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Research Article

Survival Exponents for Some Gaussian Processes

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The problem is a power-law asymptotics of the probability that a self-similar process does not exceed a fixed level during long time. The exponent in such asymptotics is estimated for some Gaussian processes, including the fractional Brownian motion (FBM) in $(-T_-, T)$, $T \geq T_- \gg 1$ and the integrated FBM in $(0, T)$, $T \gg 1$.

1. The Problem

Let $x(t), x(0) = 0$ be a real-valued stochastic process with the following asymptotics:

$$P(x(t) < 1,\, t \in \Delta_T) = T^{-\theta_x + o(1)}, \quad T \longrightarrow \infty, \tag{1.1}$$

where θ_x is the so-called *survival* exponent of $x(t)$. Below we focus on estimating θ_x for some self-similar Gaussian processes in extended intervals $\Delta_T = (0, T)$ and $(-T_-, T)$, $T \geq T_- \gg 1$. Usually the estimation of the survival exponents is based on Slepian's lemma. The estimation requires reference processes with explicit or almost explicit values of θ. Unfortunately, the list of such processes is very short. This includes the fractional Brownian motion (FBM), $w_H(t)$, of order $0 < H < 1$ both with one- and multidimensional time. According to Molchan ([1])

$$\theta_{w_H} = 1 - H \quad \text{for } \Delta_T = (0, T), \qquad \theta_{w_H} = d \quad \text{for } \Delta_T = (-T, T)^d. \tag{1.2}$$

Another important example is the integrated Brownian motion $I(t) = \int_0^t w(s)ds$ with the exponent

$$\theta_I = \frac{1}{4}, \qquad \Delta_T = (0, T), \tag{1.3}$$

(Sinai [2]).

The nature of this result is best understood in terms of a series of generalizations where the integrand is a random walk with discrete or continuous time (see, e.g., Isozaki and Watanabe [3]; Isozaki and Kotani [4]; Simon [5]; Vysotsky [6, 7]; Aurzada and Dereich [8]; Dembo et al. [9]; Denisov and Wachtel [10]. The extension of (1.3) to include the case of the integrated fractional Brownian motion, $I_H(t) = \int_0^t w_H(s)ds$, remains an important; but as yet unsolved problem.

Below we consider the survival exponents for the following Gaussian processes: $I_H(t)$, $t \in (0,T)$; $\chi_H(t) = \operatorname{sign}(t)w_H(t)$, $t \in (-T,T)$; FBM in $\Delta_T = (-T^\alpha, T)$, $0 \le \alpha \le 1$; the Laplace transform of white noise with $\Delta_T = (0,T)$; the fractional Slepian's stationary process whose correlation function is $B_{S_H}(t) = (1-|t|^{2H})_+$, $0 < H \le 1/2$.

Our approach to the estimation of θ is more or less traditional. Namely, any self-similar process $x(t)$ in $\Delta_T = (0,T)$ generates a *dual stationary process* $\tilde{x}(s) = e^{-hs}x(e^s)$, $s < \ln T := \tilde{T}$, where h is the self-similarity index of $x(t)$. For a large class of Gaussian processes, relation (1.1) induces the dual asymptotics

$$P\left(\tilde{x}(s) \le 0,\ 0 < s < \tilde{T}\right) = \exp\left(-\tilde{\theta}_x \tilde{T}(1+o(1))\right), \quad \tilde{T} \longrightarrow \infty, \tag{1.4}$$

with the same exponent $\tilde{\theta}_x = \theta_x$, [1, 11]. More generally, the dual exponent is defined by the asymptotics

$$P(x(t) \le 0,\ t \in \Delta_T \setminus (-1,1)) = \exp\left(-\tilde{\theta}_x \tilde{T}(1+o(1))\right). \tag{1.5}$$

To formulate the simplest condition for the exponents to be equal, we define one more exponent $\check{\theta}_x$ by means of the asymptotics

$$P\left(\left|t_T^*\right| \le 1\right) = T^{-\check{\theta}_x + o(1)}, \tag{1.6}$$

where t_T^* is the position of the maximum of $x(t)$ in Δ_T, that is, $x(t_T^*) = \sup(x(t),\ t \in \Delta_T)$.

Lemma 1.1 (see [1, 11]). *Let $x(t), x(0) = 0$ be a self-similar continuous Gaussian process in $\Delta_T = (-T_-, T)$, $T_- \le T$ and $(H_x(\Delta_T), \|\ \|_T)$ be the reproducing kernel Hilbert space associated with $x(t)$. Suppose that there exists such an element φ of $H_x(\Delta_T)$ that $\varphi(t) \ge 1$, $|t| > 1$ and $\|\varphi\|_T^2 = o(\ln T)$. Then θ_x, $\tilde{\theta}_x$, and $\check{\theta}_x$ can exist simultaneously only; moreover, the exponents are equal to each other.*

The equality $\theta = \tilde{\theta}$ reduces the original problem to the estimation of $\tilde{\theta}$. Nonnegativity of the correlation function of $\tilde{x}(s)$ guarantees the existence of the exponent $\tilde{\theta}$, [12]. In turn, the inequality of two correlation functions, $B_1(s) \le B_2(s)$, $B_i(0) = 1$, implies, by Slepian's lemma, the inverted inequality for the corresponding exponents: $\tilde{\theta}_1 \ge \tilde{\theta}_2$.

An essentially different approach is required to find the explicit value of θ for FBM in $\Delta_T = (-T^\alpha, T)$ and to estimate $\tilde{\theta}$ in (1.4) for the fractional Slepian process with a small parameter H.

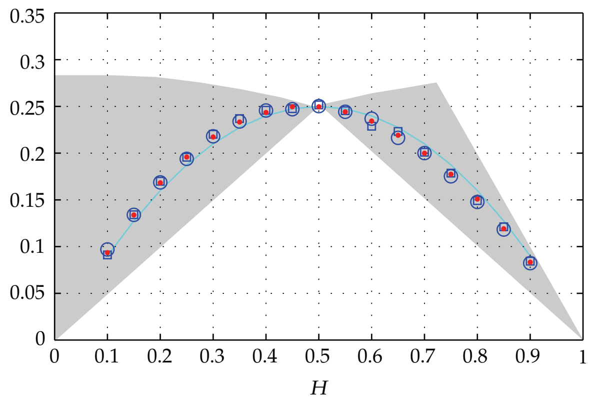

Figure 1: The survival exponents $\tilde{\theta}_{I_H}$ for the integrated fractional Brownian motion in $\Delta_T = (-T, T)$: hypothetical values (*parabolic line*), empirical estimates (*small circles, squares*), and theoretical bounds (*shaded zone* given by Proposition 2.1(b, c)). The empirical exponents are based on the model ((2.9), $\alpha(H) = 0$) in three time intervals of $\tilde{T} = \ln T : \ln(1/\varepsilon) \le \tilde{T}(1-H)H \le \ln(10/\varepsilon)$, where $\varepsilon = 0.01$, 0.003, and 0.001 (see more in [13]).

2. Examples

2.1. Integrated Fractional Brownian Motion

Consider the process

$$I_H(t) = \int_0^t w_H(s)ds, \tag{2.1}$$

where $w_H(t)$ is the fractional Brownian motion, that is, a Gaussian random process with the stationary increments: $E|w_H(t) - w_H(s)|^2 = |t-s|^{2H}$, $w_H(0) = 0$. Molchan and Khokhlov [13, 14] analyzed the exponent θ_{I_H} theoretically and numerically and formulated the following *Hypothesis*: $\theta_{I_H} = H(1-H)$ for $\Delta_T = (0, T))$ and $\theta_{I_H} = 1 - H$ for $\Delta_T = (-T, T)$.

The unexpected symmetry $\theta_{I_H} = \theta_{I_{1-H}}$ for $\Delta_T = (0, T)$ caused some doubt as to the numerical results. To support the hypothesis, Molchan [11] derived the following estimates of θ for $I_H(t)$:

$$\rho H(1-H) \le \theta^{+}_{I_H} \le \theta^{-/+}_{I_H} \le (1-H), \tag{2.2}$$

where ρ is a small constant and $(+)$ and $(-/+)$ are indicators of the intervals $\Delta_T = (0, T)$ and $\Delta_T = (-T, T)$, respectively. Note that, in the case of $H < 1/2$ and $\Delta_T = (-T, T)$, it is unknown whether the exponent exists. In such cases we have to operate with upper $\overline{\theta}$ and lower $\underline{\theta}$ exponents. Therefore, $\theta^{-/+}_{I_H}$ in (2.2) for $H < 1/2$ is any number from the interval $(\underline{\theta}, \overline{\theta})$. The relation (2.2) can be improved as follows.

Proposition 2.1. *For $\Delta_T = (0, T)$, one has*

(a) $\theta_{I_H} \ge \theta_{I_{1-H}}$, $0 < H \le 0.5$,

(b) $0.5(H \wedge \overline{H}) \le \theta_{I_H} \le \overline{H}$, $\overline{H} = 1 - H$,

(c) $\theta_{I_H} \le \sqrt{(1-(H \wedge \overline{H})^2)/12}$.

Proof. The identity of the dual exponents for $I_H(t)$ follows from [14]; the dual survival exponent exists because the dual correlation function,

$$\widetilde{B}_{I_H}(s) = (2+4H)^{-1}\left[(2+2H)\left(e^{Hs}+e^{-Hs}\right) - e^{(1+H)s} - e^{-(1+H)s} + \left(e^{s/2}-e^{-s/2}\right)^{2H+2}\right] \quad (2.3)$$

is positive. The inequality (a) is a consequence of the relation

$$\widetilde{B}_{I_H}(t) \le \widetilde{B}_{I_{1-H}}(t), \quad 0 < H \le \frac{1}{2}. \quad (2.4)$$

To prove (b, c), we use the correlation function of the process $\widetilde{I}_{1/2}(ps)$, that is,

$$\widetilde{B}_{I_{1/2}}(ps) = \frac{1}{2}\left(3\exp\left(-\frac{p|s|}{2}\right) - \exp\left(-\frac{3p|s|}{2}\right)\right), \quad (2.5)$$

and the respective exponent $\widetilde{\theta} = p/4$ (see (1.3)). The relation

$$\widetilde{B}_{I_H}(t) \le \widetilde{B}_{I_{1/2}}(pt), \quad H \ge \frac{1}{2}, \ p = 2(1-H), \quad (2.6)$$

implies $\theta_{I_H} \ge (1-H)/2$ for $H \ge 1/2$. Using (a) in addition, we come to the lower bound in (b) because $\theta_{I_H} \ge \theta_{I_{1-H}} \ge H/2$ for $H \le 1/2$.

Similarly, the relation

$$\widetilde{B}_{I_H}(t) \ge \widetilde{B}_{I_{1/2}}(pt), \quad H \le \frac{1}{2}, \ p = 2\sqrt{\frac{1-H^2}{3}}, \quad (2.7)$$

implies (c) for all H. A test of the purely analytical facts (2.4), (2.6), and (2.7) is given in the appendix. □

Remark 2.2. Proposition 2.1(a) follows from the more informative relation:

$$P\left(\widetilde{I}_H(s) \le 0, \ s \in \left(0, \widetilde{T}\right)\right) \le P\left(\widetilde{I}_{1-H}(s) \le 0, \ s \in \left(0, \widetilde{T}\right)\right). \quad (2.8)$$

This inequality is important for understanding the numerical result by Molchan and Khokhlov [13] represented in the form of empirical estimates of $\widetilde{\theta}_{I_H}$ in Figure 1. We can see that the empirical estimates show small but one-sided deviations from the hypothetical curve $\theta = H(1-H)$ before and after $H = 1/2$. The signs of these deviations are consistent with (2.8), while the amplitudes are compatible with the model

$$P\left(\widetilde{I}_H(s) \le 0, \ s \in \left(0, \widetilde{T}\right)\right) \approx C\widetilde{T}^{\alpha(H)} \exp\left(-H(1-H)\widetilde{T}\right), \quad \widetilde{T} \gg 1, \ \operatorname{sgn}\alpha(H) = \operatorname{sign}(H-0.5), \quad (2.9)$$

and $\alpha(H) = H - 0.5$ (more can be found in [13]).

2.2. The Laplace Transform of White Noise

Consider the process $L(t) = t\int_0^\infty e^{-tu} dw(u)$, where $w(u)$ is Brownian motion. The dual stationary process $\widetilde{L}(s)$ has the correlation function $\widetilde{B}_L(s) = 1/\cosh(s/2)$. Using (2.5) as a majorant of $\widetilde{B}_L(s)$, we improve the lower bound of $\widetilde{\theta}_L$ as follows.

Proposition 2.3. $3^{-1/2} \le 4\widetilde{\theta}_L \le 1$.

Proof. That the exponents for the dual processes L and $\widetilde{L}$ are equal follows from Lemma 1.1 with $\varphi(t) = t(1+\varepsilon_T)/(t+\varepsilon_T)$, where $\varepsilon_T = 1/\sqrt{\ln T}$. For indeed, $\varphi(t) = EL(t)\eta$, where $\eta = (1+\varepsilon_T^{-1})L(\varepsilon_T)$. By definition of the Hilbert space $H_L(\Delta_T)$, we have the desired estimate:

$$\|\varphi\|_T^2 = E\eta^2 = \frac{\varepsilon_T^{-1}(\varepsilon_T + 1)^2}{2} = O\left(\sqrt{\ln T}\right). \tag{2.10}$$

By (1.3) and Slepian's lemma, the relation

$$\widetilde{B}_{I_{1/2}}(t) \le \widetilde{B}_L(pt), \quad p \le 1 \tag{2.11}$$

has as a consequence the estimate $4p\widetilde{\theta}_L \le 1$. The opposite inequality

$$\widetilde{B}_{I_{1/2}}(t) \ge \widetilde{B}_L(pt), \quad p^2 \ge 3, \tag{2.12}$$

implies $4p\widetilde{\theta}_L \ge 1$. The test of ((2.11), $p = 1$) and ((2.12), $p = 2$) is very simple and yields the Li and Shao [12, 15] estimates: $0.5 < 4\widetilde{\theta}_L < 1$. The appendix contains a proof of (2.11), (2.12) for all interesting values of p: $1, 2$, and $\sqrt{3}$. □

Remark 2.4. The *dual survival* exponent of $L(t)$ is of interest as a parameter of the following asymptotic relation:

$$P\left(\sum_0^{2n} \xi_i x^i \ne 0, x \in R^1\right) = (2n)^{-4\widetilde{\theta}_L + o(1)}, \quad n \longrightarrow \infty, \tag{2.13}$$

for random polynomials with the standard Gaussian independent coefficients [16]. A continuous analogue of the polynomial on any of four intervals $0 < \pm x^{\pm 1} \le 1$ is the Laplace transform of white noise, which partially explains the appearance of $\widetilde{\theta}_L$ in the asymptotic relation (2.13). Simulations suggest $4\widetilde{\theta}_L = 0.76 \pm 0.03$, [16] and $4\widetilde{\theta}_L \approx 0.75$, [17].

2.3. Fractional Slepian's Process

We reserve this term for a Gaussian stationary process $S_H(t)$ with correlation function

$$B_{S_H}(t) = \left(1 - |t|^{2H}\right)_+, \quad 0 < H \le \frac{1}{2}, \tag{2.14}$$

because $S_{1/2}(t)$ is known as the Slepian process and $S_H(t) - S_H(0)$, $0 < t \le 1$, is equal in distribution to the fractional Brownian motion on the interval (0,1). By the Polya criterion, the fractional Slepian process exists because $B_{S_H}(t)$ is a nonincreasing and a convex function on the semiaxis. The fact of the correlation function being nonnegative guarantees the existence of $\tilde{\theta}_{S_H}$ in (1.4). $S_H(t)$ can be useful as a reference process in estimation of the survival exponents. Therefore it is important to have accurate estimates of the exponent for $S_H(t)$. The case of small H is the most interesting because it describes a transition of $S_H(t)$ to white noise. Our estimates of $\tilde{\theta}_{S_H}$ are based on two lemmas, where we use the following notation:

$$\tilde{\theta}(f, \Delta) = -|\Delta|^{-1} \log P\left(x(t) \le f(t),\, t \in \Delta\right). \tag{2.15}$$

Lemma 2.5 (see [12]). *Let $x(t)$ be a centered Gaussian stationary process with a finite nonnegative correlation function, that is, $B_x(t) \ge 0$ and $B_x(t) = 0$ for $|t| \ge T_0$. Then the limit*

$$\tilde{\theta}(a) = \lim_{T \to \infty} \tilde{\theta}(a, (0, T)), \tag{2.16}$$

exists for every $a \in R^1$. Moreover,

$$\left(1 + \frac{1}{k}\right)^{-1} \tilde{\theta}(a, k\Delta_0) \le \tilde{\theta}(a) \le \tilde{\theta}(a, k\Delta_0), \quad \Delta_0 = (0, T_0). \tag{2.17}$$

Remark 2.6. Lemma 1.1 was derived by Li and Shao [12] for the Slepian process, $S_{1/2}(t)$, but the proof remains valid for the general case. There is an explicit but very complicated formula for $\tilde{\theta}_{S_H}(0, \Delta)$ with $H = 1/2$ [18]. In case of $\Delta = (0, 2)$, this result reduces to

$$P(S_{1/2}(t) \le 0,\, t \in (0, 2)) = \frac{1}{6} - \frac{2 + \sqrt{3}}{8\pi} \tag{2.18}$$

and gives $1.336 < \tilde{\theta}_{S_{1/2}} < 2.004$.

Lemma 2.7 (see [8]). *Let $x(t)$ be a centered Gaussian process in an interval Δ with a correlation function $B(t, s)$ and $(H_x(\Delta), \|\cdot\|_\Delta)$ be the Hilbert space with the reproducing kernel $B(t, s)$ on $\Delta \times \Delta$. If $0 < \tilde{\theta}(a, \Delta) < \infty$, then*

$$\left| \sqrt{\tilde{\theta}(a + f, \Delta)} - \sqrt{\tilde{\theta}(a, \Delta)} \right| \le \frac{\|f\|_\Delta}{\sqrt{2|\Delta|}}. \tag{2.19}$$

Remark 2.8. Lemma 2.7 is a version of Proposition 1.6 from the paper by Aurzada and Dereich [8]; relation (2.19) successfully supplements the original Lemma 1.1.

Proposition 2.9. *The persistence exponent of process $S_H(t)$ has the following estimates:*

$$-(1-H)H^{-1}\ln(2H) \le \widetilde{\theta}_{S_H} \le 49H^{-2}, \tag{2.20}$$

where the left inequality holds for $0 < H \le e^{-2}/2$.

Corollary 2.10. *If $w_H^-(t) = (w_H(t) - w_H(-t))/2$ is the odd component of the fractional Brownian motion, then*

$$\widetilde{\theta}_{w_H^-} \le \frac{(7/H)^2}{2}, \quad 0 < H < 0.5. \tag{2.21}$$

Proof. The dual stationary process $\widetilde{w}_H^-$ has the correlation function

$$\widetilde{B}_{w_H^-}(t) = \left(\cosh\frac{t}{2}\right)^{2H} - \left(\sinh\frac{t}{2}\right)^{2H}, \tag{2.22}$$

which is positive. Hence the exponent $\widetilde{\theta}_{w_H^-}$ exists. The inequality

$$\widetilde{B}_{w_H^-}(2t) = (\cosh t)^{2H}\left(1 - (\tanh t)^{2H}\right) \ge (\cosh t)^{2H}\left(1 - |t|^{2H}\right)_+ \ge \widetilde{B}_{S_H}(t), \tag{2.23}$$

and Proposition 2.9 immediately imply the corollary. □

Remark 2.11. The following estimates of $\widetilde{\theta}_{w_H^-}$ are due to Krug et al. [19]:

$$\begin{aligned} \widetilde{\theta}_{w_H^-} &\ge \min\left(\frac{(1-H)^2}{H}, (1-H)2^{1/(2H)-1}\right), \quad 0 < H < 0.5, \\ \widetilde{\theta}_{w_H^-} &\le \frac{(1-H)^2}{H}, \quad 0.1549 < H < 0.5. \end{aligned} \tag{2.24}$$

For small H these estimates are one-sided only.

Remark 2.12. A considerable difference in the behavior of $\widetilde{\theta}_{w_H^-}$ and $\widetilde{\theta}_{w_H} = 1 - H$ for small H is expected. Heuristically this can be explained as follows. As $H \to 0$, the discrete processes $\widetilde{w}_H^-(k\Delta)$ and $\widetilde{w}_H(k\Delta)$ have different weak limits: $\{\xi_k\}$ and $\{\xi_k - \eta/\sqrt{2}\}$, respectively, where $\{\xi_k\}$ and η are independent standard Gaussian variables. The probability (1.4) for the limiting processes is quite different:

$$P(\xi_k < 0,\, k = 1 \div N) = 2^{-N}, \qquad P(\xi_k - \eta \le 0, k = 1 \div N) = (N+1)^{-1}. \tag{2.25}$$

Unfortunately, this argument fails to predict the behavior of $\widetilde{\theta}_{S_H}$ for small H, because the step Δ cannot be arbitrary and is a function of H.

2.4. Khanin's Problem

The survival exponent for fractional Brownian motion in the intervals $\Delta_T = (-T, T)$ is independent of the parameter H: $\theta_{w_H} = 1$. This interesting fact follows from both self-similarity of w_H and the stationarity of its increments [1].

In the case $H < 0.5$, the variables $w_H(t)$ and $w_H(-t)$ are positive correlated. Therefore, a possible power-law asymptotics

$$P(w_H(t) < 1, -w_H(-t) < 1, t \in (0, T)) = T^{-\theta + o(1)}, \tag{2.26}$$

where we change sign before $w_H(t)$ for negative t only, may have a radically different exponent compared with $\theta_{w_H} = 1$. The question of finding bounds on the exponent θ_{χ_H} for the process

$$\chi_H(t) = \operatorname{sign}(t) w_H(t), \quad \Delta_T = (-T, T), \tag{2.27}$$

was asked by K. Khanin. The next proposition contains a partial answer to this question.

Proposition 2.13. (1) *In the case* $0.5 \le H < 1$, *the exponent* θ_{χ_H} *for* $\Delta_T = (-T, T)$ *exists and admits of the following estimates:*

$$1 < \theta_{\chi_H} (1 - H)^{-1} \le 2, \quad 0.5 \le H < 1, \tag{2.28}$$

in addition, $\theta_{\chi_{1/2}} = 1$.

(2) *Let* $\underline{\theta}_{\chi_H}$ *be the lower exponent in* (2.26), *then*

$$\begin{aligned} \underline{\theta}_{\chi_H} (1 - H)^{-1} &\ge \left(H^{-1} - 1\right) \wedge 2^{1/2H - 1}, \quad 0 < H < 0.25, \\ \underline{\theta}_{\chi_H} (1 - H)^{-1} &\ge 2 \quad 0.25 < H \le 0.5. \end{aligned} \tag{2.29}$$

Remark 2.14. To clarify why $\theta_{\chi_H} / \theta_{w_H}$ is unbounded for small H in the case $\Delta_T = (-T, T)$, we consider again the limiting sequence for $w_H(k\Delta)$ as $H \to 0$. This is $\{(\xi_k - \xi_0)/\sqrt{2}\}$, where the $\{\xi_k\}$ are independent standard Gaussian variables. The probability (1.1) for the limit sequence is

$$P\left\{\xi_k < \xi_0 + \sqrt{2}, |k| \le N\right\} = (2N + 1)^{-1} l(N), \tag{2.30}$$

where $l(N)$ is a slowly varying function, whereas for the limit sequence of $\chi_H(k\Delta)$ we have

$$P\left\{\xi_{-k} - \sqrt{2} < \xi_0 < \xi_k + \sqrt{2}, 0 < k \le N\right\} \approx \sqrt{\pi} e N^{-1/2} \Phi\left(\sqrt{2}\right)^{2N}, \tag{2.31}$$

where $\Phi(x)$ is the Gaussian distribution function. As in Remark 2.12, we have nontrivial exponential asymptotics where the threshold for $\{\xi_k\}$ is constant or bounded. Indeed, if the event in (2.31) is true, then

$$|\xi_0| < \sqrt{2} + \frac{\max\left(\left|\sum_1^N \xi_{-k}\right|, \left|\sum_1^N \xi_k\right|\right)}{N} = \sqrt{2} + \frac{O_p(1)}{\sqrt{N}}. \tag{2.32}$$

2.5. *An Explicit Value of* θ_x

We have two explicit but isolated results for the fractional Brownian motion: $\theta_{w_H} = (1-H)$ for $\Delta_T = (0,T)$ and $\theta_{w_H} = 1$ for $\Delta_T = (-T,T)$. These results can be combined as follows.

Proposition 2.15. *If* $\Delta_T = (-T^\alpha, T)$, $0 \le \alpha \le 1$, *then* $\theta_{w_H} = \alpha H + (1-H)$.

Remark 2.16. The result is based on the following properties of the position t_Δ^* of the maximum of $w_H(t)$ in $\Delta = [0,1]$: t_Δ^* has a continuous probability density $f_\Delta^*(t)$ in (0, 1) and $f_\Delta^*(t) \approx O(t^{-H})$ as $t \to 0$. In the case of multidimensional time, the behavior of $f_\Delta^*(t)$, $\Delta = (0,1)^d$ near $t=0$ is a key to the survival exponent of $w_H(t)$ for $\Delta_T = (-T^\alpha, T)^d$, $0 < \alpha < 1$ and $H < 1$. By (1.2), $\theta_{w_H} = d$ in the case $\alpha = 1$, and $\theta_{w_H} = \alpha d$ in the degenerate case: $H = 1$.

3. Proofs

Proof of Proposition 2.9

Lower Bound. Let $\widetilde{w}_H(t)$ be a dual fractional Brownian motion with the parameter H, that is, a Gaussian stationary process with correlation function $\widetilde{B}_{w_H}(t) = \cosh(Ht) - 0.5(2\sinh(t/2))^{2H}$. We prove in the appendix that for $0 < H \le e^{-2}/2$,

$$\widetilde{B}_{w_H}(pt) \ge B_{S_H}(t), \quad p = -H^{-1}\ln(2H). \tag{3.1}$$

Applying Slepian's lemma, one has $\widetilde{\theta}_{S_H} \ge p(1-H)$ because $\widetilde{\theta}_{w_H} = (1-H)$.

Upper Bound. The random variable $\eta = \int_0^1 S_H(t)dt$ corresponds to an element $f_\eta(t)$ of the Hilbert space, $H_S(\Delta)$, $\Delta = (0,1)$, with the reproducing kernel $B(t,s) = 1 - |t-s|^{2H}$. By definition of $H_S(\Delta)$, we have

$$f_\eta(t) = ES_H(t)\eta = 1 - \frac{t^{1+2H} + (1-t)^{1+2H}}{1+2H},$$
$$\|f_\eta\|_\Delta^2 = E\eta^2 = H(3+2H)(1+H)^{-1}(1+2H)^{-1}. \tag{3.2}$$

It is easy to see that $f_\eta(0) \le f_\eta(t) \le f_\eta(1/2)$. Therefore,

$$H < f_\eta(t) < 2H\ln(2e), \qquad \frac{4H}{3} < \|f_\eta\|_\Delta^2 < 3H. \tag{3.3}$$

Let m_H be the median of the random variable $M = \max\{S_H(t),\ t \in \Delta\}$, where $\Delta = (0,1)$. Then

$$0.5 = P(S_H(t) < m_H,\ t \in \Delta) < P\Big(S_H(t) < m_H H^{-1} f_\eta(t),\ t \in \Delta\Big), \tag{3.4}$$

because $H^{-1} f_\eta(t) > 1$. Setting $x(t) = S_H(t)$ in Lemma 2.5 and using notation (2.15), one has

$$\begin{gathered} \tilde{\theta}\Big(m_H H^{-1} f_\eta(t), \Delta\Big) < \ln 2, \\ \sqrt{\tilde{\theta}(0,\Delta)} < \sqrt{\ln 2} + \frac{m_H H^{-1} \|f_\eta\|_\Delta}{\sqrt{2}}. \end{gathered} \tag{3.5}$$

Using Lemma 1.1 and the inequality $\|f_\eta\|_\Delta < \sqrt{3H}$, we have

$$\tilde{\theta}_{S_H} < \tilde{\theta}(0,\Delta) < \left(\sqrt{\ln 2} + m_H \sqrt{\frac{1.5}{H}}\right)^2. \tag{3.6}$$

It is well known (see, e.g., [20]) that $m_H < 4\sqrt{2} D(\Delta, \sigma/2)$, where $\sigma^2 = \max_\Delta E S_H(t)$ and D is the Dudley entropy integral related to the semimetrics on Δ: $\rho^2(t,s) = E(S_H(t) - S_H(s))^2$.

In our case $\rho(t,s) = \sqrt{2}|t-s|^H$, $\sigma = 1$ and therefore

$$m_H < \frac{c_H}{\sqrt{H}}, \tag{3.7}$$

where

$$c_H = 4\sqrt{(1-H)\ln 2} + 2^{3-H}\sqrt{\pi}\Phi\Big(-\sqrt{1-H}\ln 4\Big) < 5.36, \quad H < \frac{1}{2}, \tag{3.8}$$

and $\Phi(x)$ is the standard Gaussian distribution. Hence,

$$\tilde{\theta}_{S_H} < \left(\sqrt{\ln 2} + \frac{5.36\sqrt{1.5}}{H}\right)^2 < \left(\frac{7}{H}\right)^2. \tag{3.9}$$

Proof of Proposition 2.13

Part (1). In the case of $H \geq 0.5$, the process $\chi_H(t) = \operatorname{sign}(t) w_H(t)$ has nonnegative correlations on R^1. In the standard manner, this implies the existence of θ_{χ_H} for $\Delta_T = (-T,T)$. More precisely, starting from a self-similar 2D process $x(t) = (w_H(t), -w_H(-t))$ on R^1_+, we consider the dual 2D stationary process $\tilde{x}(t) = x(e^t)\exp(-Ht)$ whose correlation matrix has positive elements. By [12], we conclude that the exponent $\tilde{\theta}_{\chi_H}$ for $\tilde{x}(t)$ exists.

The equality $\tilde{\theta}_{\chi_H} = \theta_{\chi_H}$ *for* $\Delta_T = (-T,T)$. We will use Lemma 1.1. By the relation $\chi_H(t) = \operatorname{sign}(t) w_H(t)$, the map $\varphi(t) \mapsto \operatorname{sign}(t)\varphi(t)$ is an isometry between the Hilbert spaces

$H_{\chi_H}(\Delta_T)$ and $H_{w_H}(\Delta_T)$ associated with $\chi_H(t)$ and $w_H(t)$ on $\Delta_T = (-T, T)$, respectively. To prove the equality of the dual exponents, it is enough to find $\varphi(t) \in (H_{w_H}(R^1), \|\cdot\|_R)$ such that $\mathrm{sgn}(t)\varphi(t) \geq 1$ for $|t| \geq 1$. We can use

$$\varphi(t) = \mathrm{sgn}(t)\min(|t|, 1) = \int \left(e^{it\lambda} - 1\right)\frac{\sin\lambda}{i\pi\lambda^2}d\lambda, \tag{3.10}$$

because

$$\|\varphi\|_R^2 = k_H \int \frac{(\sin\lambda)^2}{(\pi\lambda^2)^2}|\lambda|^{1+2H}d\lambda < \infty, \tag{3.11}$$

(see [14]).

Estimation of θ_{χ_H}, $H > 1/2$. Since $E\chi_H(t)\chi_H(s) \geq 0$ for any t, s, we have, by Slepian's lemma,

$$p_T := P(w_H(t) < 1,\ -w_H(-t) < 1,\ t \in (0, T)) \geq [P(w_H(t) < 1,\ t \in (0, T))]^2. \tag{3.12}$$

Using (1.2), one has $\theta_{\chi_H} \leq 2(1 - H)$.

Obviously, $p_T \leq P(w_H(t) < 1,\ t \in (0, T))$. Therefore, $\theta_{\chi_H} \geq (1 - H)$ for any H.

Part (2). Let $0 < H \leq 1/2$, then $Ew_H(t)(-w_H(-s)) \leq 0$ for $t, s > 0$. Hence,

$$p_T \leq [P(w_H(t) < 1,\ t \in (0, T))]^2, \quad \theta_{\chi_H} \geq 2(1 - H). \tag{3.13}$$

Finally,

$$p_T \leq P(w_H(t) - w_H(-t) < 2,\ t \in (0, T)) = P\left(w_H^-(t) < 1,\ t \in (0, T)\right). \tag{3.14}$$

But then, $\underline{\theta}_{\chi_H} \geq \theta_{w_H^-}$ for all H. If $\theta_{w_H^-} = \tilde{\theta}_{w_H^-}$, then we get a lower bound of $\tilde{\theta}_{\chi_H}$ for $0 < H \leq 1/4$.

The equality $\theta_{w_H^-} = \tilde{\theta}_{w_H^-}$.Let $H_{w_H^-}(\Delta)$ and $H_{w_H}(\Delta)$ be the reproducing Kernel Hilbert spaces associated with $w_H^-(t)$ and $w_H(t)$, respectively. By the definition of $w_H^-(t)$, the map $(\varphi(t), t > 0) \mapsto (\mathrm{sign}(t)\varphi(|t|), |t| < \infty)$ is an isometric embedding of $H_{w_H^-}(R_+^1)$ in $H_{w_H}(R^1)$. To prove that the exponents are equal, it is enough to find $\varphi(t)$, $t \geq 0$ such that $\mathrm{sign}(t)\varphi(|t|) \in (H_{w_H}(R^1), \|\cdot\|_R)$, $\varphi(t) \geq 1$ for $t \geq 1$, and $\|\varphi\|_R < \infty$. As we showed above, this can be $\varphi(t) = \min(t, 1)$, $t > 0$.

Proof of Proposition 2.15

Consider the fractional Brownian motion in $\Delta_T = (-T^\alpha, T)$, $0 \leq \alpha \leq 1$. By Lemma 1.1, we can focus on the exponent related to the position of the maximum of $w_H(t)$ in Δ_T, $t^*_{\Delta_T}$.

Distribution of t^*_Δ. We remind the main properties of the distribution function, $F^*(x)$, of t^*_Δ related to the normalized interval $\Delta = (0, 1)$ (see [1, 14]):

(i) $F^*(x)$ has a continuous density $f_\Delta{}^*(x) > 0$, $0 < x < 1$ such that $(1 - x)f_\Delta{}^*(x)$ decreases and $xf_\Delta{}^*(x)$ increases on Δ;

(ii) $F^*(x)$ have the following estimates:

$$x^{1-H}l^{-1}(x) \le F^*(x) \le x^{1-H}l(x), \tag{3.15}$$

where $l(x) = \exp(c\sqrt{-\ln x})$, $c > 0$.

Due to monotonicity of $(1-x)f_\Delta^*(x)$ and $xf_\Delta^*(x)$, one has

$$(1-x)f_\Delta^*(x) \le x^{-1}\int_0^x (1-u)f_\Delta^*(u)du \le x^{-1}F^*(x), \tag{3.16}$$

$$xf_\Delta^*(x) \ge x^{-1}\int_{xq}^x uf_\Delta^*(u)du \ge q\bigl(F^*(x) - F^*(xq)\bigr), \quad 0 < q < 1. \tag{3.17}$$

By (3.15), (3.16),

$$f_\Delta^*(x) \le x^{-H}l(x)(1-x)^{-1}. \tag{3.18}$$

Using (3.15), (3.17), one has

$$f_\Delta^*(x) \ge qx^{-H}l^{-1}(x)\Bigl(1 - l(x)l(xq)q^{1-H}\Bigr). \tag{3.19}$$

If we set $q^{1-H} = l^{-2}(x)/2$, then

$$f_\Delta^*(x) \ge \frac{qx^{-H}l^{-1}(x)}{2} = c_H x^{-H}l^{-\nu_H}(x), \tag{3.20}$$

where $\nu_H = (3-H)/(1-H)$, $c_H = 2^{-(2-H)/(1-H)}$.

Distribution of $t_{\Delta_T}^*$. Let $T_1 = T_- + T$, where $T_- = T^\alpha$, then the processes $w_H(T_1\tau - T_-) - w_H(-T_-)$ and $w_H(\tau)T_1^H$ on $\Delta = (0,1)$ are equal in distribution. Hence, $t_{\Delta_T}^*$ and $T_1 t_\Delta^* - T_-$ have the same distribution as well. Therefore,

$$p_T := P\Bigl(\bigl|t_{\Delta_T}^*\bigr| \le 0.5\Bigr) = P\left(\left|t_\Delta^* - \frac{T_-}{T_1}\right| \le \frac{0.5}{T_1}\right) = T_1^{-1}f_\Delta^*\left(\frac{T_- + \varepsilon}{T_1}\right), \tag{3.21}$$

where $|\varepsilon| \le 0.5$. We have used here the existence and continuity of $f_\Delta^*(x)$.

Exponent $\breve{\vartheta}_{w_H}$. Set $\alpha = 1$. Then (3.21) implies $\lim_{T\to\infty} Tp_T = 0.5 f_\Delta^*(0.5)$.

Let $\alpha < 1$, then $(T_- + \varepsilon)/T_1 = o(1)$ as $T \to \infty$, and (3.20), (3.21) give a lower bound on p_T:

$$T_1 p_T \ge c_H \left(a_T^+\right)^{-H} l^{-\nu_H}\left(a_T^+\right). \tag{3.22}$$

Here and below $a_T^\pm = (T_- \pm 0.5)/T_1$.

Using (3.18), (3.21), we get an upper bound on p_T:

$$T_1 p_T = f_\Delta^*\left(\frac{T^\alpha + \varepsilon}{T_1}\right) \leq \frac{\left(a_T^-\right)^{-H} l\left(a_T^-\right) T_1}{T+1} \leq 2\left(a_T^-\right)^{-H} l\left(a_T^-\right). \tag{3.23}$$

By substituting $T_- = T^\alpha$, we have

$$\ln a_T^\pm = -(1-\alpha)\ln T + O\left(T^{-\beta}\right), \qquad \beta = \alpha \wedge (1-\alpha), \qquad \ln l\left(a_T^\pm\right) = O\left(\sqrt{\ln T}\right). \tag{3.24}$$

Hence,

$$-\ln p_T = (1-(1-\alpha)H)\ \ln T + O\left(\sqrt{\ln T}\right), \tag{3.25}$$

that is, $\breve{\theta}_{w_H} = \alpha H + (1-H)$.

The equality $\breve{\theta}_{w_H} = \theta_{w_H}$. Consider the Hilbert space $(H_{w_H}(R^1), \|\cdot\|_R)$ related to FBM and a function

$$\varphi(t) = \min(|t|, 1) = \int \left(e^{it\lambda} - 1\right)\left(\frac{\sin \lambda/2}{\sqrt{2\pi}\lambda/2}\right)^2 d\lambda. \tag{3.26}$$

The standard spectral representation of the kernel $Ew_H(t)w_H(s)$ and the representation (3.26) yield

$$\|\varphi\|_R^2 = k_H \int \left(\frac{\sin \lambda/2}{\sqrt{2\pi}\ \lambda/2}\right)^4 |\lambda|^{1+2H} d\lambda < \infty, \tag{3.27}$$

where $k_H = \int |e^{i\lambda} - 1|^2 |\lambda|^{-1-2H}$. Setting $\varphi_T := \{\varphi(t), t \in \Delta_T\}$, the desired statement follows from Lemma 1.1 because $\varphi_T \in (H_{w_H}(\Delta_T), \|\cdot\|_T)$ and $\|\varphi_T\|_T \leq \|\varphi\|_R$.

Appendix

Relation (2.4). ($\widetilde{B}_{I_H}(t) \leq \widetilde{B}_{I_{1-H}}(t)$).

By (2.3), one has for small and large t

$$\begin{aligned}
\widetilde{B}_{I_H}(t) &= 1 - \frac{(1-H^2)t^2}{2} + (2+4H)^{-1} t^{2+2H}(1+o(1)), \quad t \longrightarrow 0, \\
\widetilde{B}_{I_H}(t) &= (1+H)(1+2H)^{-1} e^{-Ht}\left(1-e^{-t}\right) \\
&\quad + 0.5(1+H)e^{-\overline{H}t}\left(1+O\left(e^{-t}\right)\right), \quad t \longrightarrow \infty,
\end{aligned} \tag{A.1}$$

where $\overline{H} = 1 - H$. Therefore, we have the following asymptotics for $\Delta(t) = \widetilde{B}_{I_H}(t) - \widetilde{B}_{I_{\overline{H}}}(t)$:

$$\Delta(t) = -\frac{(1-2H)t^2}{2} + O\left(t^{2+2H}\right), \quad t \longrightarrow 0,$$
$$\Delta(t) = -(1-2H)H(2+4H)^{-1}e^{-Ht} - \left(1-2\overline{H}\right)\overline{H}\left(2+4\overline{H}\right)^{-1}e^{-\overline{H}t} + O\left(e^{-t}\right), \quad t \longrightarrow \infty. \tag{A.2}$$

These relations support (2.4) both for small and large enough t. To verify (2.4) in the general case, we consider the following test function: $(2+4H)\,(2+4\overline{H})\Delta(t)\exp(-1.5t)$. Using new variables: $x = \exp(-t)$, $\alpha = 1-2H$, the test function is transformed to a function ψ on the square $S = (0,1)\times(0,1)$. Namely, $\psi = U(x,\alpha) - U(x,-\alpha)$, where

$$U(x,\alpha) = \left(4-\alpha^2\right)x^{\alpha/2}(3-\alpha)\int_0^x \left[(x-u)\left((1-u)^{1-\alpha} - u^{1-\alpha}\right) + u^{1-\alpha}\right]du. \tag{A.3}$$

We have to show that $\psi \le 0$. It is easy to see that $\psi = 0$ at the boundary of S. By (A.1), $\psi \le 0$ in vicinities of two sides of S: $x = 0$ and $x = 1$. The same is true for the other sides: $\alpha = 0$ and $\alpha = 1$ because

$$\frac{\partial\psi}{\partial\alpha}(x,0) = -4\left(1-x^2\right)\int_{1-x}^{1}\ln\left(\frac{1}{u}\right)du < 0,$$
$$\frac{\partial\psi}{\partial\alpha}(x,1) = (1-x)x^{-1/2}f(x) > 0. \tag{A.4}$$

Here

$$f(x) = -x(1-x) + x^3\ln\frac{1}{x} + \left(1-x^3\right)\ln\frac{1}{(1-x)}. \tag{A.5}$$

To verify $f(x) > 0$, $0 < x < 1$, note that $f'(x) = 3x^2(1 + v + \ln v)$, where $v = (1-x)/x$. Obviously, f' has a single zero in (0,1), that is, f has a single extreme point. But $f(0) = 0 = f(1)$ and $f(x) > 0$ for small x. Therefore, $f(x) \ge 0$, $0 < x < 1$.

Numerical testing supports the desired inequality $\psi < 0$ for interior points of S.

Comment 1. Our preliminary numerical test was concerned with points on a grid with a step of 0.005. The first derivatives of ψ are uniformly bounded from above on S. This fact helps us to find the final grid step to prove $\psi < 0$ for all interior points of S. The relevant analysis is cumbersome and so has been omitted.

Relation (2.6). ($\widetilde{B}_{I_H}(t) \le \widetilde{B}_{I_{1/2}}(pt)$, $H \ge 1/2$, $p = 2(1-H)$).

To verify the inequality $\Delta(t) = \widetilde{B}_{I_H}(t) - \widetilde{B}_{I_{1/2}}(2(1-H)t) \le 0$, we consider the following test function: $(2+4H)\Delta(t)\exp(-(1+H)t)$. Using (2.3), (2.5), and new variables ($x = \exp(-t), \alpha = 2H-1) \in S = (0,1) \times (0,1)$, we will have the following representation for the test function:

$$\begin{aligned}\psi(x,\alpha) = (3+\alpha)\left(x + x^{\alpha+2}\right) - 1 - x^{\alpha+3} + (1-x)^{\alpha+3} \\ - 3(\alpha+2)x^2 + (\alpha+2)x^{3-\alpha}.\end{aligned} \tag{A.6}$$

One has $\psi(x,\alpha) \le 0$ in vicinities of two sides of S: $x = 0$ and $x = 1$, because

$$\begin{aligned}\psi(x,\alpha) &= -\frac{(\alpha+2)(3-\alpha)x^2}{2} + O\left(x^{(\alpha+2)\wedge(3-\alpha)}\right) < 0, \quad x \longrightarrow 0, \\ \psi(x,\alpha) &= -\frac{2\alpha(1-\alpha)(3-\alpha)(1-x)^2}{2} + O\left((1-x)^3\right) \le 0, \quad x \longrightarrow 1.\end{aligned} \tag{A.7}$$

The same is true for the other sides: $\alpha = 0$ and $\alpha = 1$.

Side $\alpha = 0$

One has $\psi(x,0) = 0$ and

$$\frac{\partial \psi}{\partial \alpha}(x,0) = (1-x)\left[x(1-x) + 3x^2\ln x + (1-x)^2\ln(1-x)\right] := (1-x)\varphi_3(x) \le 0 \tag{A.8}$$

because

$$\varphi_a(x) = x(1-x) + ax^2\ln x + (1-x)^2\ln(1-x) \le 0, \quad a > 1. \tag{A.9}$$

To prove (A.9), note that $\varphi_a(0) = \varphi_a(1) = 0$ and $\varphi_a(x) = ax^2\ln x + O(x^2) \le 0$ as $x \to 0$. Hence, (A.9) holds if $\varphi_a(x)$ has a single extremum in (0,1). By

$$\varphi_a^{(4)}(x) = -2ax^{-2} - 2(1-x)^{-2} \le 0, \tag{A.10}$$

we conclude that

$$\varphi_a''(x) = (3a+1) + 2a\ln x + 2\ln(1-x), \tag{A.11}$$

is a concave function with two zeroes in (0,1), because $\varphi_a''(1/2) > 0$ and $\varphi_a''(x) \to -\infty$ as $x \to 0$ or 1.

This means that

$$\varphi_a'(x) = (a-1)x + 2ax\ln x - 2(1-x)\ln(1-x), \tag{A.12}$$

has two extremums in (0,1) only. But $\varphi'_a(0) = 0$, $\varphi'_a(1) = a - 1 > 0$, and $\varphi'_a(x) \le 0$ for small x because $\varphi''_a(x) \to -\infty$ as $x \to 0$. Hence $\varphi'_a(x)$ has a single zero in (0,1) and $\varphi_a(x)$ has a single extremum.

We have proved that $\psi(x,\alpha) \le 0$ for small α.

Side $\alpha = 1$

Here $\psi(x,1) = 0$ and

$$\frac{\partial\psi}{\partial\alpha}(x,1) = (1-x)(3-x)x^2\ln\left(\frac{1}{x}\right) + (1-x)^2\left[x + (1-x)^2\ln(1-x)\right] \ge 0, \tag{A.13}$$

because $[x + (1-x)^2\ln(1-x)] \ge x + (1-x)\ln(1-x) = -\int_0^x \ln(1-u)du \ge 0$.

Hence, $\psi(x,\alpha) = \psi'_\alpha(x,1)\,(\alpha-1)\,(1+o(1-\alpha)) \le 0$, $\alpha \to 1$.

As a result $\psi(x,\alpha) \le 0$ near the boundary of $S = (0,1)\times(0,1)$. Numerical testing supports the desired inequality $\psi < 0$ for the interior of S (see more in the Comment 1 from the appendix section "Relation (2.4)").

Relation (2.7). ($\widetilde{B}_{I_H}(t) \ge \widetilde{B}_{I_{1/2}}(pt)$, $H \le 1/2$, $p = 2\sqrt{(1-H^2)/3}$).

Let $\psi = (2+4H)(\widetilde{B}_{I_H}(t) - \widetilde{B}_{I_{1/2}}(pt))e^{-(1+H)t}$. By change of variables: $x = \exp(-t)$ and $\alpha = 2H$, we get a test function

$$\begin{aligned}\psi(x,\alpha) = {} & (2+\alpha)\left(x + x^{\alpha+1}\right) - 1 - x^{\alpha+2} + (1-x)^{\alpha+2} \\ & - 3(\alpha+1)x^{1+(\alpha+p)/2} + (\alpha+1)x^{1+(\alpha+3p)/2},\end{aligned} \tag{A.14}$$

on $S = (0,1)\times(0,1)$ and the relation between p and α is

$$3\left(\frac{p}{2}\right)^2 + \left(\frac{\alpha}{2}\right)^2 = 1. \tag{A.15}$$

One has

$$\begin{aligned}\psi(x,\alpha) &= (2+\alpha)x^{1+\alpha} - 3(1+\alpha)x^{1+(p+\alpha)/2} + O\left(x^2\right) \ge 0, \quad x \longrightarrow 0, \\ \psi(x,\alpha) &= (1-x)^{2+\alpha} + O\left((1-x)^3\right) \ge 0, \quad x \longrightarrow 1.\end{aligned} \tag{A.16}$$

In addition,

$$\psi(x,0) = x\left(2 - 3x^{3^{-1/2}} + x^{3^{1/2}}\right) \ge 0. \tag{A.17}$$

Finally, $\psi(x,1) = 0$ and

$$\frac{\partial\psi}{\partial\alpha}(x,1) = \bar{x}\left(x\bar{x} + 2x^2\ln x + \bar{x}^2\ln\bar{x}\right) = \bar{x}\varphi_2(x), \tag{A.18}$$

where $\overline{x} = 1 - x$. By (A.9), $\varphi_2(x) \le 0$.

Therefore $\psi(x, \alpha) \le 0$ near the boundary of $S = (0,1) \times (0,1)$. The numerical testing supports this conclusion for the interior of S (see more in the Comment 1 from the appendix section "Relation (2.4)").

Relations (2.11), (2.12)

Consider $\Delta(t) = \widetilde{B}_{I_{1/2}}(t) - \widetilde{B}_L(pt)$, where $\widetilde{B}_L(t) = 1/\cosh(t/2)$ and $\widetilde{B}_{I_{1/2}}(t)$ is given in (2.5). By the change of variables $x = e^{-t/2}$, we transform the test function $2(1 + e^{-pt})\Delta(t)$ to a function ψ on $(0,1)$ such that

$$\psi(x) = \left(3x - x^3\right)\left(1 + x^{2p}\right) - 4x^p. \tag{A.19}$$

Taking into account the asymptotics of ψ near 0, we come to a necessary condition for ψ to be negative, namely, $p \le 1$. Let $p = 1$, then $\psi = -(1 - x^2)^2 x \le 0$, that is, $4\theta_L \le 1$.

The Case $p > 1$. Here, $\psi \ge 0$ as $x \to 0$. An additional condition on $p > 1$ we can get from the relation $\psi \ge 0$ as $x \to 1$. One has $\psi = xQ(x)$, where

$$Q(x) = \left(3 - x^2\right)\left(1 + x^{2p}\right) - 4x^{p-1}. \tag{A.20}$$

By $Q(0) = 3$, $Q(1) = Q'(1) = 0$, we have $Q(x) = (1 - x)^2 P(x)$ and $P(1) = 0.5Q''(1) = 2(p^2 - 3)$. Thus $Q(1) \ge 0$ if $p^2 \ge 3$.

The Case $p = 2$. Here, $P(x)$ is a polynomial, $P(x) = 3 + 2x - 2x^3 - x^4$, and $P''(x) = -12x(1+x) \le 0$, that is, $P(x)$ is a concave function with $P(0) = 3$, $P(1) = 2$. Therefore, $P(x) \ge 0$ and as a result, $4\theta_L \ge 1/p = 0.5$.

Consider $p = \sqrt{3}$. One has $Q(x) = 8(1-x)^3(1+o(1))$, $x \to 1$ and $Q(0) = 3 > 0$. Therefore, $Q(x) \ge 0$, if $Q(x)$ is convex, that is, $Q''(x) \ge 0$. To verify this property, note that

$$\begin{aligned} 0.5x^2 Q''(x) &= 2(3p - 5)x^{p-1} + 3(6 - p)x^{2p} - x^2 - (7 + 3p)x^{2+2p} \\ &= (7 + 3p)x^{2p}\left(1 - x^2\right) + \rho x^{p-1} + (1 - \rho)x^{2p} - x^2 := \varphi(x), \end{aligned} \tag{A.21}$$

where $\rho = 6p - 10$.

Obviously, $\varphi(x) \ge 0$ if $px^{\alpha-1} - x^2 \ge 0$. This holds for $0< x < x_0 = 0.478$.

For $x > x_0$,

$$\rho x^{p-1} + (1 - \rho)x^{2p} - x^2 \ge \left(\rho + (1 - \rho)x_0^{p+1}\right)x^{p-1} - x^2. \tag{A.22}$$

The right part here is positive for $x < 0.55$, that is, $\varphi(x) \ge 0$ for $x \le 0.5$.

Let $x > 0.5$. Then

$$\begin{aligned} \varphi(x) &\ge (7 + 3p)2^{-2p}\left(1 - x^2\right) + \rho x^{\alpha-1} + (1 - \rho)x^{2p} - x^2 \\ &= C - (C + 1)x^2 + \rho x^{p-1} + (1 - \rho)x^{2p} := u(x), \end{aligned} \tag{A.23}$$

where $C = (7+3p)2^{-2p}$. We have $u(0) = C$, $u(1) = 0$ and

$$\begin{aligned} u'(x) &= -2(C+1)x + \rho(p-1)x^{p-2} + 2(1-\rho)px^{2p-1} \\ &= -\left(C+1-2(1-\rho)px^{2p-2}\right)x - \left((C+1)x^{3-p} - \rho(p-1)\right)x^{p-2}. \end{aligned} \tag{A.24}$$

It is easy to see that both terms in parentheses are positive on $(0.5, 1)$.

Thus, $u(x)$ decreases to $u(1) = 0$. This means that $Q''(x) \geq 0$.

Relation (3.1). ($\widetilde{B}_{w_H}(pt) \geq B_{S_H}(t)$, $pH = -\ln(2H)$, $0 < H < e^{-2}/2$).

The difference between the correlation functions is

$$\Delta(t) = \left(\cosh(Hpt) - 0.5\left(2\sinh\left(\frac{pt}{2}\right)\right)^{2H}\right) - \left(1-|t|^{2H}\right)_{+}. \tag{A.25}$$

Let $t > 1$, then $\Delta(t) = \widetilde{B}_{w_H}(pt) \geq 0$.

Let $2H < t < 1$. It is enough to show that the first term, φ, in the following representation:

$$\Delta(t) = \left[0.5e^{-Hpt} - 1 + t^{2H}\right] + 0.5e^{Hpt}\left(1-\left(1-e^{-pt}\right)^{2H}\right) := \varphi + R, \tag{A.26}$$

is nonnegative. Setting $Hp = -\ln(2H)$, $\alpha = 2H$ one has

$$\varphi(t) = 0.5\alpha^{t} + t^{\alpha} - 1. \tag{A.27}$$

Let us show that φ is decreasing. In this case φ is positive because $\varphi(1) = \alpha/2$.

We have

$$\varphi'(t) = \alpha^{t}\left(-0.5\ln\left(\frac{1}{\alpha}\right) + \psi(t)\right), \tag{A.28}$$

where $\psi(t) = \alpha^{1-t}/t^{1-\alpha}$. The function $\psi(t)$ has a single extreme point in the interval: $t^{*} = (1-\alpha)/\ln(1/\alpha)$. But $\psi(t^{*}) = \min$, because $\psi(t)$ decreases near $t = \alpha$:

$$\psi(\alpha) = 1, \quad \psi'(\alpha) = \frac{(\alpha\ln(e/\alpha) - 1)}{\alpha} \leq 0 \quad \text{for } 0 < \alpha < 1. \tag{A.29}$$

Hence, $\psi(t) \leq \max(\psi(\alpha), \psi(1)) = 1$. As a result,

$$\varphi'(t) \leq \alpha^{t}\left(-0.5\ln\left(\frac{1}{\alpha}\right) + 1\right) \leq 0. \tag{A.30}$$

The last inequality holds for $0 < \alpha < e^{-2}$, so we have

$$\Delta(t) \geq 0, \qquad 2H < t < 1 \quad \text{for } 0 < \alpha < e^{-2}. \tag{A.31}$$

Let $0 < t < 2H$. Use

$$\Delta(t) = \cosh(Hpt) - 1 + t^{2H}\left[1 - 0.5\left(2t^{-1}\sinh\left(\frac{pt}{2}\right)\right)^{2H}\right], \tag{A.32}$$

then $\Delta(t) \geq 0$ if

$$2^{1/(2H)} \geq \max_{(0,2H)}\left(2t^{-1}\sinh\left(\frac{pt}{2}\right)\right) = H^{-1}\sinh(pH) = (2H)^{-2} - 1. \tag{A.33}$$

This inequality holds for $0 < 2H < 1/4$.

Combining the above inequalities, we get (3.1) for $2H \leq e^{-2} \wedge 1/4$.

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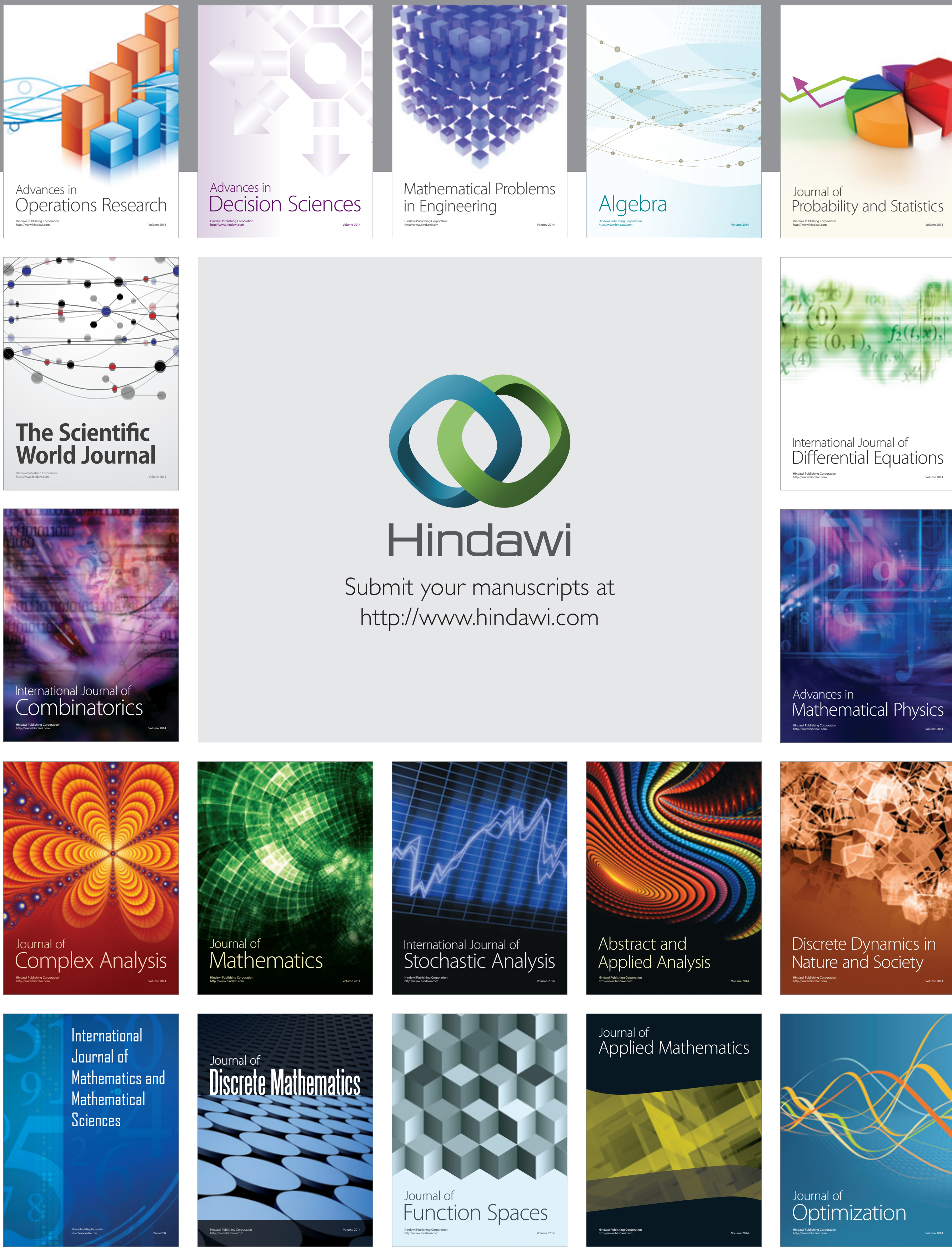
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