Research Article

Regime-Switching Risk: To Price or Not to Price?

Tak Kuen Siu

Department of Applied Finance and Actuarial Studies and the Centre for Financial Risk, Faculty of Business and Economics, Macquarie University, Sydney, NSW 2109, Australia

Correspondence should be addressed to Tak Kuen Siu, ktksiu2005@gmail.com

Should the regime-switching risk be priced? This is perhaps one of the important "normative" issues to be addressed in pricing contingent claims under a Markovian, regime-switching, Black-Scholes-Merton model. We address this issue using a minimal relative entropy approach. Firstly, we apply a martingale representation for a double martingale to characterize the canonical space of equivalent martingale measures which may be viewed as the largest space of equivalent martingale measures to incorporate both the diffusion risk and the regime-switching risk. Then we show that an optimal equivalent martingale measure over the canonical space selected by minimizing the relative entropy between an equivalent martingale measure and the real-world probability measure does not price the regime-switching risk. The optimal measure also justifies the use of the Esscher transform for option valuation in the regime-switching market.

1. Introduction

Regime-switching models are one of the major classes of models for economic and financial dynamics. They allow the flexibility that model parameters can change over time according to an underlying state process, which is usually modeled by a Markov chain. The idea of regime switching has a long history in engineering though it also appeared in some early works in statistics and econometrics. Quandt [1] and Goldfeld and Quandt [2] adopted regime-switching regression models to investigate nonlinearity in economic data. Tong [3–5] introduced the idea of probability switching in nonlinear time series analysis when the field was at its embryonic stage. Hamilton [6] popularized the application of (Markovian) regime-switching models in economics and econometrics. Since then, much attention has been paid to various applications of Markovian regime-switching models in economics and finance. There is a considerable amount of works on option valuation in Markovian regime-switching models. Some early works include Naik [7], Guo [8], Buffington and Elliott [9],

Elliott et al. [10], and Siu [11]. This topic is of practical importance since many models used in practice cannot incorporate the impact of changing economic conditions, which may lead to inaccurate valuation results. From the theoretical perspective, option valuation in Markovian regime-switching models is a challenging issue due to the presence of the regime-switching risk, which is attributed to changing economic conditions and is described by the modulating Markov chain in a Markovian regime-switching model. The market in a Markovian regime-switching model is, in general, incomplete. Consequently, not all contingent claims can be perfectly hedged and there is more than one equivalent martingale measure for option valuation.

Guo [8] introduced an approach based on the completion of a Markovian regime-switching market using a set of "fictitious" securities and valued options in the completed market. A major concern about this approach is that the "fictitious" securities are not tradable in reality. Elliott et al. [10] introduced an approach based on the Esscher transform, a time-honored tool in actuarial science, for option valuation in a Markovian regime-switching model. Indeed, Gerber and Shiu [12] pioneered the use of the Esscher transform in finance, in particular in option valuation. It provides a convenient method to specify an equivalent martingale measure. Siu [13] justified the use of the Esscher transform for option valuation in a regime-switching diffusion model and a regime-switching jump-diffusion model using a game theoretic approach. In particular, the pricing kernels selected by the Esscher transform are related to the saddle points (i.e., special cases of the Nash equilibrium) of stochastic differential games.

The Esscher transform in Elliott et al. [10] does not price regime-switching risk since the probability laws of the modulating Markov chain remain unchanged after the measure change. Siu and Yang [14] considered a modified version of the Esscher transform used in Elliott et al. [10] to incorporate explicitly the intensity matrix of the Markov chain in the specification of an equivalent martingale measure. Elliott and Siu [15] and Elliott et al. [16] considered the pricing of both the diffusion risk and the regime-switching risk using a product of two density processes, one for a measure change for a diffusion process and another one for a measure change for a Markov chain. Intuitively, one would expect that the regime-switching risk should be priced. This is not unlike the situation where one should price the jump risk in a jump-diffusion model for option valuation. Nevertheless, in the original contribution by Merton [17], it was assumed that the jump risk can be diversified, so it was not priced. By the same token, in the context of regime-switching models for option valuation, it is interesting to ask whether the regime-switching risk should be priced.

In this paper, we address this question using a minimal entropy approach. In a Markovian regime-switching market, the price process of an underlying risky asset, say a share, is modeled by a Markovian, regime-switching, geometric Brownian motion modulated by a continuous-time, finite-state, Markov chain. The states of the chain can be interpreted as proxies for different levels of observable macroeconomic factors, such as gross domestic product, retail price index, and sovereign credit ratings. There are two sources of risk, namely, the diffusion risk described by a standard Brownian motion and the regime-switching risk described by the Markov chain. We first apply a version of the martingale representation for a double martingale in Elliott [18] to characterize the canonical space of equivalent martingale measures, which may be viewed as the largest space of equivalent martingale measures with respect to the enlarged filtration generated by information about the price process of the underlying risky asset and the Markov chain. This space of equivalent martingale measures is general and flexible enough to incorporate both the diffusion risk and the regime-switching risk. Then we use the minimal relative entropy approach to select an equivalent martingale

measure from the canonical space which minimizes the distance between an equivalent martingale measure and the real-world probability measure. Here the distance between the two probability measures is described by their relative entropy. We show that an optimal equivalent martingale measure over the canonical space selected by minimizing the relative entropy does not price the regime-switching risk. This result also justifies the use of the Esscher transform for option valuation in the regime-switching market proposed in Elliott et al. [10].

The rest of the paper is organized as follows. The next section describes the model dynamics. Section 3 presents a martingale representation and its use for characterising the canonical space of equivalent martingale measures in the Markovian regime-switching model. In Section 4, we determine the optimal equivalent martingale measure using the minimal relative entropy approach. The final section summarizes the paper.

2. The Model Dynamics

We consider a simplified, continuous-time economy with two primitive securities, namely, a bond and a share. These securities are tradable continuously over time in a finite time horizon $\mathcal{T} := [0, T]$, where $T < \infty$. As usual, to describe uncertainty, we consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is a real-world probability measure.

Let $\mathbf{X} := \{\mathbf{X}(t) \mid t \in \mathcal{T}\}$ be a continuous-time, finite-state, Markov chain on $(\Omega, \mathcal{F}, \mathbb{P})$ with state space $\mathcal{S} := \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_N\} \subset \mathfrak{R}^N$. We suppose that the chain $\mathbf{X}$ is observable. To facilitate the use of mathematics, as in Elliott et al. [19], we identify, without loss of generality, the state space of the chain $\mathbf{X}$ with a finite set of standard unit vectors $\mathcal{E} := \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\} \subset \mathfrak{R}^N$, where the jth component of $\mathbf{e}_i$ is the Kronecker delta δ_{ij}, for each $i, j = 1, 2, \ldots, N$. The space $\mathcal{E}$ is called the canonical state space of $\mathbf{X}$.

Suppose $\{\mathbf{A}(t) \mid t \in \mathcal{T}\}$ is the family of rate matrices, or intensity matrices, of the chain $\mathbf{X}$ under the measure $\mathbb{P}$, where, for each $t \in \mathcal{T}$, $\mathbf{A}(t) := [a_{ji}(t)]_{i,j=1,2,\ldots,N}$. The probability laws of the chain $\mathbf{X}$ under $\mathbb{P}$ are specified by $\{\mathbf{A}(t) \mid t \in \mathcal{T}\}$. For each $i, j = 1, 2, \ldots, N$ with $i \neq j$ and each $t \in \mathcal{T}$, $a_{ji}(t)$ is the instantaneous intensity of a transition of the chain $\mathbf{X}$ from state $\mathbf{e}_i$ to state $\mathbf{e}_j$ at time t. Note that for each $i, j = 1, 2, \ldots, N$ and each $t \in \mathcal{T}$,

$$
\begin{aligned}
a_{ji}(t) &\geq 0, \quad \forall i \neq j; \\
\sum_{j=1}^{N} a_{ji}(t) &= 0, \quad \text{so } a_{ii}(t) \leq 0.
\end{aligned}
\tag{2.1}
$$

Let $\mathbb{F}^{\mathbf{X}} := \{\mathcal{F}^{\mathbf{X}}(t) \mid t \in \mathcal{T}\}$ be the $\mathbb{P}$-augmentation of the natural filtration generated by the chain $\mathbf{X}$. Note that $\mathbb{F}^{\mathbf{X}}$ is right continuous. Write, for each $t \in \mathcal{T}$, $\mathbf{V}(t) := \int_0^t \mathbf{A}(u)\mathbf{X}(u-)du$. Then with the canonical state space of the chain $\mathbf{X}$, Elliott et al. [19] obtained the following semimartingale dynamics for $\mathbf{X}$:

$$
\mathbf{X}(t) = \mathbf{X}(0) + \mathbf{V}(t) + \mathbf{M}(t), \quad t \in \mathcal{T}.
\tag{2.2}
$$

Here $\mathbf{M} := \{\mathbf{M}(t) \mid t \in \mathcal{T}\}$ is an $\mathfrak{R}^N$-valued, $(\mathbb{F}^{\mathbf{X}}, \mathbb{P})$-martingale; $\mathbf{V} := \{\mathbf{V}(t) \mid t \in \mathcal{T}\}$ is a predictable process of bounded variation. Hence $\mathbf{X}$ is, indeed, a special semimartingale, and the above semimartingale decomposition is unique.

For each $t \in \mathcal{T}$, let $r(t)$ be the continuously compounded rate of interest of the bond at time t. We suppose that the interest rate $r(t)$ is modulated by the chain $\mathbf{X}$ as follows:

$$r(t) := \langle \mathbf{r}, \mathbf{X}(t) \rangle, \tag{2.3}$$

where $\mathbf{r} := (r_1, r_2, \ldots, r_N)' \in \mathfrak{R}^N$ with $r_i > 0$, for each $i = 1, 2, \ldots, N$; r_i is the interest rate of the bond when the economy is in the ith state; $\mathbf{y}'$ is the transpose of a matrix, or a vector, $\mathbf{y}$; the scalar product $\langle \cdot, \cdot \rangle$ in $\mathfrak{R}^N$ selects the component of $\mathbf{r}$ in force depending on the state of the chain $\mathbf{X}$ at a particular time.

The price process $\{B(t) \mid t \in \mathcal{T}\}$ of the bond evolves over time as follows:

$$dB(t) = r(t)B(t)dt, \quad t \in \mathcal{T}, \quad B(0) = 1. \tag{2.4}$$

Let $W := \{W(t) \mid t \in \mathcal{T}\}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ with the right-continuous, $\mathbb{P}$-completed, natural filtration $\mathbb{F}^W := \{\mathcal{F}^W(t) \mid t \in \mathcal{T}\}$. To simplify our analysis, we suppose that W and $\mathbf{X}$ are independent under $\mathbb{P}$.

For each $t \in \mathcal{T}$, let $\mu(t)$ and $\sigma(t)$ be the appreciation rate and the volatility of the share at time t, respectively. Again we assume that the chain determines $\mu(t)$ and $\sigma(t)$ as follows:

$$\begin{aligned} \mu(t) &:= \langle \boldsymbol{\mu}, \mathbf{X}(t) \rangle, \\ \sigma(t) &:= \langle \boldsymbol{\sigma}, \mathbf{X}(t) \rangle. \end{aligned} \tag{2.5}$$

Here $\boldsymbol{\mu} := (\mu_1, \mu_2, \ldots, \mu_N)' \in \mathfrak{R}^N$ with $\mu_i > r_i$ and $\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \ldots, \sigma_N)' \in \mathfrak{R}^N$ with $\sigma_i > 0$; μ_i and σ_i are the appreciation rate and the volatility of the share when the economy is in the ith state, respectively, for each $i = 1, 2, \ldots, N$.

Then under $\mathbb{P}$ the share price process $\{S(t) \mid t \in \mathcal{T}\}$ evolves over time according to the following Markovian, regime-switching, geometric Brownian motion (GBM):

$$\begin{aligned} dS(t) &= \mu(t)S(t)dt + \sigma(t)S(t)dW(t), \\ S(0) &= s > 0. \end{aligned} \tag{2.6}$$

3. Martingale Representation and Canonical Space of Equivalent Martingale Measures

In this section, we first present a martingale representation for a double martingale in Elliott [18], where the double martingale is a martingale with respect to an enlarged filtration generated by both the Brownian motion W and the Markov chain $\mathbf{X}$. By the martingale representation, any double martingale can be represented as the sum of a stochastic integral with respect to the Brownian motion W and integrals with respect to basic martingales associated with the chain $\mathbf{X}$. The martingale representation is then used to specify the canonical space of equivalent martingale measures in the Markovian regime-switching market.

Firstly, we define a set of basic martingales associated with the chain $\mathbf{X}$. For each $i, j = 1, 2, \ldots, N$ with $i \neq j$ and $t \in \mathcal{T}$, let $N_{ij}(t)$ be the number of jumps of the chain $\mathbf{X}$ from state $\mathbf{e}_i$ to state $\mathbf{e}_j$ up to time t. Write

$$M_{ij}(t) := \int_0^t \langle \mathbf{X}(s-), \mathbf{e}_i \rangle \langle d\mathbf{M}(s), \mathbf{e}_j \rangle. \tag{3.1}$$

Then

$$
\begin{aligned}
N_{ij}(t) &:= \sum_{0 < s \leq t} \langle \mathbf{X}(s-), \mathbf{e}_i \rangle \langle \mathbf{X}(s), \mathbf{e}_j \rangle \\
&= \sum_{0 < s \leq t} \langle \mathbf{X}(s-), \mathbf{e}_i \rangle \langle \Delta\mathbf{X}(s), \mathbf{e}_j \rangle \\
&= \int_0^t \langle \mathbf{X}(s-), \mathbf{e}_i \rangle \langle d\mathbf{X}(s), \mathbf{e}_j \rangle \\
&= \int_0^t \langle \mathbf{X}(s-), \mathbf{e}_i \rangle \langle \mathbf{A}(s)\mathbf{X}(s-), \mathbf{e}_j \rangle ds + \int_0^t \langle \mathbf{X}(s-), \mathbf{e}_i \rangle \langle d\mathbf{M}(s), \mathbf{e}_j \rangle \\
&= \int_0^t a_{ij}(s) I_{\{\mathbf{X}(s-)=\mathbf{e}_i\}} ds + M_{ij}(t).
\end{aligned}
\tag{3.2}
$$

Consequently, for each $i, j = 1, 2, \ldots, N$ with $i \neq j$,

$$
\begin{aligned}
M_{ij}(t) &= N_{ij}(t) - \int_0^t a_{ij}(s) I_{\{\mathbf{X}(s-)=\mathbf{e}_i\}} ds \\
&= \int_0^t \langle \mathbf{X}(s-), \mathbf{e}_i \rangle \langle d\mathbf{M}(s), \mathbf{e}_j \rangle, \quad t \in \mathcal{T},
\end{aligned}
\tag{3.3}
$$

is an $(\mathbb{F}^{\mathbf{X}}, \mathbb{P})$-martingale.

For each $i, j = 1, 2, \ldots, N$ with $i \neq j$, let $N_{ij} := \{N_{ij}(t) \mid t \in \mathcal{T}\}$. Then the chain $\mathbf{X}$ is isomorphic to the family of jump processes $\{N_{ij} \mid i, j = 1, 2, \ldots, N, i \neq j\}$. Write $M_{ij} := \{M_{ij}(t) \mid t \in \mathcal{T}\}$. Then the family of martingales $\{M_{ij} \mid i, j = 1, 2, \ldots, N, i \neq j\}$ is the set of basic martingales of the chain $\mathbf{X}$ under $\mathbb{P}$. These martingales are orthogonal, purely discontinuous, and square integrable. Furthermore, $M_{ij}(0) = 0$.

For each $t \in \mathcal{T}$, let $\mathcal{G}(t) := \mathcal{F}^W(t) \vee \mathcal{F}^{\mathbf{X}}(t)$, the minimal σ-field containing both $\mathcal{F}^W(t)$ and $\mathcal{F}^{\mathbf{X}}(t)$. Write $\mathbb{G} := \{\mathcal{G}(t) \mid t \in \mathcal{T}\}$. Before we present the martingale representation for a double martingale by Elliott [18], we have to define the following spaces of processes

Definition 3.1. $L^2(W)$ is the space of real-valued, G-predictable processes $\{\theta(t) \mid t \in \mathcal{T}\}$ such that

$$E\left[\int_0^\infty |\theta(t)|^2 d\langle W, W \rangle(t)\right] = E\left[\int_0^\infty |\theta(t)|^2 dt\right] < \infty. \tag{3.4}$$

Here E is an expectation under $\mathbb{P}$ and $\{\langle W, W \rangle(t) \mid t \in \mathcal{T}\}$ is the quadratic variation of the Brownian motion W.

Definition 3.2. For each $i, j = 1, 2, \dots, N$ with $i \neq j$, $L^1(M_{ij})$ is the space of real-valued, G-predictable processes $\{\eta_{ij}(t) \mid t \in \mathcal{T}\}$ such that

$$E\left[\int_0^\infty |\eta_{ij}(t)| dM_{ij}(t)\right] < \infty. \tag{3.5}$$

Definition 3.3. $L^2_{\text{loc}}(W)$ is the space of processes $\{\theta(t) \mid t \in \mathcal{T}\}$ such that there is an increasing sequence of stopping times $\{T_n \mid n = 1, 2, \dots\}$ with $\lim_{n \to \infty} T_n = \infty$, $\mathbb{P}$-a.s., such that for each $n = 1, 2, \dots$, $\{\theta(t) I_{\{t < T_n\}} \mid t \in \mathcal{T}\} \in L^2(W)$, where $I_{\{t < T_n\}}$ is the indicator function of the event $\{t < T_n\}$.

Note that $L^2_{\text{loc}}(W)$ is called the localization of $L^2(W)$ and that the sequence of stopping times $\{T_n \mid n = 1, 2, \dots\}$, which may depend on $\{\theta(t) \mid t \in \mathcal{T}\}$, is called the localizing sequence.

Definition 3.4. For each $i, j = 1, 2, \dots, N$ with $i \neq j$, $L^1_{\text{loc}}(M_{ij})$ is the space of processes $\{\eta_{ij}(t) \mid t \in \mathcal{T}\}$ such that there is an increasing sequence of stopping times $\{T_n^{ij} \mid n = 1, 2, \dots\}$ with $\lim_{n \to \infty} T_n^{ij} = \infty$, $\mathbb{P}$-a.s., such that for each $n = 1, 2, \dots$, $\{\eta_{ij}(t) I_{\{t < T_n^{ij}\}} \mid t \in \mathcal{T}\} \in L^1(M_{ij})$.

Again, for each $i, j = 1, 2, \dots, N$ with $i \neq j$, $L^2_{\text{loc}}(M_{ij})$ is the localization of $L^2(M_{ij})$.

We now present a martingale representation for a $(\mathbb{G}, \mathbb{P})$-(local)-martingale. This martingale representation follows directly from the martingale representation for a double martingale by Elliott [18, Theorem 5.1] where the set of basic martingales associated to a jump process in Davis [20] and Elliott [18] is replaced by the set of basic martingales $\{M_{ij} \mid i, j = 1, 2, \dots, N, i \neq j\}$ associated to the Markov chain **X** here.

Theorem 3.5. *Suppose $L := \{L(t) \mid t \in \mathcal{T}\}$ is a $(\mathbb{G}, \mathbb{P})$-(local)-martingale. Then there are unique processes $\{\tilde{\theta}(t) \mid t \in \mathcal{T}\} \in L^2_{\text{loc}}(W)$ and $\{\tilde{\eta}_{ij}(t) \mid t \in \mathcal{T}\} \in L^1_{\text{loc}}(M_{ij})$, $i, j = 1, 2, \dots, N$ with $i \neq j$, such that for each $t \in \mathcal{T}$,*

$$L(t) = L(0) + \int_0^t \tilde{\theta}(u) dW(u) + \sum_{i, j=1, i \neq j}^N \int_0^t \tilde{\eta}_{ij}(u-) dM_{ij}(u), \quad \mathbb{P}\text{-a.s.} \tag{3.6}$$

In the sequel, we shall apply Theorem 3.5 to characterize the canonical space of equivalent martingale measures. Suppose $\mathbb{Q}$ is a probability measure equivalent to the measure $\mathbb{P}$ on the σ-field $\mathcal{G}(T)$. Then there is a $\mathcal{G}(T)$-measurable, strictly positive, random variable, denoted by $\Lambda(T)$, such that

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{G}(T)} := \Lambda(T). \tag{3.7}$$

Let $\Lambda := \{\Lambda(t) \mid t \in \mathcal{T}\}$ be a process such that for each $t \in \mathcal{T}$, $\Lambda(t)$ is a right-continuous version of the conditional expectation $E[\Lambda(T) \mid \mathcal{G}(t)]$. Then by definition, Λ is a $(\mathbb{G}, \mathbb{P})$-martingale. Note that $\mathbb{Q}$ is a probability measure, so

$$\Lambda(0) = E[\Lambda(T) \mid \mathcal{G}(0)] = E[\Lambda(T)] = 1. \tag{3.8}$$

Consequently by Theorem 3.5, there are unique processes $\{\tilde{\theta}(t) \mid t \in \mathcal{T}\} \in L^2_{\text{loc}}(W)$ and $\{\tilde{\eta}_{ij}(t) \mid t \in \mathcal{T}\} \in L^1_{\text{loc}}(M_{ij}), i, j = 1, 2, \ldots, N$ with $i \neq j$, such that for each $t \in \mathcal{T}$,

$$\Lambda(t) = 1 + \int_0^t \tilde{\theta}(u) dW(u) + \sum_{i,j=1,i\neq j}^N \int_0^t \tilde{\eta}_{ij}(u-) dM_{ij}(u), \quad \mathbb{P}\text{-a.s.} \tag{3.9}$$

For each $t \in \mathcal{T}$ and $i, j = 1, 2, \ldots, N$ with $i \neq j$, let

$$\theta(t) := \frac{\tilde{\theta}(t)}{\Lambda(t-)}, \qquad \eta_{ij}(t) := \frac{\tilde{\eta}_{ij}(t)}{\Lambda(t-)}. \tag{3.10}$$

For each t, $\Lambda(t-)$ is strictly positive, $\mathbb{P}$-a.s., so the processes $\{\theta(t) \mid t \in \mathcal{T}\}$ and $\{\eta_{ij}(t) \mid t \in \mathcal{T}\}$ are well defined. It is also obvious that $\{\theta(t) \mid t \in \mathcal{T}\} \in L^2_{\text{loc}}(W)$ and $\{\eta_{ij}(t) \mid t \in \mathcal{T}\} \in L^1_{\text{loc}}(M_{ij})$. Then for each $t \in \mathcal{T}$,

$$\Lambda(t) = 1 + \int_0^t \theta(u)\Lambda(u-) dW(u) + \sum_{i,j=1,i\neq j}^N \int_0^t \eta_{ij}(u-)\Lambda(u-) dM_{ij}(u), \quad \mathbb{P}\text{-a.s.} \tag{3.11}$$

This means that the density process Λ for the measure change from $\mathbb{P}$ to $\mathbb{Q}$ has the above (double) martingale representation (3.11). Since $\mathbb{Q}$ is any arbitrary probability measure equivalent to $\mathbb{P}$ on $\mathcal{G}(T)$, the density process of any arbitrary probability measure equivalent to $\mathbb{P}$ on $\mathcal{G}(T)$ has the representation (3.11).

Let $\eta := \{\eta_{ij} \mid i, j = 1, 2, \ldots, N, i \neq j\}$, where $\eta_{ij} := \{\eta_{ij}(t) \mid t \in \mathcal{T}\} \in L^1_{\text{loc}}(M_{ij})$. Write $L^1_{\text{loc}}(M)$ for the space of such families of processes η. For each $\theta \in L^2_{\text{loc}}(W)$ and $\eta \in L^1_{\text{loc}}(M)$, we define the process $\Lambda^{\theta,\eta} := \{\Lambda^{\theta,\eta}(t) \mid t \in \mathcal{T}\}$ by

$$\Lambda^{\theta,\eta}(t) = 1 + \int_0^t \theta(u)\Lambda^{\theta,\eta}(u-) dW(u) + \sum_{i,j=1,i\neq j}^N \int_0^t \eta_{ij}(u-)\Lambda^{\theta,\eta}(u-) dM_{ij}(u), \quad \mathbb{P}\text{-a.s.} \tag{3.12}$$

For each $\theta \in L^2_{\text{loc}}(W)$ and $\eta \in L^2_{\text{loc}}(M)$, let $\mathbb{Q}^{\theta,\eta}$ be a probability measure equivalent to $\mathbb{P}$ on $\mathcal{G}(\mathcal{T})$ such that

$$\left.\frac{d\mathbb{Q}^{\theta,\eta}}{d\mathbb{P}}\right|_{\mathcal{G}(T)} := \Lambda^{\theta,\eta}(T). \tag{3.13}$$

Then the space of probability measures equivalent to $\mathbb{P}$ on $\mathcal{G}(T)$, denoted by $Q_e(\mathcal{G}(T), \mathbb{P})$, is generated by

$$Q_e(\mathcal{G}(T), \mathbb{P}) = \left\{ \mathbb{Q}^{\theta,\eta} \mid \theta \in L^2_{\text{loc}}(W), \eta \in L^1_{\text{loc}}(M) \right\}. \tag{3.14}$$

We call this the canonical space of probability measures equivalent to $\mathbb{P}$ on $\mathcal{G}(T)$.

We now restrict the space $Q_e(\mathcal{G}(T), \mathbb{P})$ to the space of equivalent (local) martingale measures using the fundamental theorem of asset pricing. This theorem was originally

developed by Harrison and Kreps [21] and Harrison and Pliska [22, 23]. It was later extended by a number of authors (see, for example, Delbaen and Schachermayer [24, 25] and the references therein). A version of this theorem states that the absence of arbitrage opportunities is (essentially) equivalent to the existence of an equivalent (local) martingale measure under which discounted asset price processes are (local)-martingales with respect to some filtration. In our case, the filtration is given by the enlarged filtration $\mathbb{G}$, which is the observed filtration.

Let $\mathcal{M}_e(\mathcal{G}(T), \mathbb{P})$ be the space of equivalent, (local), martingale measures. Then $\mathcal{M}_e(\mathcal{G}(T), \mathbb{P})$ is a subspace of $Q_e(\mathcal{G}(T), \mathbb{P})$ such that for each $\mathbb{Q}^{\theta, \eta} \in \mathcal{M}_e(\mathcal{G}(T), \mathbb{P})$, the discounted share price process is a $(\mathbb{G}, \mathbb{Q}^{\theta, \eta})$-(local)-martingale. The following theorem gives a characterization for $\mathcal{M}_e(\mathcal{G}(T), \mathbb{P})$.

Theorem 3.6. *For each $t \in \mathcal{T}$, let*

$$\theta^{\dagger}(t) = \frac{r(t) - \mu(t)}{\sigma(t)}. \tag{3.15}$$

Write $\theta^{\dagger} := \{\theta^{\dagger}(t) \mid t \in \mathcal{T}\}$. Then

$$\mathcal{M}_e(\mathcal{G}(T), \mathbb{P}) = \left\{ \mathbb{Q}^{\theta^{\dagger}, \eta} \in Q_e(\mathcal{G}(T), \mathbb{P}) \mid \eta \in L^1_{\text{loc}}(M) \right\}. \tag{3.16}$$

Proof. Let $\widetilde{S} := \{\widetilde{S}(t) \mid t \in \mathcal{T}\}$ be the discounted share price process, where $\widetilde{S}(t) = \exp(-\int_0^t r(u) du) S(t)$ for each $t \in \mathcal{T}$. Then under $\mathbb{P}$,

$$d\widetilde{S}(t) = \left(\mu(t) - r(t) \right) \widetilde{S}(t) dt + \sigma(t) \widetilde{S}(t) dW(t). \tag{3.17}$$

Note that $\widetilde{S}$ is a $(\mathbb{G}, \mathbb{Q}^{\theta, \eta})$-(local)-martingale, where $\mathbb{Q}^{\theta, \eta} \in Q_e(\mathcal{G}(T), \mathbb{P})$, if and only if $\Lambda^{\theta, \eta} \widetilde{S}$ is a $(\mathbb{G}, \mathbb{P})$-(local)-martingale. Applying Itô's differentiation rule to $\Lambda^{\theta, \eta}(t) \widetilde{S}(t)$ gives

$$\begin{aligned}
\Lambda^{\theta, \eta}(t) \widetilde{S}(t) &= \Lambda^{\theta, \eta}(0) \widetilde{S}(0) + \int_0^t \widetilde{S}(u-) d\Lambda^{\theta, \eta}(u) + \int_0^t \Lambda^{\theta, \eta}(u) d\widetilde{S}(u) + \left[\widetilde{S}, \Lambda^{\theta, \eta} \right](t) \\
&= \Lambda^{\theta, \eta}(0) \widetilde{S}(0) + \int_0^t \Lambda^{\theta, \eta}(u) \widetilde{S}(u) \left(\mu(u) - r(u) + \theta(u) \sigma(u) \right) du \\
&\quad + \int_0^t \Lambda^{\theta, \eta}(u) \widetilde{S}(u) (\theta(u) + \sigma(u)) dW(u) + \sum_{i, j = 1, i \neq j}^{N} \int_0^t \Lambda^{\theta, \eta}(u) \widetilde{S}(u) \eta_{ij}(u) dM_{ij}(u).
\end{aligned} \tag{3.18}$$

This is a $(\mathbb{G}, \mathbb{P})$-(local)-martingale if and only if for each $t \in \mathcal{T}$,

$$\mu(t) - r(t) + \theta(t) \sigma(t) = 0. \tag{3.19}$$

This proves the result. $\square$

Note that $\mathcal{M}_e(\mathcal{G}(T), \mathbb{P})$ is the largest subspace of $Q_e(\mathcal{G}(T), \mathbb{P})$ such that for each $\mathbb{Q} \in \mathcal{M}_e(\mathcal{G}(T), \mathbb{P})$, the discounted share price process $\widetilde{S}$ is a $(\mathbb{G}, \mathbb{Q})$-(local)-martingale. We call $\mathcal{M}_e(\mathcal{G}(T), \mathbb{P})$ the canonical space of equivalent martingale measures. This space is generated by the density processes $\Lambda^{\theta^\dagger, \eta}$, $\eta \in L^1_{\mathrm{loc}}(M)$, from the martingale representation of a double martingale presented in Theorem 3.5.

Note that the martingale condition alone is not sufficient to determine a unique equivalent martingale measure. In particular, $\eta \in L^1_{\mathrm{loc}}(M)$ is undetermined. Consequently, $\mathcal{M}_e(\mathcal{G}(T), \mathbb{P})$ has more than one element. In the next section 4, we shall determine a unique equivalent martingale measure which is optimal in some sense over all of the equivalent martingale measures in the canonical space $\mathcal{M}_e(\mathcal{G}(T), \mathbb{P})$.

4. The Minimal Relative Entropy Approach

In this section we shall employ the minimal relative entropy approach to select an equivalent martingale measure from the canonical space $\mathcal{M}_e(\mathcal{G}(T), \mathbb{P})$. The idea of entropy has a long history in physics. Boltzmann [26] was the first to introduce the concept of entropy in thermodynamics, where the thermodynamic entropy was introduced and its historical probabilistic interpretation was given. The concept of entropy also plays a fundamental role in statistics. In particular, Akaike [27] provided a statistical characterization of entropy based on a multinomial distribution and introduced the famous Akaike information criterion (AIC) by linking the maximization of likelihood function with the maximization of entropy, (see also Tong [28] for further discussion). Nowadays, AIC is a standard tool for model identification and selection in statistical science.

The concept of entropy also plays an important role in mathematical finance. Miyahara [29] was the first to introduce the minimal entropy martingale measure (MEMM) approach to select an equivalent martingale measure in an incomplete market. Nowadays, the MEMM approach has become one of the major approaches for option valuation in an incomplete market. The basic idea of the MEMM approach is to select an equivalent martingale measure so as to minimize the "distance" between an equivalent martingale measure and a real-world probability measure described by their relative entropy. Consequently, the MEMM is the equivalent martingale measure which is closest to the real-world probability measure. Indeed, the MEMM can be related to a known tool in actuarial science, namely, the Esscher transform (see Bühlmann et al. [30]). The Esscher transform was first introduced to finance, in particular, option valuation, by the seminal work of Gerber and Shiu [12]. For details about the MEMM approach for option valuation, interested readers may refer to works by Miyahara [31, 32], Fujiwara and Miyahara [33], and Schweizer [34]. In addition to option valuation, the concept of relative entropy has been considered in risk measurement. For example, the entropic risk measure is introduced by assuming that the penalty function of a convex risk measure is a relative entropy (see, for example, Föllmer and Schied [35]).

Before presenting the MEMM approach for pricing regime-switching risk, we first give the dynamics of the share price process S and the Markov chain $\mathbf{X}$ under an equivalent martingale measure $\mathbb{Q}^{\theta^\dagger, \eta} \in \mathcal{M}_e(\mathcal{G}(T); \mathbb{P})$.

Consider the following process $Z := \{Z(t) \mid t \in \mathcal{T}\}$ defined by putting the following:

$$Z(t) := \int_0^t \theta^\dagger(u) dW(u) + \sum_{i,j=1, i \neq j}^N \int_0^t \eta_{ij}(u) dM_{ij}(u). \tag{4.1}$$

Then the density process $\Lambda^{\theta^\dagger,\eta}$ associated with the measure $\mathbb{Q}^{\theta^\dagger,\eta}$ can be written as follows:

$$\Lambda^{\theta^\dagger,\eta}(t) = 1 + \int_0^t \Lambda^{\theta^\dagger,\eta}(u-)dZ(u). \tag{4.2}$$

Consequently, by Elliott [36, see Theorem 13.5], we have, for each $t \in \mathcal{T}$, $\mathbb{P}$-a.s., that

$$\begin{aligned}
\Lambda^{\theta^\dagger,\eta}(t) &= \mathcal{E}(Z)(t) \\
&= \exp\left(Z(t) - \frac{1}{2}\langle Z^c, Z^c\rangle(t)\right) \prod_{0<u\leq t}(1 + \Delta Z(u))e^{-\Delta Z(u)},
\end{aligned} \tag{4.3}$$

where $\mathcal{E}(Z) := \{\mathcal{E}(Z)(t) \mid t \in \mathcal{T}\}$ is the stochastic exponential of the process Z; $Z^c := \{Z^c(t) \mid t \in \mathcal{T}\}$ is the continuous part of the process Z; $\{\langle Z^c, Z^c\rangle(t) \mid t \in \mathcal{T}\}$ is the predictable quadratic variation process of Z^c; $\Delta Z(t) := Z(t) - Z(t-)$, the jump at time t.

Then for each $t \in \mathcal{T}$,

$$\begin{aligned}
\Lambda^{\theta^\dagger,\eta}(t) = \exp\Bigg(&\int_0^t \theta^\dagger(u)dW(u) - \frac{1}{2}\int_0^t \left(\theta^\dagger(u)\right)^2 du + \sum_{i,j=1, i\neq j}^N \int_0^t \ln\left(1 + \eta_{ij}(u)\right)dM_{ij}(u) \\
&+ \sum_{i,j=1, i\neq j}^N \int_0^t \left[\ln\left(1 + \eta_{ij}(u)\right) - \eta_{ij}(u)\right]a_{ij}(u)I_{\{\mathbf{X}(u-)=\mathbf{e}_i\}}du\Bigg).
\end{aligned} \tag{4.4}$$

Note that the density process $\Lambda^{\theta^\dagger,\eta}$ is strictly positive $\mathbb{P}$-a.s. if and only if $\eta(t) > -1$, $\mathbb{P}$-a.s., for each $t \in \mathcal{T}$. Consequently, we must impose the condition that $\eta(t) > -1$, $\mathbb{P}$-a.s., for each $t \in \mathcal{T}$.

By the Girsanov theorem for jump processes, (see Elliott [36], Chapter 13 therein), under the measure $\mathbb{Q}^{\theta^\dagger,\eta}$, the process $W^{\theta^\dagger,\eta} := \{W^{\theta^\dagger,\eta}(t) \mid t \in \mathcal{T}\}$ defined by putting the following:

$$W^{\theta^\dagger,\eta}(t) := W(t) - \int_0^t \theta^\dagger(u)du, \qquad t \in \mathcal{T}, \tag{4.5}$$

is a standard Brownian motion with respect to $\mathbb{G}$. This implies that under $\mathbb{Q}^{\theta^\dagger,\eta}$, the share price process is governed by

$$dS(t) = r(t)S(t)dt + \sigma(t)S(t)dW^{\theta^\dagger,\eta}(t). \tag{4.6}$$

Furthermore, for each $i,j = 1,2,\ldots,N$ with $i \neq j$, the process $M_{ij}^{\theta^\dagger,\eta} := \{M_{ij}^{\theta^\dagger,\eta}(t) \mid t \in \mathcal{T}\}$ defined by putting the following:

$$M_{ij}^{\theta^\dagger,\eta}(t) := N_{ij}(t) - \int_0^t \left(1 + \eta_{ij}(u)\right)a_{ij}(u)I_{\{\mathbf{X}(u-)=\mathbf{e}_i\}}du, \tag{4.7}$$

is a $(\mathbb{G}, \mathbb{Q}^{\theta^\dagger, \eta})$-martingale. In other words, $(1 + \eta_{ij}(t)) a_{ij}(t) I_{\{\mathbf{X}(t-)=\mathbf{e}_i\}}$ is the intensity at time t of the point process N_{ij} under $\mathbb{Q}^{\theta^\dagger, \eta}$.

For each $i, j = 1, 2, \ldots, N$ with $i \neq j$ and each $t \in \mathcal{T}$, let

$$a_{ij}^\eta(t) := (1 + \eta_{ij}(t)) a_{ij}(t). \tag{4.8}$$

We then take

$$a_{ii}^\eta(t) := - \sum_{j=1, j \neq i}^{N} a_{ij}^\eta(t), \tag{4.9}$$

so that

$$\sum_{j=1}^{N} a_{ij}^\eta(t) = 0. \tag{4.10}$$

Note that $\eta(t) > -1$, $\mathbb{P}$-a.s., for each $t \in \mathcal{T}$, so $a_{ij}^\eta(t) \geq 0$, $i \neq j$, and $a_{ii}(t) \leq 0$.

We now define a family of matrices, $\mathbf{A}^\eta(t) := [a_{ij}^\eta(t)]_{i,j=1,2,\ldots,N}$, $t \in \mathcal{T}$. Then under $\mathbb{Q}^{\theta^\dagger, \eta}$, the chain $\mathbf{X}$ has the family of matrices $\mathbf{A}^\eta(t) := [a_{ij}^\eta(t)]_{i,j=1,2,\ldots,N}$, $t \in \mathcal{T}$. Consequently, under $\mathbb{Q}^{\theta^\dagger, \eta}$, the dynamics of the chain $\mathbf{X}$ become

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \mathbf{A}^\eta(u) \mathbf{X}(u-) du + \mathbf{M}^{\theta^\dagger, \eta}(t). \tag{4.11}$$

Here $\mathbf{M}^{\theta^\dagger, \eta} := \{ \mathbf{M}^{\theta^\dagger, \eta}(t) \mid t \in \mathcal{T} \}$ is an $\mathfrak{R}^N$-valued, $(\mathbb{G}, \mathbb{Q}^{\theta^\dagger, \eta})$-martingale such that

$$M_{ij}^{\theta^\dagger, \eta}(t) = \int_0^t \langle \mathbf{X}(u-), \mathbf{e}_i \rangle \langle \mathbf{e}_j, d\mathbf{M}^{\theta^\dagger, \eta}(u) \rangle. \tag{4.12}$$

To determine η, we adopt the MEMM approach. The relative entropy between $\mathbb{Q}^{\theta^\dagger, \eta}$ and $\mathbb{P}$ is defined by

$$\mathcal{R}\left(\mathbb{Q}^{\theta^\dagger, \eta}, \mathbb{P} \right) := \mathrm{E} \left[\left(\frac{d\mathbb{Q}^{\theta^\dagger, \eta}}{d\mathbb{P}} \right) \ln \left(\frac{d\mathbb{Q}^{\theta^\dagger, \eta}}{d\mathbb{P}} \right) \right]. \tag{4.13}$$

Our object is to determine η so as to minimize $\mathcal{R}(\mathbb{Q}^{\theta^\dagger, \eta}, \mathbb{P})$. That is, to solve the following optimization problem:

$$\min_\eta \mathcal{R}\left(\mathbb{Q}^{\theta^\dagger, \eta}, \mathbb{P} \right). \tag{4.14}$$

By a version of the Bayes' rule and the definition of $d\mathbb{Q}^{\theta^\dagger,\eta}/d\mathbb{P}$, we have the following:

$$
\begin{aligned}
\mathcal{R}\left(\mathbb{Q}^{\theta^\dagger,\eta},\mathbb{P}\right) &= \mathrm{E}^{\theta^\dagger,\eta}\left[\ln\left(\frac{d\mathbb{Q}^{\theta^\dagger,\eta}}{d\mathbb{P}}\right)\right] \\
&= \mathrm{E}^{\theta^\dagger,\eta}\left[\int_0^T \theta^\dagger(t)dW(t) - \frac{1}{2}\int_0^T \left(\theta^\dagger(t)\right)^2 dt + \sum_{i,j=1,i\neq j}^N \int_0^T \ln(1+\eta_{ij}(t))dM_{ij}(t)\right. \\
&\qquad \left. + \sum_{i,j=1,i\neq j}^N \int_0^T \left[\ln(1+\eta_{ij}(t)) - \eta_{ij}(t)\right]a_{ij}(t)I_{\{\mathbf{X}(t-)=\mathbf{e}_i\}}dt\right] \\
&= \mathrm{E}^{\theta^\dagger,\eta}\left[\int_0^T \theta^\dagger(t)dW^{\theta^\dagger,\eta}(t) + \frac{1}{2}\int_0^T \left(\theta^\dagger(t)\right)^2 dt + \sum_{i,j=1,i\neq j}^N \int_0^T \ln(1+\eta_{ij}(t))dM^{\theta^\dagger,\eta}(t)\right. \\
&\qquad \left. + \sum_{i,j=1,i\neq j}^N \int_0^T \left[(1+\eta_{ij}(t))\ln(1+\eta_{ij}(t)) - \eta_{ij}(t)\right]a_{ij}(t)I_{\{\mathbf{X}(t-)=\mathbf{e}_i\}}dt\right] \\
&= \mathrm{E}^{\theta^\dagger,\eta}\left[\sum_{i,j=1,i\neq j}^N \int_0^T \left[(1+\eta_{ij}(t))\ln(1+\eta_{ij}(t)) - \eta_{ij}(t)\right]a_{ij}(t)I_{\{\mathbf{X}(t-)=\mathbf{e}_i\}}dt\right. \\
&\qquad \left. + \frac{1}{2}\int_0^T \left(\theta^\dagger(t)\right)^2 dt\right].
\end{aligned}
\tag{4.15}
$$

Here $\mathrm{E}^{\theta^\dagger,\eta}$ is an expectation under $\mathbb{Q}^{\theta^\dagger,\eta}$. The last equality follows from the fact that both $W^{\theta^\dagger,\eta}$ and $M^{\theta^\dagger,\eta}$ are $(\mathbb{G},\mathbb{Q}^{\theta^\dagger,\eta})$-martingales.

For each $(t,\omega) \in \tau \times \Omega$, let

$$
K(t,\omega) := \sum_{i,j=1,i\neq j}^N \left((1+\eta_{ij}(t,\omega))\ln(1+\eta_{ij}(t,\omega)) - \eta_{ij}(t,\omega)\right).
\tag{4.16}
$$

If we can find an $\eta^\dagger := \{\eta_{ij}^\dagger \mid i,j=1,2,\ldots,N, i\neq j\}$ such that for each $(t,\omega) \in \tau \times \Omega$,

$$
\eta^\dagger(t,\omega) := \arg-\min_\eta K(t,\omega),
\tag{4.17}
$$

then $\eta^\dagger$ minimizes the relative entropy $\mathcal{R}(\mathbb{Q}^{\theta^\dagger,\eta},\mathbb{P})$ over η.

For each $i,j = 1,2,\ldots,N$ with $i\neq j$ and each $(t,\omega) \in \tau \times \Omega$, differentiating $K(t,\omega)$ with respect to $\eta_{ij}(t,\omega)$ and setting the derivative equal to zero give the following:

$$
\ln(1+\eta_{ij}(t,\omega)) = 0, \qquad (t,\omega) \in \tau \times \Omega.
\tag{4.18}
$$

This then implies that for each $(t,\omega) \in \tau \times \Omega$,

$$
\eta_{ij}(t,\omega) = 0.
\tag{4.19}
$$

Consequently,

$$\Lambda^{\theta^\dagger,\eta^\dagger}(t) = \exp\left(\int_0^t \left(\frac{r(u)-\mu(u)}{\sigma(u)}\right)dW(u) - \frac{1}{2}\int_0^t \left(\frac{r(u)-\mu(u)}{\sigma(u)}\right)^2 du\right). \qquad (4.20)$$

In this case, $\mathbf{A}^{\eta^\dagger}(t) = \mathbf{A}(t)$, $t \in \mathcal{T}$. In other words, the probability laws of the chain $\mathbf{X}$ remains unchanged when changing the measures from $\mathbb{P}$ to $\mathbb{Q}^{\theta^\dagger,\eta^\dagger}$. Therefore, if we wish to select an equivalent martingale measure from the canonical space $\mathcal{M}_e(\mathcal{G}(T), \mathbb{P})$ using the minimal relative entropy as the criterion, it is optimal that we do not price the regime-switching risk. Indeed this also justifies the version of the Esscher transform adopted in Elliott et al. [10] for option valuation in a Markovian regime-switching market.

5. Conclusion

Using the concept of relative entropy, we have addressed the question about whether regime-switching risk should be priced. We first applied a martingale representation theorem for a double martingale to identify the canonical space of equivalent martingale measures in the Markovian, regime-switching, Black-Scholes-Merton market. This canonical space may be viewed as the largest space of equivalent martingale measures in the regime-switching market. We then selected an optimal equivalent martingale measure from this canonical space by minimizing the "distance" between an equivalent martingale measure and the real-world probability measure, where the "distance" is described by their relative entropy. It turned out that the optimal equivalent martingale measure does not price the regime-switching risk which further justifies the Esscher transform used in Elliott et al. [10] for option valuation in a Markovian regime-switching market.

References

[1] R. E. Quandt, "The estimation of the parameters of a linear regression system obeying two separate regimes," *Journal of the American Statistical Association*, vol. 53, pp. 873–880, 1958.

[2] S. M. Goldfeld and R. E. Quandt, "A Markov model for switching regressions," *Journal of Econometrics*, vol. 1, no. 1, pp. 3–15, 1973.

[3] H. Tong, "On a threshold model," in *Patten Recognition and Signal Processing*, C. H. Chen, Ed., vol. 29 of *NATO ASI Series E*, pp. 575–586, Sijthoff, The Netherlands, 1978.

[4] H. Tong, "A view on nonlinear time series model building," in *Time Series*, O. D. Anderson, Ed., pp. 41–56, North-Holland, Amsterdam, The Netherlands, 1980.

[5] H. Tong, *Threshold Models in NonlinearTime Series Analysis*, Springer, Berlin, Germany, 1983.

[6] J. D. Hamilton, "A new approach to the economic analysis of nonstationary time series and the business cycle," *Econometrica*, vol. 57, no. 2, pp. 357–384, 1989.

[7] V. Naik, "Option valuation and hedging strategies with jumps in the volatility of asset returns," *Journal of Finance*, vol. 48, pp. 1969–1984, 1993.

[8] X. Guo, "Information and option pricings," *Quantitative Finance*, vol. 1, no. 1, pp. 38–44, 2001.

[9] J. Buffington and R. J. Elliott, "American options with regime switching," *International Journal of Theoretical and Applied Finance*, vol. 5, no. 5, pp. 497–514, 2002.

[10] R. J. Elliott, L. L. Chan, and T. K. Siu, "Option pricing and Esscher transform under regime switching," *Annals of Finance*, vol. 1, no. 4, pp. 423–432, 2005.

[11] T. K. Siu, "Fair valuation of participating policies with surrender options and regime switching," *Insurance: Mathematics & Economics*, vol. 37, no. 3, pp. 533–552, 2005.

[12] H. U. Gerber and E. S. W. Shiu, "Option pricing by Esscher transforms," *Transactions of the Society of Actuaries*, vol. 46, pp. 99–191, 1994.

[13] T. K. Siu, "A game theoretic approach to option valuation under Markovian regime-switching models," *Insurance: Mathematics & Economics*, vol. 42, no. 3, pp. 1146–1158, 2008.

[14] T. K. Siu and H. Yang, "Option pricing when the regime-switching risk is priced," *Acta Mathematicae Applicatae Sinica*, vol. 25, no. 3, pp. 369–388, 2009.

[15] R. J. Elliott and T. K. Siu, "Pricing and hedging contingent claims with regime switching risk," *Communications in Mathematical Sciences*, vol. 9, no. 2, pp. 477–498, 2011.

[16] R. J. Elliott, T. K. Siu, and A. Badescu, "On pricing and hedging options in regime-switching models with feedback effect," *Journal of Economic Dynamics & Control*, vol. 35, no. 5, pp. 694–713, 2011.

[17] R. C. Merton, "Option pricing when underlying stock returns are discontinuous," *Journal of Financial Economics*, vol. 3, pp. 125–144, 1976.

[18] R. J. Elliott, "Double martingales," *Z.Wahrscheinlichkeitstheorie und Verw.Gebiete*, vol. 34, no. 1, pp. 17–28, 1976.

[19] R. J. Elliott, L. Aggoun, and J. B. Moore, *Hidden Markov Models: Estimation and Control*, Springer, New York, NY, USA, 1994.

[20] M. H. A. Davis, "The representation of martingales of jump processes," *SIAM Journal on Control and Optimization*, vol. 14, no. 4, pp. 623–638, 1976.

[21] J. M. Harrison and D. M. Kreps, "Martingales and arbitrage in multiperiod securities markets," *Journal of Economic Theory*, vol. 20, no. 3, pp. 381–408, 1979.

[22] J. M. Harrison and S. R. Pliska, "Martingales and stochastic integrals in the theory of continuous trading," *Stochastic Processes and their Applications*, vol. 11, no. 3, pp. 215–260, 1981.

[23] J. M. Harrison and S. R. Pliska, "A stochastic calculus model of continuous trading: complete markets," *Stochastic Processes and their Applications*, vol. 15, no. 3, pp. 313–316, 1983.

[24] F. Delbaen and W. Schachermayer, "A general version of the fundamental theorem of asset pricing," *Mathematische Annalen*, vol. 300, no. 3, pp. 463–520, 1994.

[25] F. Delbaen and W. Schachermayer, "The fundamental theorem of asset pricing for unbounded stochastic processes," *Mathematische Annalen*, vol. 312, no. 2, pp. 215–250, 1998.

[26] L. Boltzmann, "Uber die Beziehung zwischen dem zweiten Hauptsatze der mechanischen Warmtheorie und der Wahrscheinlichkeitsrechnung respectiv den Satzen uber das Warmegleichgemicht," *Wiener Berichte*, vol. 76, pp. 373–435, 1877.

[27] H. Akaike, "On the likelihood of a time series model," *The Statistican*, vol. 21, pp. 217–235, 1978.

[28] H. Tong, *Non-Linear Time Series: A Dynamical System Approach*, Oxford University Press, Oxford, UK, 1990.

[29] Y. Miyahara, "Canonical martingale measures of incomplete assets markets," in *Proceedings of the 7th Japan-Russia Symposium, Probability Theory and Mathematical Statistics*, S. Watanabe, M. Fukushima, Y. V. Prohorov, and A. N. Shiryaev, Eds., pp. 343–352, Tokyo, Japan, 1996.

[30] H. Bühlmann, F. Delbaen, P. Embrechts, and A. N. Shiryaev, "No-arbitrage, change of measure and conditional Esscher transforms," *CWI Quarterly*, vol. 9, no. 4, pp. 291–317, 1996.

[31] Y. Miyahara, "Minimal entropy martingale measures of jump type price processes in incomplete assets markets," *Asia-Pacific Financial Markets*, vol. 6, no. 2, pp. 97–113, 1999.

[32] Y. Miyahara, "Geometric lévy process & MEMM: pricing model and related estimation problems," *Asia-Pacific Financial Markets*, vol. 8, no. 1, pp. 45–60, 2001.

[33] T. Fujiwara and Y. Miyahara, "The minimal entropy martingale measures for geometric Lévy processes," *Finance and Stochastics*, vol. 7, no. 4, pp. 509–531, 2003.

[34] M. Schweizer, "The minimal entropy martingale measure," in *Encyclopedia of Quantitative Finance*, R. Cont, Ed., pp. 1195–1200, John Wiley & Sons, New York, NY, USA, 2010.

[35] H. Föllmer and A. Schied, "Convex and coherent risk measures," in *Encyclopedia of Quantitative Finance*, R. Cont, Ed., pp. 355–363, John Wiley & Sons, New York, NY, USA, 2010.

[36] R. J. Elliott, *Stochastic Calculus and Applications*, Springer, Berlin, Germany, 1982.