t)\right\rangle = \sum_{\ell=1}^{\infty}\Bigg\{ &\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\left\langle u, A^1_{\ell,k}\left(S^h(t)\right)\right\rangle \dot{h}^1_{\ell,k}(t)dt \\
&+\sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\left\langle u, A^2_{\ell,k}\left(S^h(t)\right)\right\rangle \dot{h}^2_{\ell,k}(t)dt\Bigg\}.
\end{aligned}
\tag{4.21}
$$

The uniqueness is deduced from similar estimates. □

Lemma 4.3. *For any $N > 0$, the set $\{S^h : \|h\|_{\mathscr{H}} \le N\}$ is relatively compact in $C_x([0,T], S^d)$.*

Proof. By the Ascoli-Arzela lemma, we need to show that $\{S^h : \|h\|_{\mathscr{H}} \le N\}$ is uniformly bounded and equicontinuous. The first fact is obvious, because $\|S^h\| = 1$ for any $h \in \mathscr{H}$. Next, we will show that $\{S^h : \|h\|_{\mathscr{H}} \le N\}$ is equicontinuous.

Let $\{u_i,\ i = 1, \ldots, d+1\}$ be an orthonormal basis of R^{d+1}, and by Proposition A.4 in [1] and Lemma 4.2, we have

$$
\begin{aligned}
\left|\left\langle S^h(t) - S^h(s), u_i\right\rangle\right|^2 &= \Bigg|\int_s^t \sum_{\ell=1}^{\infty}\Bigg[\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\left\langle A^1_{\ell,k}\left(S^h(u)\right), u_i\right\rangle \dot{h}^1_{\ell,k}(u)du \\
&\qquad +\sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\left\langle A^2_{\ell,k}\left(S^h(u)\right), u_i\right\rangle \dot{h}^2_{\ell,k}(u)du\Bigg]\Bigg|^2 \\
&\le \int_s^t \sum_{\ell=1}^{\infty}\left(\frac{da_\ell}{D_{\ell,1}}\sum_{k=1}^{D_{\ell,1}}\left\langle A^1_{\ell,k}\left(S^h(u)\right), u_i\right\rangle^2 + \frac{db_\ell}{D_{\ell,2}}\sum_{k=1}^{D_{\ell,2}}\left\langle A^2_{\ell,k}\left(S^h(u)\right), u_i\right\rangle^2\right)du \\
&\quad \times \int_s^t \sum_{\ell=1}^{\infty}\left(\sum_{k=1}^{D_{\ell,1}}\left|\dot{h}^1_{\ell,k}(u)\right|^2 + \sum_{k=1}^{D_{\ell,2}}\left|\dot{h}^2_{\ell,k}(u)\right|^2\right)du \\
&= \int_s^t \sum_{\ell=1}^{\infty}(a_\ell + b_\ell)\sin^2\theta_u du \times \int_s^t \sum_{\ell=1}^{\infty}\left(\sum_{k=1}^{D_{\ell,1}}\left|\dot{h}^1_{\ell,k}(u)\right|^2 + \sum_{k=1}^{D_{\ell,2}}\left|\dot{h}^2_{\ell,k}(u)\right|^2\right)du \\
&\le \sum_{\ell=1}^{\infty}(a_\ell + b_\ell)\|h\|^2_{\mathscr{H}}|t-s|,
\end{aligned}
\tag{4.22}
$$

where $\theta_t = d(S^h(t), u_i)$. Thus,

$$
\left|S^h(t) - S^h(s)\right|^2 = \sum_{i=1}^{d+1}\left|\left\langle S^h(t) - S^h(s), u_i\right\rangle\right|^2 \le (d+1)\sum_{\ell=1}^{\infty}(a_\ell + b_\ell)\|h\|^2_{\mathscr{H}}|t-s|,
\tag{4.23}
$$

which finishes the proof. □

Lemma 4.4. *The mapping $h \to S^h$ is continuous from $\{h : \|h\|_{\mathscr{H}} \le N\}$ with respect to the topology on Ω into $C_x([0,T], S^d)$.*

Proof. Let $h_n \in \mathscr{H}$ with $\|h_n\|_{\mathscr{H}} \leq N$ and assume that h_n converges to h in Ω, then $h_n \to h$ weakly in $\mathscr{H}$. By Lemma 4.2, $\{S^{h_n},\ n \geq 1\}$ is relatively compact. Let $g \in C_x([0,T], S^d)$ be a limit of any convergent subsequence of $\{S^{h_n},\ n \geq 1\}$. We will finish the proof the lemma by showing that $g = S^h$. Now, for simplicity, we drop the subindex k

$$S^{h_n}(t) = x + \int_0^t \sum_{\ell=1}^{\infty}\left[\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}} A^1_{\ell,k}\left(S^{h_n}(u)\right)\dot{h}^1_{n,\ell,k}(u) + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}} A^2_{\ell,k}\left(S^{h_n}(u)\right)\dot{h}^2_{n,\ell,k}(u)\right]du,$$
$$S^{h}(t) = x + \int_0^t \sum_{\ell=1}^{\infty}\left[\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}} A^1_{\ell,k}\left(S^{h}(u)\right)\dot{h}^1_{\ell,k}(u) + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}} A^2_{\ell,k}\left(S^{h}(u)\right)\dot{h}^2_{\ell,k}(u)\right]du. \tag{4.24}$$

It is sufficient to show that $S^{h_n} \Rightarrow S^h$ in $C_x([0,T], S^d)$.

Write $S^{h_n}(t) - S^h(t) = I_3 - I_4$ with I_3, I_4 being given by

$$\begin{aligned} I_3 &= \int_0^t \sum_{\ell=1}^{\infty}\left[\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\left(A^1_{\ell,k}\left(S^{h_n}(u)\right) - A^1_{\ell,k}\left(S^h(u)\right)\right)\dot{h}^1_{n,\ell,k}(u)\right.\\ &\qquad\left.+\sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\left(A^2_{\ell,k}\left(S^{h_n}(u)\right) - A^2_{\ell,k}\left(S^h(u)\right)\right)\dot{h}^2_{n,\ell,k}(u)\right]du,\\ I_4 &= \int_0^t \sum_{\ell=1}^{\infty}\left[\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}} A^1_{\ell,k}\left(S^h(u)\right)\left(\dot{h}^1_{n,\ell,k}(u) - \dot{h}^1_{\ell,k}(u)\right)\right.\\ &\qquad\left.+\sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}} A^2_{\ell,k}\left(S^h(u)\right)\left(\dot{h}^2_{n,\ell,k}(u) - \dot{h}^2_{\ell,k}(u)\right)\right]du. \end{aligned} \tag{4.25}$$

Let $\theta_t = d(S^h(t), S^{h_n}(t))$, and by Proposition A.4 in [1] and Lemma 4.2, we have

$$\begin{aligned} I_3 &\leq \int_0^t \left[\sum_{\ell=1}^{\infty}\left(\frac{da_\ell}{D_{\ell,1}}\sum_{k=1}^{D_{\ell,1}}\left|A^1_{\ell,k}\left(S^{h_n}(u)\right) - A^1_{\ell,k}\left(S^h(u)\right)\right|^2 + \frac{db_\ell}{D_{\ell,2}}\sum_{k=1}^{D_{\ell,2}}\left|A^2_{\ell,k}\left(S^{h_n}(u)\right) - A^2_{\ell,k}\left(S^h(u)\right)\right|^2\right)\right]^{1/2}\\ &\quad\times\left[\sum_{\ell=1}^{\infty}\left(\sum_{k=1}^{D_{\ell,1}}\left|\dot{h}^1_{n,\ell,k}(u)\right|^2 + \sum_{k=1}^{D_{\ell,2}}\left|\dot{h}^2_{n,\ell,k}(u)\right|^2\right)\right]^{1/2} du\\ &\leq \|h_n\|_{\mathscr{H}}\left(\int_0^t \sum_{\ell=1}^{\infty}[2da_\ell - 2a_\ell(d-1+\cos\theta_u)\gamma_\ell(\cos\theta_u) - \cos\theta_u \sin^2\theta_u \gamma'_\ell(\cos\theta_u)\right.\\ &\qquad\left.+2db_\ell - 2b_\ell\left(d\cos\theta_u\gamma_\ell(\cos\theta_u) - \sin^2\theta_u\gamma'_\ell(\cos\theta_u)\right)]\,du\right)^{1/2}\\ &\leq \|h_n\|_{\mathscr{H}}\left(\int_2^t 2\left[daG(0) - a\left((d-1+\cos^2\theta_u)G(\theta_u) + \cos\theta_u\sin\theta_u G'(\theta)\right)\right.\right. \end{aligned}$$

$$
\begin{aligned}
&\qquad +dbG(0) - b\big(d\cos\theta_u G(\theta_u) + \sin\theta_u G'(\theta_u)\big)\big]\,du\Bigg)^{1/2} \\
&\leq \left(\int_0^t C\theta_u^2 \log\frac{2\pi}{\theta_u}\,du\right)^{1/2}.
\end{aligned}
\tag{4.26}
$$

Let

$$
\begin{aligned}
f^1_{\ell,k}(v) &= \int_0^v \sqrt{\frac{da_\ell}{D_{\ell,1}}} A^1_{\ell,k}\Big(S^h(u)\Big) I_{[0,t]}(u)\,du, \\
f^2_{\ell,k}(v) &= \int_0^v \sqrt{\frac{db_\ell}{D_{\ell,2}}} A^2_{\ell,k}\Big(S^h(u)\Big) I_{[0,t]}(u)\,du.
\end{aligned}
\tag{4.27}
$$

Then, $f = ((f^1_{\ell,k}(v))_{\ell\geq 1,\ 1\leq k\leq D_{\ell,1}}, (f^2_{\ell,k}(v))_{\ell\geq 1,\ 1\leq k\leq D_{\ell,2}}) \in \mathscr{H}$, because of

$$
\begin{aligned}
\|f\|_{\mathscr{H}} &= \sum_{\ell=1}^{\infty}\left[\int_0^t \frac{da_\ell}{D_{\ell,1}} \sum_{k=1}^{D_{\ell,1}} \left|A^1_{\ell,k}\Big(S^h(u)\Big)\right|^2 du + \int_0^t \frac{da_\ell}{D_{\ell,2}} \sum_{k=1}^{D_{\ell,2}} \left|A^2_{\ell,k}\Big(S^h(u)\Big)\right|^2 du\right] \\
&= \sum_{\ell=1}^{\infty} \int_0^t (a_\ell + b_\ell)\,du < \infty.
\end{aligned}
\tag{4.28}
$$

Therefore,

$$
\begin{aligned}
I_4 = \langle f, h_n - h\rangle_{\mathscr{H}} &= \int_0^t \sum_{\ell=1}^{\infty}\Bigg[\sqrt{\frac{da_\ell}{D_{\ell,1}}} \sum_{k=1}^{D_{\ell,1}} A^1_{\ell,k}\Big(S^h(u)\Big)\Big(\dot{h}^1_{n\ell,k}(u) - \dot{h}^1_{\ell,k}(u)\Big)du \\
&\qquad + \sqrt{\frac{db_\ell}{D_{\ell,2}}} \sum_{k=1}^{D_{\ell,2}} A^2_{\ell,k}\Big(S^h(u)\Big)\Big(\dot{h}^2_{n\ell,k}(u) - \dot{h}^2_{\ell,k}(u)\Big)\,du\Bigg] \\
&\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned}
\tag{4.29}
$$

Combining above estimates,

$$
\theta_t = d\Big(S^h(t), S^{h_n}(t)\Big) \leq \frac{\pi}{2}\left[\left(\int_0^t C\theta_u^2 \log\frac{2\pi}{\theta_u}\,du\right)^{1/2} + \left|\langle f, h_n - h\rangle_{\mathscr{H}}\right|\right].
\tag{4.30}
$$

Hence,

$$
\theta_t^2 \leq 2C\int_0^t \theta_u^2 \log\frac{2\pi}{\theta_u}\,du + 2\left|\langle f, h_n - h\rangle_{\mathscr{H}}\right|^2.
\tag{4.31}
$$

This implies

$$
\theta_t \leq C_4\left|\langle f, h_n - h\rangle_{\mathscr{H}}\right|^{e^{-C_5 t}},
\tag{4.32}
$$

which yields

$$S^{h_n} \Longrightarrow S^h \quad \text{as } n \Longrightarrow \infty. \tag{4.33}$$

□

Lemma 4.5. *$I(f)$ is a good rate function.*

Proof. For any $a > 0$,

$$\{I(f) \leq a\} = \left\{ S^h, \frac{1}{2}\|h\|_{\mathscr{H}}^2 \leq a \right\} = S^h\left(\|h\|_{\mathscr{H}} \leq \sqrt{2a}\right). \tag{4.34}$$

The subset $\{\|h\|_{\mathscr{H}} \leq \sqrt{2a}\}$ is a compact set in Ω and $h \to S^h$ is a continuous map for any a. Therefore, $\{I(f) \leq a\}$ is a compact set for any a. So, $I(f)$ is a good rate function. □

5. The Proof of Theorem 3.2

Let $x_t^{n,\varepsilon}$ be the solution to

$$dx_t^{n,\varepsilon} = \varepsilon \sum_{\ell=1}^{2^n} \left\{ \sqrt{\frac{da_\ell}{D_{\ell,1}}} \sum_{k=1}^{D_{\ell,1}} A_{\ell,k}^1(x_t^{n,\varepsilon}) \circ dB_{\ell,k}^1(t) + \sqrt{\frac{db_\ell}{D_{\ell,2}}} \sum_{k=1}^{D_{\ell,2}} A_{\ell,k}^2(x_t^{n,\varepsilon}) \circ dB_{\ell,k}^2(t) \right\}, \quad x_0^{n,\varepsilon} = x. \tag{5.1}$$

We first have the following proposition.

Proposition 5.1. *For any $\delta > 0$,*

$$\lim_{n\to\infty} \limsup_{\varepsilon\to 0} \varepsilon^2 \log P\left(\sup_{0\leq t\leq T} \left| x_t^{\varepsilon} - x_t^{n,\varepsilon} \right| > \delta \right) = -\infty. \tag{5.2}$$

Proof. Let $\theta_n^\varepsilon(t) = d(x_t^{n,\varepsilon}, x_t^\varepsilon)$. Using the similar estimates as that in [1] (see pages 582–585), there exists a real-valued Brownian motion $W_n(t)$ such that

$$d\theta_n^\varepsilon(t) = -\varepsilon\sigma_n(t)dW_n(t) - \varepsilon^2 B_n(t)dt, \tag{5.3}$$

where $\sigma_n(t) = \sigma_n(\theta_n^\varepsilon(t))$, $B_n(t) = B_n(\theta_n^\varepsilon(t))$ are defined as in Lemma 4.2.

Let $\xi_n(t) = (\theta_n^\varepsilon)^2(t)$, we have

$$\begin{aligned} d\xi_n(t) &= 2\theta_n^\varepsilon(t)d\theta_n^\varepsilon(t) + d\theta_n^\varepsilon(t)d\theta_n^\varepsilon(t) \\ &= -2\varepsilon\theta_n^\varepsilon(t)\sigma_n(\theta_n^\varepsilon(t))dW_n(t) + \varepsilon^2\left(\sigma_n^2(\theta_n^\varepsilon(t)) - 2\theta_n^\varepsilon(t)B_n(\theta_n^\varepsilon(t))\right)dt, \\ d\xi_n(t)d\xi_n(t) &= 4\varepsilon^2\theta_n^\varepsilon(t)^2\sigma_n^2(\theta_n^\varepsilon(t))dt. \end{aligned} \tag{5.4}$$

Introduce the function $\psi_\rho : (0, e^{-1}) \rightarrow R$ by

$$\psi_\rho(\xi) = \int_0^\xi \frac{ds}{s\log(2\pi/s) + \rho}. \tag{5.5}$$

Then, for any $0 < \xi < 1$,

$$\psi_\rho(\xi) \uparrow \psi_0(\xi) = \int_0^\xi \frac{ds}{s\log(2\pi/s)} = +\infty, \tag{5.6}$$

as $\rho \rightarrow 0$.

Define for $\lambda > 0$,

$$\Phi_{\rho,\lambda}(\xi) = e^{\lambda\psi_\rho(\xi)}. \tag{5.7}$$

We have

$$\begin{aligned}
\Phi'_{\rho,\lambda}(\xi)\left(\xi\log\frac{2\pi}{\xi} + \rho\right) &= \lambda\Phi_{\rho,\lambda}(\xi), \\
\Phi''_{\rho,\lambda}(\xi) &= \lambda^2\Phi_{\rho,\lambda}(\xi)\frac{1}{\xi\log(2\pi/\xi) + \rho} + \lambda\Phi_{\rho,\lambda}(\xi)\frac{1 - \log(2\pi/\xi)}{(\xi\log(2\pi/\xi) + \rho)^2} \\
&\leq \lambda^2\Phi_{\rho,\lambda}(\xi)\frac{1}{(\xi\log(2\pi/\xi) + \rho)^2}, \quad \text{if } \xi \leq \mathrm{e}^{-1}.
\end{aligned} \tag{5.8}$$

Without loss of generality, we may assume $\delta < e^{-1}$. Define $\tau_n = \inf\{t \geq 0,\ \theta_n^\varepsilon(t) > \delta\}$. By Itö formula, we have

$$\begin{aligned}
\Phi_{\rho,\lambda}(\xi_n(t\wedge\tau_n)) &= 1 + \int_0^{t\wedge\tau_n} \Phi'_{\rho,\lambda}(\xi_n(s))d\xi_n(s) + \frac{1}{2}\int_0^{t\wedge\tau_n} \Phi''_{\rho,\lambda}(\xi_n(s))d\xi_n(s)d\xi_n(s) \\
&= 1 + 2\varepsilon\int_0^{t\wedge\tau_n} \Phi'_{\rho,\lambda}(\xi_n(s))(-\theta_n^\varepsilon(s)\sigma_n(\theta_n^\varepsilon(s)))dW_n(s) \\
&\quad + \varepsilon^2\int_0^{t\wedge\tau_n} \Phi'_{\rho,\lambda}(\xi_n(s))\left(\sigma_n^2(\theta_n^\varepsilon(s))\right)ds \\
&\quad - 2\varepsilon^2\int_0^{t\wedge\tau_n} \Phi'_{\rho,\lambda}(\xi_n(s))(\theta_n^\varepsilon(s)B_n(\theta_n^\varepsilon(s)))ds \\
&\quad + 2\varepsilon^2\int_0^{t\wedge\tau_n} \Phi''_{\rho,\lambda}(\xi_n(s))\left(\theta_n^\varepsilon(s)^2\sigma_n^2(\theta_n^\varepsilon(s))\right)ds.
\end{aligned} \tag{5.9}$$

Using Lemma 4.2, $\exists N$ such that $n \geq N$,

$$\frac{\lambda\sigma_n^2(\theta_n^\varepsilon(s))}{\xi_n(s)\log(2\pi/\xi_n(s))+\rho} \leq \frac{\lambda C\left(\theta_n^\varepsilon(s)^2\log(2\pi/\theta_n^\varepsilon(s))+2^{-n}\right)}{(\theta_n^\varepsilon(s))^2\log\left(2\pi/\theta_n^\varepsilon(s)^2\right)+\rho} \leq \lambda C_1,$$

$$\frac{\lambda\theta_n^\varepsilon(s)(-B_n(\theta_n^\varepsilon(s)))}{\xi_n(s)\log(2\pi/\xi_n(s))+\rho} \leq \frac{\lambda C\left(\theta_n^\varepsilon(s)^2\log(2\pi/\theta_n^\varepsilon(s))+2^{-n}\right)}{(\theta_n^\varepsilon(s))^2\log\left(2\pi/\theta_n^\varepsilon(s)^2\right)+\rho} \leq \lambda C_2, \tag{5.10}$$

$$\frac{2\lambda^2\theta_n^\varepsilon(s)^2\sigma_n^2(\theta_n^\varepsilon(s))}{\left(\xi_n\log(2\pi/\xi_n)+\rho\right)^2} \leq \frac{2\lambda^2\theta_n^\varepsilon(s)^2\left(\theta_n^\varepsilon(s)^2\log(2\pi/\theta_n^\varepsilon(s))+2^{-n}\right)}{\left(\theta_n^\varepsilon(s)^2\log\left(2\pi/\theta_n^\varepsilon(s)^2\right)+\rho\right)^2} \leq 2\lambda^2 C_3.$$

Therefore, it follows from (5.9) that

$$E\left[\Phi_{\rho,\lambda}(\xi_n(t\wedge\tau_n))\right] \leq 1+\varepsilon^2 C_4\left(\lambda^2+\lambda\right)E\int_0^t \Phi_{\rho,\lambda}(\xi_n(s\wedge\tau_n))ds, \tag{5.11}$$

which implies that

$$E\left[\Phi_{\rho,\lambda}(\xi_n(t\wedge\tau_n))\right] \leq E^{C_4(\lambda^2+\lambda)\varepsilon^2 t}. \tag{5.12}$$

Since

$$E\left[\Phi_{\rho,\lambda}(\xi_n(1\wedge\tau_n))\right] \geq E\left[\Phi_{\rho,\lambda}(\xi_n(1\wedge\tau_n)),\ \tau_n \leq 1\right] = e^{\lambda\psi_\rho(\delta^2)}P(\tau_n \leq 1), \tag{5.13}$$

we have

$$P\left(\sup_{0\leq t\leq 1}\theta_n^\varepsilon(t) > \delta\right) = P(\tau_n \leq 1) \leq e^{-\lambda\psi_\rho(\delta^2)}e^{C(\lambda^2+\lambda)\varepsilon^2}. \tag{5.14}$$

Taking $\lambda = 1/\varepsilon^2$, we obtain that

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0}\varepsilon^2\log P\left(\sup_{0\leq t\leq 1}\theta_n^\varepsilon(t) > \delta\right) \leq -\psi_\rho\left(\delta^2\right)+C \longrightarrow -\infty. \tag{5.15}$$

Let $\rho \to 0$ to get (5.2). The proof is complete. □

Define

$$I_n(f) = \inf\left\{\frac{1}{2}\|h^n\|_{\mathscr{H}}^2, S^{n,h}(t) = f\right\}, \tag{5.16}$$

where

$$h^n = \left(\left(h^1_{\ell,k} \right)_{1 \le \ell \le 2^n,\ 1 \le k \le D_{\ell,1}}, \left(h^2_{\ell,k} \right)_{1 \le \ell \le 2^n,\ 1 \le k \le D_{\ell,2}} \right), \tag{5.17}$$

$$\|h^n\|^2_{\mathscr{H}} = \sum_{\ell=1}^{2^n} \left\{ \sum_{k=1}^{D_{\ell,1}} \int_0^T \left| \dot{h}^1_{\ell,k}(t) \right|^2 dt + \sum_{k=1}^{D_{\ell,2}} \int_0^T \left| \dot{h}^2_{\ell,k}(t) \right|^2 dt \right\}. \tag{5.18}$$

It is obvious that

$$I_n(f) \ge I(f). \tag{5.19}$$

Proof of Theorem 3.2. For any closed subset $C \subset C_x([0,T], S^d)$ and $\delta > 0$,

$$\begin{aligned} P(x^\varepsilon \in C) &\le P(\|x^\varepsilon - x^{n,\varepsilon}\| \le \delta,\ x^\varepsilon \in C) + P(\|x^\varepsilon - x^{n,\varepsilon}\| \ge \delta,\ x^\varepsilon \in C) \\ &\le P(x^{n,\varepsilon} \in C_\delta) + P(\|x^\varepsilon - x^{n,\varepsilon}\| \ge \delta), \end{aligned} \tag{5.20}$$

where

$$C_\delta = \left\{ f \in C_x\left([0,T], S^d\right), \inf_{g \in C} \|f - g\| \le \delta \right\}. \tag{5.21}$$

Therefore,

$$\begin{aligned} \limsup_{\varepsilon \to 0} \varepsilon^2 \log P(x^\varepsilon \in C) &\le \limsup_{\varepsilon \to 0} \varepsilon^2 \log P(x^{n,\varepsilon} \in C_\delta) \vee \limsup_{\varepsilon \to 0} \varepsilon^2 \log P(\|x^\varepsilon - x^{n,\varepsilon}\| > \delta) \\ &\le \left(-\inf_{f \in C_\delta} I_n(f) \vee \limsup_{\varepsilon \to 0} \varepsilon^2 \log P(\|x^\varepsilon - x^{n,\varepsilon}\| > \delta) \right) \\ &\le \left(-\inf_{f \in C_\delta} I(f) \right) \vee \limsup_{\varepsilon \to 0} \varepsilon^2 \log P(\|x^\varepsilon - x^{n,\varepsilon}\| > \delta). \end{aligned} \tag{5.22}$$

Let $n \to \infty$ to get

$$\limsup_{\varepsilon \to 0} \varepsilon^2 \log P(x^\varepsilon \in C) \le -\inf_{f \in C_\delta} I(f) \longrightarrow -\inf_{f \in C} I(f) \quad \text{as } \delta \longrightarrow 0, \tag{5.23}$$

which gives the upper bound of Theorem 3.2(i).

Let $G \subset C_x([0,T], S^d)$ be an open subset. Take $f \in G$ with $I(f) < \infty$. Then, there exists $h \in \mathscr{H}$ such that

$$f = S^h, \qquad I(f) = \frac{1}{2}\|h\|_{\mathscr{H}}. \tag{5.24}$$

Let $f^n = S^{n,h^n}$, h^n be defined as (5.17). Then, $f^n \Rightarrow f$ as $n \to \infty$ and also $I_n(f^n) \le (1/2)\|h^n\|_{\mathscr{H}}$. Choose $\delta > 0$ such that $B_f(2\delta) = \{g \in C_x([0,T], S^d), \|f - g\| \le 2\delta\} \subset G$. Then, there exists $N > 0$ such that for $n > N$,

$$\|f^n - f\| < \delta, \qquad B_{f^n}(\delta) \subset G. \tag{5.25}$$

Therefore,

$$\begin{aligned} P\left(x^{n,\varepsilon} \in B_{f^n}(\delta)\right) &\le P\left(\|x^{n,\varepsilon} - x^{\varepsilon}\| < \delta,\ x^{n,\varepsilon} \in B_{f^n}(\delta)\right) + P\left(\|x^{n,\varepsilon} - x^{\varepsilon}\| \ge \delta,\ x^{n,\varepsilon} \in B_{f^n}(\delta)\right) \\ &\le P\left(x^{\varepsilon} \in B_f(2\delta)\right) + P\left(\left\|x^{n,\delta} - x^{\varepsilon}\right\| \ge \delta\right) \\ &\le P(x^{\varepsilon} \in G) + P(\|x^{n,\varepsilon} - x^{\varepsilon}\| \ge \delta). \end{aligned} \tag{5.26}$$

Thus,

$$\begin{aligned} -\frac{1}{2}\|h^n\|_{\mathscr{H}}^2 \le -I_n(f^n) &\le \liminf_{\varepsilon \to 0} \varepsilon^2 \log P\left(x^{n,\varepsilon} \in B_{f^n}(\delta)\right) \\ &\le \liminf_{\varepsilon \to 0} \varepsilon^2 \log P(x^{\varepsilon} \in G) \vee \limsup_{\varepsilon \to 0} \varepsilon^2 \log P(\|x^{n,\varepsilon} - x^{\varepsilon}\| \ge \delta). \end{aligned} \tag{5.27}$$

Let $n \to \infty$ to obtain

$$\liminf_{\varepsilon \to 0} \varepsilon^2 \log P(x^{\varepsilon} \in G) \ge -\frac{1}{2}\|h\|_{\mathscr{H}} = -I(f). \tag{5.28}$$

Because f is arbitrary,

$$\liminf_{\varepsilon \to 0} \varepsilon^2 \log P(x^{\varepsilon} \in G) \ge -\inf_{f \in G} I(f), \tag{5.29}$$

we complete the proof of Theorem 3.2. □

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