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Research Article

On Stochastic Equations with Measurable Coefficients Driven by Symmetric Stable Processes

V. P. Kurenok

Department of Electrical and Systems Engineering, Washington University in St. Louis, One Brookings Drive 1, St. Louis, MO 63130-4899, USA

Correspondence should be addressed to V. P. Kurenok, kurenokv@ese.wustl.edu

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We consider a one-dimensional stochastic equation $dX_t = b(t, X_{t-})dZ_t + a(t, X_t)dt$, $t \geq 0$, with respect to a symmetric stable process Z of index $0 < \alpha \leq 2$. It is shown that solving this equation is equivalent to solving of a 2-dimensional stochastic equation $dL_t = B(L_{t-})dWt$ with respect to the semimartingale $W = (Z, t)$ and corresponding matrix B. In the case of $1 \leq \alpha < 2$ we provide new sufficient conditions for the existence of solutions of both equations with measurable coefficients. The existence proofs are established using the method of Krylov's estimates for processes satisfying the 2-dimensional equation. On another hand, the Krylov's estimates are based on some analytical facts of independent interest that are also proved in the paper.

1. Introduction

Let Z be a one-dimensional symmetric stable process of index $0 < \alpha \leq 2$ with $Z_0 = 0$. In this paper we will study the existence of solutions of the equation

$$dX_t = b(t, X_{t-})dZ_t + a(t, X_t)dt, \quad t \geq 0,\ X_0 = x_0 \in \mathbb{R}, \tag{1.1}$$

where $a, b : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ are measurable functions. The existence of solutions is understood in weak sense. In the case of $1 \leq \alpha \leq 2$, the coefficients a and b are assumed to be only measurable satisfying additionally some conditions of boundness.

Two important particular cases of (1.1) are the equations

$$dX_t = b(t, X_{t-})dZ_t, \quad t \geq 0, \ X_0 = x_0 \in \mathbb{R}, \tag{1.2}$$

$$dX_t = dZ_t + a(t, X_t)dt, \quad t \geq 0, \ X_0 = x_0 \in \mathbb{R}. \tag{1.3}$$

If $\alpha = 2$, then Z is a Brownian motion, and this case has been extensively studied by many authors. The multidimensional analogue of (1.1) with only measurable (instead of continuous) coefficients was first studied by Krylov [1] who proved the existence of solutions assuming the boundness of a and b and nondegeneracty of b. The approach he used was based on L_p-estimates for stochastic integrals of processes X satisfying (1.1). Later, the results of Krylov were generalized to the case of nonbounded coefficients in various directions. We mention here only the results of Rozkosz and Slomiński [2, 3] who replaced, in particular, the assumption of boundness by the assumption of at most linear growth of the coefficients. The linear growth condition guaranteed the existence of nonexploding solutions. The case of exploding solutions was studied in [4] under assumptions of some local integrability of the coefficients a and b.

In the one-dimensional case with $\alpha = 2$, the results are even stronger. For example, for the time-independent case of the coefficients Engelbert and Schmidt obtained very general existence and uniqueness results in [5]. For the case of the time-independent equation (1.2), one had found even sufficient and necessary conditions for the existence and uniqueness (in general, exploding) solutions [6]. The time-dependent equation (1.2) was studied by several authors; we mention here [2, 7] only.

There is less known in the case $\alpha < 2$. The time-independent equation (1.1) with $1 < \alpha < 2$ was considered in [8] using the method of L_2-estimates for stable stochastic integrals with drift. To our knowledge, (1.1) in its general form and with measurable coefficients has not been studied except the particular cases (1.2) and (1.3). Thus, (1.2) in the case of $b(t, x) = b(x)$ with arbitrary index α was studied by Zanzotto in [9] where he, in particular, generalized the results of Engelbert and Schmidt to the case of $1 < \alpha \leq 2$. The time-dependent equation (1.2) with the index $1 < \alpha < 2$ was treated in [10] using the method of Krylov's estimates combined with the time change method. The time change method was also used in [11] where one obtained the sufficient conditions for the existence of solutions for the case of $0 < \alpha < 2$ different from those in [10].

On another hand, the time-independent case of (1.3), that is when $a(t, x) = a(x)$, was studied by Tanaka et al. in [12]. One obtained there the sufficient existence and uniqueness conditions assuming the drift coefficient a to be bounded plus satisfying some additional conditions depending on the case whether $0 < \alpha < 1$, $\alpha = 1$, or $1 < \alpha < 2$. The method used by them was a purely analytical one relying on some properties of homogeneous Markov processes X satisfying (1.3). More recently, Portenko [13] obtained a new existence result for the time-independent equation (1.3) for the case of $1 < \alpha < 2$ assuming the function a to be integrable on $\mathbb{R}$ of the power $p > 1/(\alpha - 1)$. The general case of (1.3) with $1 < \alpha < 2$ was studied in [14] assuming a being bounded.

The goal of this paper is to prove the existence of solutions of (1.1).

The paper is organized as follows. In Section 2 we recall the definitions and basic facts needed in the forthcoming sections. We also show that the existence of solutions of (1.1) is equivalent to the existence of solutions of a 2-dimensional stochastic equation driven by the semimartingale $W_t = (Z_t, t)$ with a corresponding matrix B. The approach is based on time change method. Section 3 is devoted to obtaining of various estimates. First, we will derive an analytic estimate for the value function associated with the control problem determined by solutions of the 2-dimensional equation. Using this estimate, we prove some variants of Krylov's estimates for solutions of the 2-dimensional equation. The results of Section 3 apply to the case with $1 \le \alpha \le 2$. Finally, in Section 4 we prove the existence of solutions of (1.1) combining the ideas of time change method with the results of Section 3.

2. Preliminaries and Time Change Method

We shall denote by $\mathbf{D}_{[0,\infty)}(\mathbb{R})$ the Skorokhod space, that is, the set of all real-valued functions $z : [0,\infty) \to \mathbb{R}$ with right-continuous trajectories and with finite left limits (also called *cádlag* functions). For simplicity, we shall write $\mathbf{D}$ instead of $\mathbf{D}_{[0,\infty)}(\mathbb{R})$. We will equip $\mathbf{D}$ with the σ-algebra $\mathfrak{D}$ generated by the Skorokhod topology. Under $\mathbf{D}^n$ we will understand the n-dimensional Skorokhod space defined as $\mathbf{D}^n = \mathbf{D} \times \cdots \times \mathbf{D}$ with the corresponding σ-algebra $\mathfrak{D}^n$ being the direct product of n one-dimensional σ-algebras $\mathfrak{D}$.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space carrying a process Z with $Z_0 = 0$, and let $\mathbb{F} = (\mathcal{F}_t)$ be a filtration on $(\Omega, \mathcal{F}, \mathbf{P})$. The notation $(Z, \mathbb{F})$ means that Z is adapted to the filtration $\mathbb{F}$. We call $(Z, \mathbb{F})$ a symmetric stable process of index $\alpha \in (0,2]$ if trajectories of Z belong to $\mathbf{D}$ and

$$\mathbf{E}\left(e^{i\xi(Z_t - Z_s)} \mid \mathcal{F}_s\right) = e^{-(t-s)|\xi|^{\alpha}} \tag{2.1}$$

for all $t > s \ge 0$ and $\xi \in \mathbb{R}$. If $\alpha = 2$, Z is a process of Brownian motion with the variance $2t$. For $\alpha = 1$ we have a Cauchy process with unbounded second moment. In general, $\mathbf{E}|Z_t|^{\beta} < \infty$ for $\beta < \alpha$. The explicit form of the probability density function is known only for three values of $\alpha \in \{1/2, 1, 2\}$.

For all $0 < \alpha \le 2$, Z is a Markov process and can be characterized in terms of analytic characteristics of Markov processes. First, for any function $f \in L^{\infty}(\mathbb{R})$ and $t \ge 0$, we can define the operator

$$(P_t f)(x) := \int_{\Omega} f(x + Z_t) d\mathbf{P}(\omega), \tag{2.2}$$

where $L^{\infty}(\mathbb{R})$ is the Banach space of functions $f : \mathbb{R} \to \mathbb{R}$ with the norm $\|f\|_{\infty} = \operatorname{ess\,sup} |f(x)|$. The family $(P_t)_{t \ge 0}$ is called the family of convolution operators associated with Z. Formally, for a suitable class of functions $g(x)$, let

$$(\mathcal{L}g)(x) = \lim_{t \downarrow 0} \frac{(P_t g)(x) - g(x)}{t}, \tag{2.3}$$

called the infinitesimal generator of the process Z.

On another hand, in the case of $\alpha \in (0,2)$, Z is a purely discontinuous Markov process that can be described by its Poisson jump measure (jump measure of Z on interval $[0,t]$) defined as

$$\mu(U \times [0,t]) = \sum_{s \le t} 1_U (Z_s - Z_{s-}), \tag{2.4}$$

the number of times before the time t that Z has jumps whose size lies in the set U. The compensating measure of μ, say ν, is given (see, e.g., [15, Propostion 13.9],) by

$$\nu(U) = \mathbf{E}\mu(U \times [0,1]) = \int_U \frac{1}{|x|^{1+\alpha}} dx. \tag{2.5}$$

It is known that for $\alpha < 2$

$$(\mathcal{L}g)(x) = \int_{\mathbb{R}\setminus\{0\}} \left[g(x+z) - g(x) - \mathbf{1}_{\{|z|<1\}} g'(x) z\right] \frac{c_\alpha}{|z|^{1+\alpha}} dz \tag{2.6}$$

for any $g \in C^2$, where C^2 is the set of all bounded and twice continuously differentiable functions $g : \mathbb{R} \to \mathbb{R}$ and c_α is a suitable constant. In contrary to the case of $\alpha \in (0,2)$, the infinitesimal generator of a Brownian motion process ($\alpha = 2$) is the Laplacian, that is, the second derivative operator.

We notice also that the use of Fourier transform can simplify calculations when working with infinitesimal generator $\mathcal{L}$. Let $g \in L_1(\mathbb{R}^2)$ and

$$\widehat{g}(\xi_1, \xi_2) := \int_{\mathbb{R}^2} e^{iz_1\xi_1 + iz_2\xi_2} g(z_1, z_2) dz_1 dz_2 \tag{2.7}$$

be the Fourier transform of g. Clearly, the function $\widehat{g}(\xi_1, \xi_2)$ can be seen as the result of taking the Fourier transform from the function $g(z_1, z_2)$ first in one variable and then in another (in any order). The following facts will be used later (cf. [14, Proposition 2.1]).

Proposition 2.1. *Let $\mathcal{L}$ be the infinitesimal generator of a symmetric stable process Z. We have he following.*

(i) *Assume that $g \in C^2(\mathbb{R})$ and $\mathcal{L}g \in L_1(\mathbb{R})$. Then*

$$\widehat{(\mathcal{L}g)}(\xi) = -|\xi|^\alpha \widehat{g}(\xi). \tag{2.8}$$

(ii) *Let g be absolutely continuous on every compact subset of $\mathbb{R}$ and $g' \in L_1(\mathbb{R})$. Then*

$$\widehat{g'}(\xi) = -i\xi \widehat{g}(\xi). \tag{2.9}$$

Finally, let us discuss how one can construct a solution of (1.1) for any $\alpha \in (0,2]$ using the time change method. By the definition, a process T is called a $\mathbb{F}$-time change if it is an increasing right-continuous process with $T_0 = 0$ such that T_t is a $\mathbb{F}$-stopping time for any $t \geq 0$ (cf. [15, chapter 6]). Define $A_t =: \inf\{s \geq 0 : T_s > t\}$ called the right-continuous inverse process to T. It follows that A is an increasing process starting at zero. Moreover, A is a $\mathbb{F}$-adapted process if and only if T is a $\mathbb{F}$-time change.

We shall here also recall the concept of exploding solutions for (1.1). Let $(\widehat{\mathbb{R}}, \mathcal{B}(\widehat{\mathbb{R}}))$ be the one-point compactification $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\Delta\}$ of $\mathbb{R}$ equipped with the σ-algebra $\mathcal{B}(\widehat{\mathbb{R}})$ of its Borel subsets. For any function $z : [0, +\infty) \to \widehat{\mathbb{R}}$ we set

$$\tau_\Delta(z) = \inf\{t \geq 0 : z(t) = \Delta\} \tag{2.10}$$

called the explosion time of the trajectory z and define $\widehat{\mathbf{D}}([0, +\infty))$ (or simply $\widehat{\mathbf{D}}$) to be the Skorohod space of exploding functions $z : [0, +\infty) \to \widehat{\mathbb{R}}$ such that z is right-continuous with finite left-hand limits on the interval $[0, \tau_\Delta(z))$ and $z(t) = \Delta$ whenever $t \geq \tau_\Delta(z)$.

We say that a stochastic process $(X, \mathbb{F})$, defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ and with trajectories in $\widehat{\mathbf{D}}$, is a *weak solution* of (1.1) with initial state $x_0 \in \mathbb{R}$ if there exists a symmetric stable process Z with respect to the filtration $\mathbb{F}$ such that $Z_0 = 0$ and

$$X_t = x_0 + \int_0^t b(s, X_{s-}) dZ_s + \int_0^t a(s, X_s) ds \quad \text{on } \{t < \tau_\Delta(X)\} \ \mathbf{P}\text{-a.s.} \tag{2.11}$$

for all $t \geq 0$, where $\tau_\Delta(X)$ is called the *explosion time* of X. Since Z is a semimartingale for all $0 < \alpha \leq 2$, the stochastic integral in (2.11) can be defined for all appropriate integrands via semimartingale integration theory.

If $\tau_\Delta(X) = \infty$, then X is called a *nonexploding solution*, otherwise—*exploding solution* with the explosion time $\tau_\Delta(X)$.

Let Z be a symmetric stable process Z of index $\alpha \in (0, 2]$ defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and $x_0 \in \mathbb{R}$ an arbitrary value. We introduce the matrix B defined as

$$B := \begin{pmatrix} 0 & |b|^{-\alpha} \\ 1 & a|b|^{-\alpha} \end{pmatrix} \tag{2.12}$$

and set $W_t := (Z_t, t)$, $L_t = (A_t, Y_t)$, $\overline{x}_0 := (0, x_0)$.

Consider the 2-dimensional equation

$$L_t = \overline{x}_0 + \int_0^t B(L_{s-}) dW_s, \tag{2.13}$$

which, if written componentwise, is equivalent to the following two one-dimensional equations:

$$A_t = \int_0^t |b|^{-\alpha}(A_s, Y_s) ds, \tag{2.14}$$

$$Y_t = x_0 + Z_t + \int_0^t a|b|^{-\alpha}(A_s, Y_s) ds. \tag{2.15}$$

Notice that the process W is a semimartingale; hence (2.13) can be seen as a stochastic differential equation with respect to a semimartingale.

Moreover, $\det(B) = -|b|^{-\alpha}$ so that the matrix B is nondegenerate since $b^{-1} := 1/b$ and $b \neq \infty$ by the definition of the coefficient b. We also see that A is a strictly increasing nonnegative process such that $A_0 = 0$. Let $A_\infty := \lim_{t\to\infty} A_t$. The properties of A imply that the right inverse to A process T is a continuous process defined on the interval $[0, A_\infty)$.

Proposition 2.2. *Assume that there exist constants $\delta_1 > 0$ and $\delta_2 > 0$ such that $\delta_1 \leq |b| \leq \delta_2$. Then,* (1.1) *has a solution if and only if* (2.13) *has a solution.*

Proof. We notice that the assumptions on the coefficient b imply that the solutions of both equations are nonexploding.

Suppose first that X is a solution of (1.1) which means that (2.11) is satisfied. The integrals on the right side of (2.11) are well defined and are **P**-a.s. finite for all $t \geq 0$. Let

$$T_t = \int_0^t |b|^{\alpha}(s, X_s)ds, \qquad A_t = \inf\{s \geq 0 : T_s > t\}. \tag{2.16}$$

It can be easily verified that the process A satisfies the relation

$$A_t = \int_0^t |b|^{-\alpha}(A_s, X_{A_s})ds. \tag{2.17}$$

By its definition, the process T is $\mathbb{F}$-adapted so that its right-inverse process A is a $\mathbb{F}$-time change process defined for all $t \geq 0$. We notice that (A_t) is a global time change (that is, $A_t \in [0, \infty)$ for all $t \geq 0$) because $T_\infty = \lim_{t\uparrow\infty} T_t = \infty$. Now define

$$Y_t = X_{A_t}, \qquad \mathcal{G}_t = \mathcal{F}_{A_t}. \tag{2.18}$$

Applying the time change $t \to A_t$ to the semimartingale X in (2.11) (see [16, Chapter 10]) and using the change of variables rule in Lebesgue-Stieltjes integral (see ch. 0, (4.9) in [17]) yield

$$Y_t = x_0 + \int_0^{A_t} b(s, X_{s-})dZ_s + \int_0^t a(A_s, Y_s)dA_s. \tag{2.19}$$

It remains to notice that the process

$$\widetilde{Z}_t := \int_0^{A_t} b(s, X_{s-})dZ_s \tag{2.20}$$

is nothing but a symmetric stable process of the index α (see [18], Theorem 3.1). Hence $L = (A, Y)$ is a solution of (2.13).

Now, let $L = (A, Y)$ be a solution of (2.13) defined on a probability space $(\Omega, \mathcal{G}, \mathbf{P})$ with a filtration $\mathbb{G}$, where $\widetilde{Z}$ is a symmetric stable process adapted to $\mathbb{G}$. Let

$$X_t = Y_{T_t}, \qquad \mathcal{F}_t = \mathcal{G}_{T_t} \tag{2.21}$$

for all $t \geq 0$ where T is the right inverse to A. It follows $A_\infty = \infty$ so that T is a global time change. By applying the time change $t \to T_t$ to the semimartingale Y in (2.15) we obtain

$$\widetilde{Z}_{T_t} = X_t - x_0 - \int_0^t a(s, X_s) ds. \tag{2.22}$$

Using the standard arguments of time change in stochastic integrals with respect to symmetric stable processes (see, e.g., [11]), we conclude that there exists a symmetric stable process (defined on the same probability space as $\widetilde{Z}$) such that

$$\widetilde{Z}_{T_t} = \int_0^t b(s, X_{s-}) dZ_s, \tag{2.23}$$

what finishes the proof. □

Actually, as Proposition 2.2 indicates, to prove the existence of solutions of (1.1), we need only to assume that (2.13) has a solution. In this sense the assumptions on the coefficient b required in Proposition 2.2 can be slightly relaxed.

Corollary 2.3. *Let L be a solution of* (2.13), *where $x_0 \in \mathbb{R}$, and there exists a constant $\delta_1 > 0$ such that $|b| > \delta_1$. Then there exists a (possibly, exploding) solution X of* (1.1).

Proof. By assumptions, there exist a solution A of (2.14) and a process Y satisfying (2.15), both adapted to the same filtration $\mathbb{F}$. For any $t \geq 0$, let

$$T_t = \inf\{s \geq 0 : A_s > t\} \tag{2.24}$$

be the right inverse of the process A. By A_∞ and T_∞ we denote the limits of processes A and T as $t \to \infty$, respectively. Clearly, A and T are strictly increasing and continuous processes defined on intervals $[0, T_\infty)$ and $[0, A_\infty)$, respectively. In particular, we have that $A_{T_t} = t \wedge A_\infty$ and $T_{A_t} = t \wedge T_\infty$ for all $t \geq 0$. We notice further that T is a $\mathbb{F}$-time change, finite on $[0, A_\infty)$ and equal to infinity for $t \geq A_\infty$. Define $X_t = x_0 + Y_{T_t}$ for all $t \in [0, A_\infty)$ and $X_t = \Delta$ for all $t \geq A_\infty$. Also let $\mathcal{H}_t = \mathcal{F}_{A_t}$ for all $t \geq 0$. Our goal is to show that the process $(X, \mathbb{H})$ is a solution of (1.1).

By making a time change in the relation (2.14), we obtain for all $t \geq 0$

$$\begin{aligned} T_t &= \int_0^{T_t} |b|^\alpha (A_s, x_0 + Y_s) |b|^{-\alpha} (A_s, x_0 + Y_s) ds = \int_0^{T_t} |b|^\alpha (A_s, x_0 + Y_s) dT_s \\ &= \int_0^{A_{T_t}} |b|^\alpha (s, x_0 + Y_{T_s}) ds = \int_0^{t \wedge A_\infty} |b|^\alpha (s, X_s) ds \quad \mathbf{P}\text{-a.s.} \end{aligned} \tag{2.25}$$

Applying the same time change to processes in (2.15) yields

$$X_{t\wedge A_\infty} = x_0 + Z_{T_t} + \int_0^{t\wedge A_\infty} a(s, X_s)ds. \tag{2.26}$$

Let us look at the process Z_{T_t}. By properties of stable integrals (see, e.g., [11, Proposition 4.3] or [18, Theorem 3.1]), there exists a symmetric stable process $(\overline{Z}, \mathbb{H})$ of the same index α stopped at A_∞ such that

$$\overline{Z}_t = \int_0^{T_t} b^{-1}(A_s, x_0 + Y_{s-})dZ_s, \quad t \geq 0. \tag{2.27}$$

From the last relation and time change properties in stochastic integrals with respect to a semimartingale (see, e.g., [16, Theorem 10.19]), it follows that

$$\overline{Z}_t = \int_0^{T_t} b^{-1}(A_s, x_0 + Y_{s-})dZ_s = \int_0^{t} b^{-1}(s, X_{s-})dZ_{T_s}, \quad t < A_\infty, \ \mathbf{P}\text{-a.s.} \tag{2.28}$$

Now, the relation (2.25) yields that, for all $t < A_\infty$, the integral $\int_0^t |b|^\alpha(s, X_s)ds$ is finite hence there exists the stochastic integral $\int_0^t b(s, X_{s-})d\overline{Z}_s$ (see, e.g., [11, Proposition 4.3]). Using (2.28), we obtain then

$$\int_0^t b(s, X_{s-})d\overline{Z}_s = \int_0^t b(s, X_{s-})b^{-1}(s, X_{s-})dZ_{T_s} = Z_{T_t}, \quad t < A_\infty, \ \mathbf{P}\text{-a.s.} \tag{2.29}$$

Enlarging the probability space, we can assume that $\overline{Z}$ is extended to a full symmetric stable process of index α. The last relation combined with (2.26) verifies that X is a solution of (2.11), possibly, exploding in A_∞. □

Corollary 2.4. *Let L be a solution of* (2.13) *with $x_0 \in \mathbb{R}$, and assume that there exist constants $\delta_1 > 0$ and $\delta_2 > 0$ such that $\delta_1 \leq |b(t,x)| \leq \delta_2$ for all (t,x). Then there exists a nonexploding solution X of* (1.1).

3. Some Estimates

Let δ_1, δ_2, and K be strictly positive constants and Z a symmetric stable process of index $0 < \alpha \leq 2$ defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration $\mathbb{F}$. By $\mathcal{D}_1$ and $\mathcal{D}_2$ we denote the classes of all $\mathbb{F}$-predictable one-dimensional processes r_t and γ_t, respectively, such that $\delta_1 \leq r_t \leq \delta_2$ and $|\gamma_t| \leq K$.

For any $(t,x) \in \mathbb{R}^2$, $\lambda > 0$, and any nonnegative, measurable function $f \in C_0^\infty(\mathbb{R}^2)$ ($C_0^\infty(\mathbb{R}^2)$ denotes the class of all infinitely many times differentiable real-valued functions with compact support defined on $\mathbb{R}^2$) defines the value function $v(t,x)$ as

$$v(t,x) = \sup_{r\in\mathcal{D}_1, \gamma\in\mathcal{D}_2} \mathbf{E}\int_0^\infty e^{-\lambda s} f\left(s + \varphi_s^r, x + X_s^\gamma\right)ds, \tag{3.1}$$

where the processes ψ^r and X^γ are given by

$$\psi_t^r = \int_0^t r_s ds, \qquad X_t^\gamma = Z_t + \int_0^t \gamma_s ds. \tag{3.2}$$

Then, for the value function v and the process (ψ^r, X^γ), the Bellman principle of optimality can be formulated as follows (cf. [1]): for any $[0,\infty)$-valued $\mathbb{F}$-stopping time τ it holds

$$v(t,x) = \sup_{r\in\mathcal{I}_1,\gamma\in\mathcal{I}_2} \mathbf{E}\left\{\int_0^\tau e^{-\lambda s} f\left(t+\psi_s^r, x+X_s^\gamma\right)ds + e^{-\lambda\tau} v\left(t+\psi_\tau^r, x+X_\tau^\gamma\right)\right\}. \tag{3.3}$$

Using standard arguments, one can derive from the principle above the corresponding Bellman equation (r and γ are deterministic)

$$\sup_{\delta_1\le r\le\delta_2} \sup_{|\gamma|\le K} \{r v_t(t,x) + \mathcal{L}v(t,x) - \lambda v(t,x) + \gamma v_x(t,x) + f(t,x)\} = 0, \tag{3.4}$$

which holds a.e. in $\mathbb{R}^2$. Here v_t and v_x denote the partial derivatives of the function $v(t,x)$ in t and x, respectively.

Define $Q = \{(t,x) : v_t(t,x) > 0\}$. Then, the Bellman equation is equivalent to two equations

$$\begin{aligned} \delta_2 v_t + \mathcal{L}v - \lambda v + K|v_x| + f = 0 \quad &\text{on } Q, \\ \delta_1 v_t + \mathcal{L}v - \lambda v + K|v_x| + f = 0 \quad &\text{on } Q^c. \end{aligned} \tag{3.5}$$

Lemma 3.1. *Let $1 < \alpha \le 2$ and $\delta_2/\delta_1 \in [1, 2+\sqrt{2})$. Then, for all $(t,x) \in \mathbb{R}^2$, it holds*

$$v(t,x) \le N\|f\|_2 := N\left(\int_{\mathbb{R}^2} f^2(s,y)ds\,dy\right)^{1/2}, \tag{3.6}$$

where the constant N depends on δ_1, K, and α only.

Proof. For any function $h : \mathbb{R}^2 \to \mathbb{R}$ such that $h \in L_1(\mathbb{R}^2)$ and any $\varepsilon > 0$ we define

$$h^{(\varepsilon)}(t,x) = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} h(t,x) q\left(\frac{t-s}{\varepsilon}, \frac{x-y}{\varepsilon}\right) ds\,dy \tag{3.7}$$

to be the ε-convolution of h with a smooth function q such that $q \in C_0^\infty(\mathbb{R}^2)$ and $\int_{\mathbb{R}^2} q(t,x)dt\,dx = 1$.

For $\varepsilon > 0$, let

$$f^{(\varepsilon)} := \begin{cases} \lambda v^{(\varepsilon)} - \delta_2 v_t^{(\varepsilon)} - \mathcal{L}v^{(\varepsilon)} - K\left|v_x^{(\varepsilon)}\right| & \text{on } Q, \\ \lambda v^{(\varepsilon)} - \delta_1 v_t^{(\varepsilon)} - \mathcal{L}v^{(\varepsilon)} - K\left|v_x^{(\varepsilon)}\right| & \text{on } Q^c. \end{cases} \tag{3.8}$$

It follows that

$$\begin{aligned}\delta_2 v_t^{(\varepsilon)} + Lv^{(\varepsilon)} - \lambda v^{(\varepsilon)} &= \left(\delta_2 v_t^{(\varepsilon)} + Lv^{(\varepsilon)} - \lambda v^{(\varepsilon)}\right)\mathbf{1}_Q + \left(\delta_2 v_t^{(\varepsilon)} + Lv^{(\varepsilon)} - \lambda v^{(\varepsilon)}\right)\mathbf{1}_{Q^c} \\ &= -\left(f^{(\varepsilon)} + K\left|v_x^{(\varepsilon)}\right|\right) + (\delta_2 - \delta_1)v_t^{(\varepsilon)}\mathbf{1}_{Q^c}\end{aligned} \tag{3.9}$$

so that

$$\left(\delta_2 v_t^{(\varepsilon)} + \mathcal{L}v^{(\varepsilon)} - \lambda v^{(\varepsilon)}\right)^2 \le 4\left(\left(f^{(\varepsilon)}\right)^2 + K^2\left(v_x^{(\varepsilon)}\right)^2\right) + 2(\delta_2 - \delta_1)^2\left(v_t^{(\varepsilon)}\right)^2. \tag{3.10}$$

Obviously, $f^{(\varepsilon)}$ is square integrable, and (3.5) implies that $f^{(\varepsilon)} \to f$ as $\varepsilon \downarrow 0$ a.s. in $\mathbb{R}^2$.

Now, applying Proposition 2.1, the Parseval identity and integration by parts to the inequality

$$\begin{aligned}&\int_{\mathbb{R}^2}\left(\delta_2 v_t^{(\varepsilon)} + \mathcal{L}v^{(\varepsilon)}(t,x) - \lambda v^{(\varepsilon)}(t,x)\right)^2 dt\,dx \\ &\quad\le 4\int_{\mathbb{R}^2}\left(\left(f^{(\varepsilon)}\right)^2(t,x) + K^2\left(v_x^{(\varepsilon)}\right)^2(t,x)\right)dt\,dx + 2(\delta_2 - \delta_1)^2\int_{\mathbb{R}^2}\left(v_t^{(\varepsilon)}\right)^2(t,x)dt\,dx\end{aligned} \tag{3.11}$$

yields

$$\begin{aligned}&\int_{\mathbb{R}^2}\left|\widehat{v}^{(\varepsilon)}(\zeta,\xi)\right|^2\left([|\xi|^\alpha + \lambda]^2 + \delta_2^2|\zeta|^2\right)d\zeta\,d\xi \\ &\quad\le 4\int_{\mathbb{R}^2}\left(\left|\widehat{f}^{(\varepsilon)}(\zeta,\xi)\right|^2 + K^2|\xi|^2\left|\widehat{v}^{(\varepsilon)}(\zeta,\xi)\right|^2\right)d\zeta\,d\xi + 2(\delta_2 - \delta_1)^2\int_{\mathbb{R}^2}|\zeta|^2\left|\widehat{v}^{(\varepsilon)}(\zeta,\xi)\right|^2 d\zeta\,d\xi.\end{aligned} \tag{3.12}$$

Let $\delta := \delta_2^2 - 2(\delta_2 - \delta_1)^2$. It follows from the assumptions that $\delta > 0$. The inequality (3.12) can be rewritten then as

$$\begin{aligned}&\int_{\mathbb{R}^2}\left|\widehat{v}^{(\varepsilon)}(\zeta,\xi)\right|^2\left(\left[|\xi|^\alpha + \lambda\right]^2 + \delta|\zeta|^2\right)\,d\zeta\,d\xi \\ &\quad\le 4\int_{\mathbb{R}^2}\left|\widehat{f}^{(\varepsilon)}(\zeta,\xi)\right|^2 d\xi\,d\zeta + 4K^2\int_{\mathbb{R}^2}|\xi|^2\left|\widehat{v}^{(\varepsilon)}(\zeta,\xi)\right|^2 d\zeta\,d\xi.\end{aligned} \tag{3.13}$$

One sees easily that there exists a constant $\lambda_0 > 0$ such that

$$\left[|\xi|^\alpha + \lambda_0\right]^2 + \delta|\zeta|^2 \ge \left[|\xi|^\alpha + \lambda_0\right]^2 \ge 8K^2\xi^2 \tag{3.14}$$

for all $(\zeta,\xi) \in \mathbb{R}^2$.

Combining the inequalities (3.12) and (3.14), we obtain for all $\lambda \geq \lambda_0$

$$\frac{1}{2}\int_{\mathbb{R}^2}\left|\widehat{v}^{(\varepsilon)}(\zeta,\xi)\right|^2\left(\left[|\xi|^\alpha+\lambda\right]+\delta|\zeta|^2\right)d\zeta\,d\xi \leq 4\int_{\mathbb{R}^2}\left|\widehat{f}^{(\varepsilon)}(\zeta,\xi)\right|^2 d\zeta\,d\xi. \tag{3.15}$$

Let

$$N_1 := \int_{\mathbb{R}^2}\frac{d\zeta d\xi}{\left(|\xi|^\alpha+\lambda\right)^2+\delta|\zeta|^2}. \tag{3.16}$$

Clearly, the constant N_1 is finite and depends on δ_1, δ_2, K, and α only.

Using the estimate (3.15) and the inverse Fourier transform yields for all $(t,x) \in \mathbb{R}^2$ and $\lambda \geq \lambda_0$

$$\begin{aligned}\left(v^{(\varepsilon)}(t,x)\right)^2 &\leq \frac{N_1}{4\pi^2}\int_{\mathbb{R}^2}\left|\widehat{v}^{(\varepsilon)}(\zeta,\xi)\right|^2\left(\left[|\xi|^\alpha+\lambda\right]^2+\delta|\zeta|^2\right)d\zeta\,d\xi \\ &\leq \frac{N_1}{\pi^2}\int_{\mathbb{R}^2}\left(f^{(\varepsilon)}(s,z)\right)^2 ds\,dz.\end{aligned} \tag{3.17}$$

The result follows then by taking the limit $\varepsilon \to 0$ in the above inequality and using the Lebesgue-dominated convergence theorem. □

Now, let $L = (A,Y)$ be a solution of (2.13), and there exist constants $K > 0$ and $\delta > 0$ such that the following assumptions are satisfied:

$$\begin{aligned}&|a(t,x)| \leq K|b(t,x)|^\alpha \quad \forall\ (t,x) \in [0,\infty)\times\mathbb{R}, \\ &\delta \leq |b|^{-\alpha}(t,x) \leq \beta\delta \quad \text{where } \beta \in \left[1, 2+\sqrt{2}\right),\ (t,x) \in [0,\infty)\times\mathbb{R}.\end{aligned} \tag{3.18}$$

We are interested in L_2-estimates of the form

$$\mathbf{E}\int_0^\infty e^{-\lambda u} f(t_0+A_u, x_0+Y_u)du \leq N\|f\|_2, \tag{3.19}$$

where $(t_0,x_0) \in \mathbb{R}^2, \lambda > 0$.

Theorem 3.2. *Assume that* $1 < \alpha \leq 2$, L *is a solution of* (2.13), *and the assumptions* (3.18) *hold. Then, for any* $(t_0,x_0) \in \mathbb{R}^2$, $\lambda \geq \lambda_0$, *and any measurable function* $f : \mathbb{R}^2 \to [0,\infty)$, *the estimate* (3.19) *is satisfied where the constant* N *depends on* δ, K, *and* α *only.*

Proof. Assume first that $f \in C_0^\infty(\mathbb{R}^2)$ so that there is a solution v of (3.5) satisfying the inequality (3.6). By taking the ε-convolution on both sides of (3.5), we obtain for all $\delta \leq r \leq \beta\delta$ and $|\gamma| \leq K$

$$r v_t^{(\varepsilon)} + \mathcal{L}v^{(\varepsilon)} - \lambda v^{(\varepsilon)} + \gamma\left|v_x^{(\varepsilon)}\right| + f^{(\varepsilon)} \leq 0. \tag{3.20}$$

Then, for $s \geq 0$, applying the Itô's formula to the expression

$$v^{(\varepsilon)}(t_0 + A_s, x_0 + Y_s)e^{-\lambda s} \tag{3.21}$$

yields

$$\begin{aligned}
&\mathbf{E}v^{(\varepsilon)}(t_0 + A_s, x_0 + Y_s)e^{-\lambda s} - v^{(\varepsilon)}(t_0, x_0) \\
&\quad = \mathbf{E}\int_0^s e^{-\lambda u}\Big[v_t^{(\varepsilon)}|b|^{-\alpha}(A_u, Y_u) + \mathcal{L}v^{(\varepsilon)} - \lambda v^{(\varepsilon)} + a|b|^{-\alpha}(A_u, Y_u)v_x^{(\varepsilon)}\Big](t_0 + A_u, x_0 + Y_u)du \\
&\quad \leq -\mathbf{E}\int_0^s e^{-\lambda u} f^{(\varepsilon)}(t_0 + A_u, x_0 + Y_u)du.
\end{aligned} \tag{3.22}$$

By Lemma 3.1

$$\mathbf{E}\int_0^s e^{-\lambda u} f^{(\varepsilon)}(t_0 + A_u, x_0 + Y_u)du \leq \sup_{t_0, x_0} v^{(\varepsilon)}(t_0, x_0) \leq N\left\|f^{(\varepsilon)}\right\|_2. \tag{3.23}$$

It remains to pass to the limit in the above inequality letting $\varepsilon \to 0$, $s \to \infty$ and using the Fatou's lemma.

The inequality (3.19) can be extended in a standard way first to any function $f \in L_2(\mathbb{R})$ and then to any nonnegative, measurable function using the monotone class theorem arguments (see, e.g., [19, Theorem 20]). □

Now, for arbitrary but fixed $t > 0$, $m \in \mathbb{N}$, define

$$\|f\|_{2,m,t} = \left(\int_0^t \int_{-m}^m |f(s,x)|^2 dx\, ds\right)^{1/2} \tag{3.24}$$

to be the L_2-norm of f on $[0,t] \times [-m,m]$. Applying (3.19) to the function $\overline{f}(s,x) = f(s,x)\mathbf{1}_{[0,t]\times[-m,m]}(s,x)$, we obtain the following local version of Krylov's estimates.

Corollary 3.3. *Let $1 < \alpha \leq 2$ and $L = (A, Y)$ be a solution of* (2.13). *Suppose that the conditions* (3.18) *are satisfied. Then, for any $t \geq 0$, $m \in \mathbb{N}$, and any nonnegative measurable function f, it follows that*

$$\mathbf{E}\int_0^{t\wedge\tau_m(Y)} f(A_u, Y_u)du \leq N\|f\|_{2,m,t}, \tag{3.25}$$

where $\tau_m(Y) := \inf\{t \geq 0 : |Y_t| \geq m\}$ and N is a constant depending on δ, K, α, m, and t only.

From Theorem 3.2 and Corollary 3.3 we also obtain the following.

Corollary 3.4. *Let (A, Y) be a solution of* (2.13) *with $\alpha = 1$. If the assumptions* (3.18) *are satisfied with arbitrary δ and $K < 1/2\sqrt{2}$, then the estimate* (3.25) *holds.*

4. Existence of Solutions

Now we turn our attention to the existence of solutions of (1.1) and (2.13). Since the case of $\alpha = 2$ is well studied, we restrict ourselves to the case $\alpha < 2$.

Theorem 4.1. *Let $1 \le \alpha < 2$ and assume that the assumptions* (3.18) *are satisfied where the constant K is arbitrary for $1 < \alpha < 2$ and $K < 1/2\sqrt{2}$ for $\alpha = 1$. Then, for any $x_0 \in \mathbb{R}$, there exists a (nonexploding) solution of* (1.1).

Proof. We first prove the existence of solutions of the equation (2.13).

It follows from the assumptions that the coefficient B is bounded. Hence we can find a sequence of functions q_n and $p_n, n \ge 1$, such that they are globally Lipshitz continuous and uniformly bounded by the constant $\min\{2\delta, 2K\delta\}$. Moreover, $q_n \to a|b|^{-\alpha}$ and $p_n \to |\mathrm{b}|^{-\alpha}$ as $n \to \infty$ pointwise and in $\|\cdot\|_{2,t,m^-}$ norm for all $t > 0$, $m \in \mathbb{N}$. For any $n = 1, 2, \ldots$, (2.13) has a unique strong solution (see, e.g., [20, Theorem 9.1]). That is, for any fixed symmetric stable process Z defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, there exists a sequence of processes $L^n = (A^n, Y^n)$, $n = 1, 2, \ldots$, such that

$$L_t^n = \overline{x}_0 + \int_0^t B_n(s, L_s^n) dW_s, \quad t \ge 0, \ \mathbf{P}\text{-a.s.}, \tag{4.1}$$

where

$$B_n := \begin{pmatrix} 0 & p_n \\ 1 & q_n \end{pmatrix} \tag{4.2}$$

or, written componentwise,

$$\begin{aligned} A_t^n &= \int_0^t p_n(A_s^n, Y_s^n) ds, \\ Y_t^n &= x_0 + Z_t + \int_0^t q_n(A_s^n, Y_s^n) ds. \end{aligned} \tag{4.3}$$

Let

$$S_t^n := \int_0^t q_n(A_s^n, Y_s^n) ds, \tag{4.4}$$

so that

$$Y^n = x_0 + Z + S^n, \quad n \ge 1. \tag{4.5}$$

Now we claim that the sequence of processes $H^n := (Y^n, S^n, A^n, Z)$, $n \geq 1$, is tight in the sense of weak convergence in $(\mathbf{D}^4, \mathfrak{D}^4)$. Due to the Aldous' criterion ([21]), we have only to show that

$$\lim_{l\to\infty} \limsup_{n\to\infty} \mathbf{P}\left(\sup_{0\leq s\leq t} \|H_s^n\| > l \right) = 0 \tag{4.6}$$

for all $t \geq 0$ and

$$\limsup_{n\to\infty} \mathbf{P}\left(\left\| H^n_{t\wedge(\tau^n + r_n)} - H^n_{t\wedge\tau^n} \right\| > \varepsilon \right) = 0 \tag{4.7}$$

for all $t \geq 0$, $\varepsilon > 0$, every sequence of $\mathbb{F}$-stopping times τ^n, and every sequence of real numbers r_n such that $r_n \downarrow 0$. We use $\|\cdot\|$ to denote the Euclidean norm of a vector.

But both conditions are clearly satisfied because of the uniform boundness of the coefficients q_n and p_n for all $n \geq 1$.

Since the sequence $\{H^n\}$ is tight, there exists a subsequence $\{n_k\}$, $k = 1, 2, \ldots$, a probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbf{P}})$ and the process $\overline{H}$ on it with values in $(\mathbf{D}^4, \mathfrak{D}^4)$ such that H^{n_k} converges weakly (in distribution) to the process $\overline{H}$ as $k \to \infty$. For simplicity, let $\{n_k\} = \{n\}$.

According to the embedding principle of Skorokhod (see, e.g., [20, Theorem 2.7]), there exists a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}})$ and the processes $\widetilde{H} = (\widetilde{Y}, \widetilde{S}, \widetilde{A}, \widetilde{Z})$, $\widetilde{H}^n = (\widetilde{Y}^n, \widetilde{S}^n, \widetilde{A}^n, \widetilde{Z}^n)$, $n = 1, 2, \ldots$, on it such that

(i) $\widetilde{H}^n \to \widetilde{H}$ as $n \to \infty \widetilde{\mathbf{P}}$-a.s.

(ii) $\widetilde{H}^n = H^n$ in distribution for all $n = 1, 2, \ldots$

Using standard measurability arguments ([1, chapter 2]), one can prove that the processes $\widetilde{Z}^n$ and $\widetilde{Z}$ are symmetric stable processes of the same index as the processes Z^n with respect to the augmented filtrations $\widetilde{\mathbb{F}}^n$ and $\widetilde{\mathbb{F}}$ generated by processes $\widetilde{H}^n$ and $\widetilde{H}$, respectively.

Using the properties (i), (ii), and (4.1), one can show (cf. [1, chapter 2]) that

$$\begin{aligned} \widetilde{Y}_t^n &= x_0 + \widetilde{Z}_t^n + \int_0^t q_n\left(\widetilde{A}_s^n, \widetilde{Y}_s^n\right) ds, \quad t \geq 0, \ \widetilde{\mathbf{P}}\text{-a.s.}, \\ \widetilde{A}_t^n &= \int_0^t p_n\left(\widetilde{A}_s^n, \widetilde{Y}_s^n\right) ds, \quad t \geq 0, \ \widetilde{\mathbf{P}}\text{-a.s.} \end{aligned} \tag{4.8}$$

On the other hand, the same properties and the quasileft continuity of the the processes $\widetilde{H}^n$ yield

$$\lim_{n\to\infty} \widetilde{A}_t^n = \widetilde{A}_t, \quad \lim_{n\to\infty} \widetilde{Y}_t^n = \widetilde{Y}_t \quad t \geq 0, \ \widetilde{\mathbf{P}}\text{-a.s.} \tag{4.9}$$

Therefore, in order to show that the process $\widetilde{L} = (\widetilde{A}, \widetilde{Y})$ is a solution of (2.13), it suffices to verify that, for all $t \geq 0$,

$$\lim_{n\to\infty}\int_0^t p_n\left(\widetilde{A}_s^n, \widetilde{Y}_s^n\right)ds = \int_0^t |b|^{-\alpha}\left(\widetilde{A}_s, \widetilde{Y}_s\right)ds \quad \widetilde{\mathbf{P}}\text{-a.s.},$$
$$\lim_{n\to\infty}\int_0^t q_n\left(\widetilde{A}_s^n, \widetilde{Y}_s^n\right)ds = \int_0^t \left[a|b|^{-\alpha}\right]\left(\widetilde{A}_s, \widetilde{Y}_s\right)\, ds \quad \widetilde{\mathbf{P}}\text{-a.s.} \tag{4.10}$$

The following fact can be proven similar as [14, Lemma 4.2].

Lemma 4.2. *For any Borel measurable function $f : \mathbb{R}^2 \to [0,\infty)$ and any $t \geq 0$, there exists a sequence $m_k \in (0,\infty)$, $k = 1, 2, \ldots$ such that $m_k \uparrow \infty$ as $k \to \infty$ and it holds*

$$\widetilde{\mathbf{E}}\int_0^{t\wedge\tau_{m_k}(\widetilde{Y})} f\left(\widetilde{A}_s, \widetilde{Y}_s\right)ds \leq N\|f\|_{2,m_k,t}, \tag{4.11}$$

where the constant N depends on δ, K, α, t, and m_k only.

Without loss of generality, we can assume in Lemma 4.2 that $\{m_k\} = \{m\}$. Now, to prove (31), it is enough to verify that for all $t \geq 0$ and $\varepsilon > 0$ we have

$$\lim_{n\to\infty}\widetilde{\mathbf{P}}\left(\left|\int_0^t p_n\left(\widetilde{A}_s^n, \widetilde{Y}_s^n\right)ds - \int_0^t |b|^{-\alpha}\left(\widetilde{A}_s, \widetilde{Y}_s\right)ds\right| > \varepsilon\right) = 0. \tag{4.12}$$

In order to prove (4.12) we estimate for a fixed $k \in \mathbb{N}$

$$\begin{aligned}
&\widetilde{\mathbf{P}}\left(\left|\int_0^t p_n\left(\widetilde{A}_s^n, \widetilde{Y}_s^n\right)ds - \int_0^t |b|^{-\alpha}\left(\widetilde{A}_s, \widetilde{Y}_s\right)ds\right| > \varepsilon\right)\\
&\quad\leq \widetilde{\mathbf{P}}\left(\left|\int_0^t p_k\left(\widetilde{A}_s^n, \widetilde{Y}_s^n\right)ds - \int_0^t p_k\left(\widetilde{A}_s, \widetilde{Y}_s\right)ds\right| > \frac{\varepsilon}{3}\right) + \widetilde{\mathbf{P}}\left(\left|\int_0^{t\wedge\tau_m(\widetilde{Y}^n)} [p_k - p_n]\left(\widetilde{A}_s^n, \widetilde{Y}_s^n\right)ds\right| > \frac{\varepsilon}{3}\right)\\
&\qquad + \widetilde{\mathbf{P}}\left(\left|\int_0^{t\wedge\tau_m(\widetilde{Y})} \left[p_k - |b|^{-\alpha}\right]\left(\widetilde{A}_s, \widetilde{Y}_s\right)ds\right| > \frac{\varepsilon}{3}\right) + \widetilde{\mathbf{P}}\left(\tau_m\left(\widetilde{Y}^n\right) < t\right) + \widetilde{\mathbf{P}}\left(\tau_m\left(\widetilde{Y}\right) < t\right)\\
&\quad= J_{n,k}^1 + J_{n,k,m}^2 + J_{k,m}^3 + \widetilde{\mathbf{P}}\left(\tau_m\left(\widetilde{Y}^n\right) < t\right) + \widetilde{\mathbf{P}}\left(\tau_m\left(\widetilde{Y}\right) < t\right).
\end{aligned} \tag{4.13}$$

By Chebyshev's inequality and Lebesgue bounded convergence theorem, $J^1_{n,k} \to 0$ as $n \to \infty$. To show that $J^2_{n,k,m} \to 0$ as $n \to \infty$ and $J^3_{k,m} \to 0$ as $k \to \infty$, we use first the Chebyshev's inequality and then Corollary 3.3 and Lemma 4.2, respectively, to estimate

$$
\begin{aligned}
J^2_{n,k,m} &\le \frac{3}{\varepsilon} N \| p_k - p_n \|_{2,m,t}, \\
J^3_{k,m} &\le \frac{3}{\varepsilon} N \| p_k - |b|^{-\alpha} \|_{2,m,t},
\end{aligned}
\tag{4.14}
$$

where the constant N depends on K_1, K_2, α, m, and t only. Obviously, $\|p_n - |b|^{-\alpha}\|_{2,m,t} \longrightarrow 0$ as $n \to \infty$ implying that the right-hand sides in (4.14) converge to 0 by letting first $n \to \infty$ and then $k \to \infty$.

Because of the property $\tau_m(\widetilde{Y}^n) \longrightarrow \tau_m(\widetilde{Y})$ as $n \to \infty$ $\widetilde{\mathbf{P}}$-a.s.,

$$
\widetilde{\mathbf{P}}\left(\tau_m\left(\widetilde{Y}^n\right) < t\right) \longrightarrow \widetilde{\mathbf{P}}\left(\tau_m\left(\widetilde{Y}\right) < t\right) \quad \text{as } n \longrightarrow \infty \tag{4.15}
$$

for all $m \in \mathbb{N}$, $t > 0$. Therefore, the last two terms can be made arbitrarily small by choosing large enough m for all n due to the fact that the sequence of processes $\widetilde{Y}^n$ satisfies the property (4.6). This proves (4.12). The proof of (4.10) is similar, and we omit the details.

We have shown that $\widetilde{L} = (\widetilde{A}, \widetilde{Y})$ is a solution of (2.13). To finish the proof of the theorem, it is enough to use Corollary 2.4 that implies that the process $\widetilde{X}_t = \widetilde{Y}_{\widetilde{T}_t}$ will be a (nonexploding) solution of (1.1). □

Remark 4.3. If $\beta = 1$, then the existence conditions of Theorem 4.1 coincide with those found in [14] where (1.1) with $b = 1$ was considered.

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