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*Research Article*

# A Stability Result for Stochastic Differential Equations Driven by Fractional Brownian Motions

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We study the stability of the solutions of stochastic differential equations driven by fractional Brownian motions with Hurst parameter greater than half. We prove that when the initial conditions, the drift, and the diffusion coefficients as well as the fractional Brownian motions converge in a suitable sense, then the sequence of the solutions of the corresponding equations converge in Hölder norm to the solution of a stochastic differential equation. The limit equation is driven by the limit fractional Brownian motion and its coefficients are the limits of the sequence of the coefficients.

## 1. Introduction and Main Result

Suppose that $B^H = (B_t^H)_{0\le t\le T}$ is an $m$-dimensional fractional Brownian motion (fBm in short) with Hurst parameter $H$ defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbf{P})$. We mean that the components $B^{H,j}$, $j = 1,\dots,m$ are independent centered Gaussian processes with the covariance function

$$R_H(s,t) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right). \tag{1.1}$$

If $H = 1/2$, then $B^H$ is clearly a Brownian motion. Since for any $p \ge 1$, $\mathbf{E}|B_t^{H,j} - B_s^{H,j}|^p = c_p|t-s|^{pH}$ the processes $B^{H,j}$ have $\alpha$-Hölder continuous paths for all $\alpha \in (0,H)$ (see [1] for further information about fBm).

In this paper we fix $1/2 < H < 1$ and we consider the solution $(X_t)_{0\le t\le T}$ of the following stochastic differential equation (abbreviated by SDE from now on) on $\mathbf{R}^d$

$$X_t^i = x^i + \sum_{j=1}^{m} \int_0^t \sigma^{i,j}(X_s) dB_s^{H,j} + \int_0^t b^i(X_s) ds, \quad 0 \le t \le T, \tag{1.2}$$

$i = 1, \dots, d$, $x \in \mathbf{R}^d$ is the initial value of the process $X$.

Under suitable assumptions on $\sigma$, the processes $\sigma(X)$ and $B^H$ have trajectories which are Hölder continuous of order strictly larger than $1/2$ so we can use the integral introduced by Young in [2]. The stochastic integral in (1.2) is then a path-wise Riemann-Stieltjes integral. A first result on the existence and uniqueness of a solution of such an equation was obtained in [3] using the notion of $p$-variation. The theory of rough paths introduced by Lyons in [3] was used by Coutin and Qian in order to prove an existence and uniqueness result for (1.2) (see [4]). The Riemann-Stieltjes integral appearing in (1.2) can be expressed as a Lebesgue integral using a fractional integration by parts formula (see Zähle [5]). Using this formula Nualart and Răşcanu have established in [6] the existence of a unique solution for a class of general differential equations that includes (1.2). Later on, the regularity in the sense of Malliavin calculus and the absolute continuity of the law of the random variables $X_t$ have been investigated in [7–10].

In order to obtain moment bounds on the solution of (1.2), we have to estimate the corresponding determinist differential equation very carefully. Indeed, an exponential of the Hölder norm of the fBm may appear and by Fernique's theorem, it is well known that such exponential moment does not always exist. This fact will be specified in Section 2. Thanks to a technical trick due to Hairer and Pillai in [11] (see also [12]), some estimations that are compatible with exponential moments are now available. This is the starting point of this short communication: first we will estimate the difference between two solutions of SDEs with different coefficients. We will endeavor ourselves to give some bounds that are suitable for stability results.

Now we present the kind of results we are interested in and so we need further notations. For a differentiable function $\varphi$ from $\mathbf{R}^d$ to $\mathbf{R}^q$, we denote (if the following quantities do exist) $\|\varphi\|_\infty = \max_{j=1,\dots,q} \|\varphi^j\|_\infty$ and

$$\|\varphi\|_{\mathbf{C}^1} = \|\varphi\|_\infty + \max_{i=1,\dots,d} \max_{j=1,\dots,q} \left\| \frac{\partial \varphi^j}{\partial x_i} \right\|_\infty . \tag{1.3}$$

The space $\mathbf{C}^1$ is the space of continuously differentiable functions $\varphi$ such that $\|\varphi\|_{\mathbf{C}^1} < \infty$. The space $\mathbf{C}^2$ is defined in a similar way. For $0 < \lambda < 1$ and $0 \le a < b \le T$, we denote by $\mathbf{C}^\lambda(a, b; \mathbf{R}^d)$ the space of $\lambda$-Hölder continuous functions $f : [a, b] \to \mathbf{R}^d$, equipped with the norm

$$\|f\|_\lambda := \|f\|_{a,b,\infty} + \|f\|_{a,b,\lambda}, \tag{1.4}$$

where

$$\|f\|_{a,b,\infty} = \sup_{a\le r\le b} |f(r)|, \qquad \|f\|_{a,b,\lambda} = \sup_{a\le r\le s\le b} \frac{|f(s) - f(r)|}{|s - r|^\lambda}. \tag{1.5}$$

We simply write $\mathbf{C}^\lambda(a, b)$ when $d = 1$.

The main result of this work is the following theorem.

**Theorem 1.1.** *Let* $(B^{H,n})_{n\geq 1}$ *be a sequence of fractional Brownian motions defined on* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbf{P})$. *Let* $(x_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$, *and* $(\sigma_n)_{n\geq 1}$ *be some sequences in* $\mathbf{R}^d$, $\mathbf{C}^1$, *and* $\mathbf{C}^2$. *One considers the sequence* $(X^n)_{n\geq 1}$ *of processes such that for any* $n \geq 1$, $(X_t^n)_{0\leq t\leq T}$ *is the unique solution of*

$$X_t^n = x_n + \int_0^t \sigma_n(X_s^n)dB_s^{H,n} + \int_0^t b_n(X_s^n)ds, \quad 0 \leq t \leq T. \tag{1.6}$$

*If there exists*

(i) $x \in \mathbf{R}^d$, $b \in \mathbf{C}^1$ *and* $\sigma \in \mathbf{C}^2$ *such that*

$$\lim_{n\to\infty} \{|x_n - x| + \|b_n - b\|_\infty + \|\sigma - \sigma_n\|_{\mathbf{C}^1}\} = 0; \tag{1.7}$$

(ii) *a fractional Brownian motion* $B^H$ *defined on* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbf{P})$ *such that for* $\beta \in (1/2, H)$ *and* $p \geq 1$ *one has*

$$\lim_{n\to\infty} \mathbf{E}\left(\left\|B^{H,n} - B^H\right\|_{0,T,\beta}^p\right) = 0; \tag{1.8}$$

*then*

$$\lim_{n\to\infty} \mathbf{E}\left(\|X^n - X\|_\beta^p\right) = 0, \tag{1.9}$$

*where* $(X_t)_{0\leq t\leq T}$ *is the solution of* (1.2) *with the coefficients* $x$, $b$, *and* $\sigma$ *and driven by the fBm* $B^H$.

As usual in the theory of SDEs driven by fBm, the above theorem will be the counterpart of a deterministic result on ordinary differential equations driven by Hölder continuous functions. More precisely, Theorem 1.1 will be a consequence of an estimation on the Hölder norm of the difference of two solutions of rough differential equations. This result is interesting in itself and it is the subject of Section 2. It is precisely stated in Proposition 2.3 but we present here a brief description of the result we have obtained. We consider for some $\beta \in (1/2, 1)$ two deterministic rough functions $g$ and $\tilde{g}$ in $\mathbf{C}^\beta$ and two deterministic differential equations

$$\begin{aligned} x_t &= x_0 + \int_0^t b(x_s)ds + \int_0^t \sigma(x_s)dg_s, \\ \tilde{x}_t &= \tilde{x}_0 + \int_0^t \tilde{b}(\tilde{x}_s)ds + \int_0^t \tilde{\sigma}(\tilde{x}_s)d\tilde{g}_s, \end{aligned} \tag{1.10}$$

where all the coefficients are smooth. Then we will prove that there exists a constant $C$ such that

$$\begin{aligned} \|x - \tilde{x}\|_\beta \leq{}& C\left(|x_0 - \tilde{x}_0| + \left\|b - \tilde{b}\right\|_\infty + \|\sigma - \tilde{\sigma}\|_{\mathbf{C}^1}\|g\|_{0,T,\beta} + \|g - \tilde{g}\|_{0,T,\beta}\right) \\ &\times \left(1 + \|g\|_{0,T,\beta}\right)^{2/\beta} \exp\left\{C\|g\|_{0,T,\beta}^{1/\beta}\right\}. \end{aligned} \tag{1.11}$$

Since our upper bound is explicit, this estimate can be viewed as a refinement of Theorems 11.3 and 11.6 of [13]. Nevertheless we strength the fact that many results have been obtained in the theory of fractional SDEs thanks to rough paths theory. We may prove the above results by rough paths techniques but we adopt the simplest context of Young's integral. The stability with respect to the driving noise is a reformulation of the continuity of the Itô map and this is well known. A weaker stability result with respect to the initial condition is proved in [14]. The stability with respect to all the coefficients in (1.2) is new to our knowledge.

The paper is organized as follows. In Section 2 we present the case of deterministic differential equations driven by Hölder continuous function and we state an estimation on the difference between the solutions of such equations (see Proposition 2.3). Theorem 1.1 will be a straightforward consequence of this work on deterministic differential equations. Finally some proofs are gathered in Section 3 and in the appendix.

## 2. Deterministic Differential Equation Driven by Rough Functions

This section deals with deterministic differential equations driven by Hölder's continuous functions. These equations are the one satisfied by the trajectories of the solution of (1.2). Our aim is to prove an estimate for the difference of two solutions of deterministic differential equations driven by two different Hölder continuous functions. In [8, Theorem 3.3], such estimates are proved but are unfortunately unusable in our context (see the discussion below). Proposition 2.3 hereafter will strengthen the result of [8].

Suppose that $f \in \mathbf{C}^{\lambda}(a,b)$ and $g \in \mathbf{C}^{\mu}(a,b)$ with $\lambda + \mu > 1$. From [2], the Riemann-Stieltjes integral $\int_a^b f dg$ exists. In [5], the author provides an explicit expression for the integral $\int_a^b f dg$ in terms of fractional derivatives. In order to give some precisions, we consider $\alpha > 0$ being such that $\beta > 1 - \alpha$. Supposing that the following limit exists and is finite, we denote $g_{b-}(t) = g(t) - \lim_{\varepsilon \downarrow 0} g(b-\varepsilon)$. Then the Riemann-Stieltjes integral can be expressed as

$$\int_a^b f_t dg_t = (-1)^{\alpha} \int_a^b \left(D_{a+}^{\alpha} f\right)(t) \left(D_{b-}^{1-\alpha} g_{b-}\right)(t) dt, \tag{2.1}$$

where

$$\begin{aligned} D_{a+}^{\alpha} f(t) &= \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(t)}{(t-a)^{\alpha}} + \alpha \int_a^t \frac{f(t)-f(s)}{(t-s)^{\alpha+1}} ds \right), \\ D_{b-}^{\alpha} g_{b-}(t) &= \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \left( \frac{g(t)-g(b)}{(b-t)^{\alpha}} + \alpha \int_t^b \frac{g(t)-g(s)}{(s-t)^{\alpha+1}} ds \right). \end{aligned} \tag{2.2}$$

We refer to [15] for further details on fractional operators. The following useful lemma is now classical. Its proof is postponed in the appendix.

**Lemma 2.1.** *Let $f$ and $g$ in $\mathbf{C}^{\beta}(s,t)$ with $1/2<\beta<1$ and $0\le s<t\le T$, then there exists a universal constant $\kappa$ such that*

$$\left|\int_s^t f_r\,dg_r\right|\le\frac{\kappa}{\beta-1/2}\|g\|_{0,T,\beta}\left[\|f\|_{s,t,\infty}(t-s)^{\beta}+\|f\|_{s,t,\beta}(t-s)^{2\beta}\right].\tag{2.3}$$

Set $1/2<\beta<1$ and let $g,\tilde{g}\in\mathbf{C}^{\beta}(0,T;\mathbf{R}^m)$. We will work with the following deterministic differential equations on $\mathbf{R}^d$:

$$\begin{aligned}x_t^i&=x_0^i+\int_0^t b^i(x_s)ds+\sum_{j=1}^m\int_0^t\sigma^{i,j}(x_s)dg_s^j,\\ \tilde{x}_t^i&=\tilde{x}_0^i+\int_0^t \tilde{b}^i(\tilde{x}_s)ds+\sum_{j=1}^m\int_0^t\tilde{\sigma}^{i,j}(\tilde{x}_s)d\tilde{g}_s^j,\end{aligned}\tag{2.4}$$

for $0\le t\le T$, $i=1,\ldots,d$ and $x_0,\tilde{x}_0\in\mathbf{R}^d$.

We introduce the following assumptions on the coefficients of the above equations. For a function $\varphi$ from $\mathbf{R}^d$ to $\mathbf{R}^q$, $\nabla\varphi$ denotes the matrix of first order derivatives and $\nabla^2\varphi$ denotes its Hessian.

(H1) There exists some positive constants $b_0$, $b_1$, $\tilde{b}_0$, $\tilde{b}_1$ such that $\|b\|_\infty\le b_0$, $\|\tilde{b}\|_\infty\le\tilde{b}_0$, $\|\nabla b\|_\infty\le b_1$, and $\|\nabla\tilde{b}\|_\infty\le\tilde{b}_1$.

(H2) There exists some positive constants $c_0$, $c_1$, $c_2$, $\tilde{c}_0$, $\tilde{c}_1$, and $\tilde{c}_2$ such that $\|\sigma\|_\infty\le c_0$, $\|\nabla\sigma\|_\infty\le c_1$, $\|\nabla^2\sigma\|_\infty\le c_2$, $\|\tilde{\sigma}\|_\infty\le\tilde{c}_0$, $\|\nabla\tilde{\sigma}\|_\infty\le\tilde{c}_1$, and $\|\nabla^2\tilde{\sigma}\|_\infty\le\tilde{c}_2$.

It is proved in [6, Theorem 5.1] that if $1-\beta<\alpha<1/2$, each of the above equations has a unique $(1-\alpha)$-Hölder continuous solution. The estimates on the solution $(x_t)_{0\le t\le T}$ obtained in [6] were improved in [8]. Let us recall the following observations concerning Theorem 3 in Hu and Nualart [8] (see also [11] for similar comments). It has been proved in [8] that if $b=0$ and $\sigma$ is twice continuously differentiable with bounded second order derivatives, then there exists a constant $k$ that depends on $T$, $\beta$, and $\sigma$ such that

$$\begin{aligned}\|x-\tilde{x}\|_{0,T,\infty}\le k\exp\Big\{&k\|g\|_{0,T,\beta}^{1/\beta}\left(1+\|x\|_{0,T,\beta}+\|\tilde{x}\|_{0,T,\beta}\right)^{1/\beta}\\ &\left(1+\|x\|_{0,T,\beta}\right)\|g-\tilde{g}\|_{0,T,\beta}\Big\}.\end{aligned}\tag{2.5}$$

Replacing $g$ and $\tilde{g}$ by the trajectories of the fractional Brownian motions $B^H$ and $\tilde{B}^H$, we obtain estimation in the supremum norm for the difference of the processes $X$ and $\tilde{X}$ satisfying

$$X_t=x_0+\int_0^t\sigma(X_s)dB_s^H,\qquad \tilde{X}_t=x_0+\int_0^t\sigma\left(\tilde{X}_s\right)d\tilde{B}_s^H.\tag{2.6}$$

Thus the estimation (2.5) holds almost-surely and one have to take expectation. For this purpose we use the following Fernique's type result on the exponential moments of the Hölder norm of the fBm.

**Lemma 2.2.** *Let $T > 0$, $1/2 < \beta < H < 1$. Then for any $\alpha < 1/(128(2T)^{2(H-\beta)})$*

$$\mathbf{E}\left[\exp\left(\alpha\left\|B^H\right\|_{0,T,\beta}^2\right)\right] \leq \left(1 - 128\alpha(2T)^{2(H-\beta)}\right)^{-1/2}. \tag{2.7}$$

One refers to [16] for a proof of this lemma. Lemma 2.2 implies the following integrability property for $X$ (or $\widetilde{X}$): for any $\lambda > 0$ and $\gamma < 2H$,

$$\mathbf{E}\left[\exp\left(\lambda\|X\|_{0,T,\infty}^{\gamma}\right)\right] < \infty. \tag{2.8}$$

Inequality (2.8) together with (2.5) will unfortunately be useless since the quantities $\|X\|_{0,T,\infty}^{1/\beta}$ and $\|B\|_{0,T,\infty}^{1/\beta}$ appear in a multiplicative way in the exponent in the right hand side of (2.5). Indeed, Young's inequality yields $\|X\|_{0,T,\infty}^{1/\beta}\|B\|_{0,T,\infty}^{1/\beta} \leq c\|X\|_{0,T,\infty}^{p/\beta} + c\|B\|_{0,T,\infty}^{q/\beta}$ and if one imposes $p/\beta < 2$ then necessarily $q/\beta > 2$ and the finiteness of the expression such as

$$\mathbf{E}\left[\exp\left(k\|B\|_{0,T,\infty}^{1/\beta}\|X\|_{0,T,\infty}^{1/\beta}\right)\right] \tag{2.9}$$

cannot be deduced from Lemma 2.2 and (2.8) (in fact we do not even know if this expectation is finite). Hence we need a suitable estimate to obtain moment bounds on the quantities we are interested in. Such investigations have been carried out in [11, 12] but we need some nontrivial modifications to handle the difference $x-\tilde{x}$ when $x$ and $\tilde{x}$ are the solutions of (2.4).

Therefore the next result is a strengthening of Theorem 3.3 in [8] and is based on the method used in [11, Lemma 3.2]. It may also be viewed as a refinement of Theorems 11.3 and 11.6 of [13].

**Proposition 2.3.** *Let $T$ be fixed and let $g$ and $\tilde{g}$ be Hölder continuous of order $1/2 < \beta < 1$. Under (H1) and (H2), there exists a constant $C$ that depends only on $T$, $\beta$, $b_0$, $\tilde{b}_0$, $c_0$, $\tilde{c}_0$, $c_1$, $\tilde{c}_1$, $c_2$, and $\tilde{c}_2$ such that*

$$\begin{aligned}\|x - \tilde{x}\|_\beta \leq C&\left(|x_0 - \tilde{x}_0| + \left\|b - \tilde{b}\right\|_\infty + \|\sigma - \tilde{\sigma}\|_{C^1}\|g\|_{0,T,\beta} + \|g - \tilde{g}\|_{0,T,\beta}\right)\\ &\times\left(1 + \|g\|_{0,T,\beta}\right)^{2/\beta}\exp\left\{C\|g\|_{0,T,\beta}^{1/\beta}\right\}.\end{aligned} \tag{2.10}$$

It is worth to notice that a careful reading of the proof shows that $C$ depends continuously on its parameters. This is important when we apply this proposition to stability properties of stochastic differential equations.

## 3. Proofs

The subject of this section is the proof of Proposition 2.3 and Theorem 1.1. We follow the arguments developed in the proof of [8, Theorem 3.2], [11, Lemma 3.2] and we give some precisions. We restrict ourselves to the case $d = m = 1$ for simplicity. Thus for a function $\varphi$, we denote $\varphi'$ its derivative and $\varphi''$ its second order derivative. Moreover $c_\beta := \kappa/(\beta - 1/2)$ designates the constant in (2.3).

### 3.1. A Preliminary Lemma

We will need the following lemma whose proof is borrowed from [11]. For the sake of completeness and to give some information on the constants that are involved in the statement, we briefly recall the arguments that are used in the proof.

**Lemma 3.1.** *There exists an explicit constant $M_1$ that depends on $T$, $b_0$, $b_1$, $c_0$, $c_1$, and $c_\beta$ such that*

$$\|x\|_{0,T,\infty} \leq T(1 + |x_0|)e^{2M_1 b_1 T}\left(2M_1\left(1 + \|g\|_{0,T,\beta}\right)\right)^{1/\beta}. \tag{3.1}$$

*Proof.* Let $0 \leq s \leq t \leq T$. Since $|b(x_r)| \leq |b(x_r) - b(x_s)| + |b(x_s)|$ and $|b(x_s)| \leq b_0 + b_1|x_s|$, we have

$$\left|\int_s^t b(x_r)dr\right| \leq b_0(t-s) + b_1|x_s|(t-s) + b_1\|x\|_{s,t,\beta}(t-s)^{1+\beta}. \tag{3.2}$$

Clearly $\|\sigma(x_\cdot)\|_{s,t,\beta} \leq c_1\|x\|_{s,t,\beta}$, thus Inequality (2.3) yields

$$\begin{aligned} \|x\|_{s,t,\beta} &\leq b_0(t-s)^{1-\beta} + b_1|x_s|(t-s)^{1-\beta} + b_1\|x\|_{s,t,\beta}(t-s) \\ &\quad + c_\beta\|g\|_{0,T,\beta}\left[c_0 + c_1\|x\|_{s,t,\beta}(t-s)^\beta\right] \\ &\leq M_1\left\{1 + \|g\|_{0,T,\beta} + b_1|x_s|(t-s)^{1-\beta} + \left(1 + \|g\|_{0,T,\beta}\right)\|x\|_{s,t,\beta}(t-s)^\beta\right\} \end{aligned} \tag{3.3}$$

with $M_1 = \max(1, T^{1-\beta}\|b\|_{C^1}, c_\beta\|\sigma\|_{C^1})$. We denote

$$\Delta_1 = \left\{2M_1\left(1 + \|g\|_{0,T,\beta}\right)\right\}^{-1/\beta} \tag{3.4}$$

and when $(t-s) \leq \Delta_1$ we may write

$$\|x\|_{s,t,\beta} \leq 2M_1\left(1 + \|g\|_{0,T,\beta} + b_1|x_s|\Delta_1^{1-\beta}\right). \tag{3.5}$$

Since $\|x\|_{s,t,\infty} \leq |x_s| + \|x\|_{s,t,\beta}(t-s)^{\beta}$, we obtain for $(t-s) \leq \Delta_1$ that

$$\|x\|_{s,t,\infty} \leq |x_s|(1 + 2M_1 b_1 \Delta_1) + 1. \tag{3.6}$$

By induction with $N = T/\Delta_1$ it follows that

$$\begin{aligned} \|x\|_{0,T,\infty} &\leq |x_0|(1 + 2M_1 b_1 \Delta_1)^N + \sum_{k=1}^{N}(1 + 2M_1 b_1 \Delta_1)^k \\ &\leq \left(\frac{T}{\Delta_1}\right)(1 + |x_0|)(1 + 2M_1 b_1 \Delta_1)^{T/\Delta_1} \end{aligned} \tag{3.7}$$

and since for any $a > 0$, $(1 + x/a)^a \leq e^x$, we have $(1 + 2M_1 b_1 \Delta_1)^{T/\Delta_1} \leq e^{2M_1 b_1 T}$ and we finally deduce (3.1).

As noticed in [11], if $b$ is the null function, then $b_1 = 0$ and the last above argument fails. In this case it is impossible to obtain a bound without any exponential of the quantity $\|g\|_{0,T,\beta}$. □

*Remark 3.2.* When $(t-s) \leq \Delta_1$, we substitute (3.1) into (3.5) and we deduce the estimate

$$\|x\|_{s,t,\beta} \leq M_2 \left(1 + \|g\|_{0,T,\beta}\right), \tag{3.8}$$

where

$$M_2 = 2M_1 \left(1 + b_1 T(1 + |x_0|)e^{2M_1 b_1 T}\right). \tag{3.9}$$

In the sequel, we naturally denote $\widetilde{M}_1$, $\tilde{\Delta}_1$, and $\widetilde{M}_2$ the corresponding quantities that are related to (7).

### 3.2. Proof of Proposition 2.3

*Proof.* Let $0 \leq s \leq t \leq T$. We write

$$x_t - \tilde{x}_t - (x_s - \tilde{x}_s) = I_1 + I_2 + I_3 + I_4 + I_5, \tag{3.10}$$

where

$$I_1 = \int_s^t [b(x_r) - b(\widetilde{x}_r)]dr,$$
$$I_2 = \int_s^t \left[b(\widetilde{x}_r) - \widetilde{b}(\widetilde{x}_r)\right]dr,$$
$$I_3 = \int_s^t [\sigma(x_r) - \sigma(\widetilde{x}_r)]dg_r, \tag{3.11}$$
$$I_4 = \int_s^t [\sigma(\widetilde{x}_r) - \widetilde{\sigma}(\widetilde{x}_r)]dg_r,$$
$$I_5 = \int_s^t \widetilde{\sigma}(\widetilde{x}_r)d[g_r - \widetilde{g}_r].$$

Since for any $s \le r \le t$, $|x_r - \widetilde{x}_r| \le \|x - \widetilde{x}\|_{s,t,\beta} + |x_s - \widetilde{x}_s|$, we may write

$$|I_1| \le b_1\left\{\|x - \widetilde{x}\|_{s,t,\beta}(t-s)^{1+\beta} + |x_s - \widetilde{x}_s|(t-s)\right\} \tag{3.12}$$

and clearly

$$|I_2| \le \left\|b - \widetilde{b}\right\|_\infty (t-s). \tag{3.13}$$

We use (2.3) to obtain

$$|I_4| \le c_\beta\|g\|_{0,T,\beta}\left\{\|\sigma - \widetilde{\sigma}\|_\infty(t-s)^\beta + \left\|\sigma' - \widetilde{\sigma}'\right\|_\infty\|\widetilde{x}\|_{s,t,\beta}(t-s)^{2\beta}\right\},$$
$$|I_5| \le c_\beta\|g - \widetilde{g}\|_{0,T,\beta}\left\{\|\widetilde{\sigma}\|_\infty(t-s)^\beta + \left\|\widetilde{\sigma}'\right\|_\infty\|\widetilde{x}\|_{s,t,\beta}(t-s)^{2\beta}\right\}. \tag{3.14}$$

Then by (3.8) we have for $(t-s) \le \widetilde{\Delta}_1$:

$$|I_4| \le M_3\|g\|_{0,T,\beta}\|\sigma - \widetilde{\sigma}\|_{C^1}(t-s)^\beta,$$
$$|I_5| \le M_3\|g - \widetilde{g}\|_{0,T,\beta}\|\widetilde{\sigma}\|_{C^1}(t-s)^\beta, \tag{3.15}$$

where $M_3 = c_\beta(1 + \widetilde{b}_1 T(1 + |\widetilde{x}_0|)e^{2\widetilde{M}_1\widetilde{b}_1 T})$.

The term $I_3$ is a little bit more elaborate. First we have

$$\begin{aligned} &|\sigma(x_r) - \sigma(\tilde{x}_r) - \sigma(x_{r'}) + \sigma(\tilde{x}_{r'})| \\ &\quad \le \|\sigma'\|_\infty |x_r - \tilde{x}_r - x_{r'} + \tilde{x}_{r'}| + \|\sigma''\|_\infty |x_r - x_{r'}|(|x_r - \tilde{x}_r| + |x_{r'} - \tilde{x}_{r'}|) \\ &\quad \le c_1 \|x - \tilde{x}\|_{s,t,\beta}(t-s)^\beta + 2c_2\|x\|_{s,t,\beta}\Big(\|x - \tilde{x}\|_{s,t,\beta}(t-s)^\beta + |x_s - \tilde{x}_s|\Big)(t-s)^\beta \end{aligned} \tag{3.16}$$

and thus

$$\|\sigma(x.) - \sigma(\tilde{x}.)\|_{s,t,\beta} \le c_1\|x - \tilde{x}\|_{s,t,\beta} + 2c_2\|x\|_{s,t,\beta}\Big(\|x - \tilde{x}\|_{s,t,\beta}(t-s)^\beta + |x_s - \tilde{x}_s|\Big). \tag{3.17}$$

Since

$$\|\sigma(x.) - \sigma(\tilde{x}.)\|_{s,t,\infty} \le \|\sigma'\|_\infty \|x - \tilde{x}\|_{s,t,\beta}(t-s)^\beta + \|\sigma'\|_\infty |x_s - \tilde{x}_s|. \tag{3.18}$$

Inequality (2.3) yields

$$\begin{aligned} |I_3| \le c_\beta \|g\|_{0,T,\beta}\Big\{ & c_1\|x - \tilde{x}\|_{s,t,\beta}(t-s)^{2\beta} + c_1|x_s - \tilde{x}_s|(t-s)^\beta \\ & + \Big[c_1\|x - \tilde{x}\|_{s,t,\beta} + 2c_2\|x\|_{s,t,\beta}\Big(\|x - \tilde{x}\|_{s,t,\beta}(t-s)^\beta + |x_s - \tilde{x}_s|\Big)\Big](t-s)^{2\beta}\Big\} \end{aligned} \tag{3.19}$$

and by (3.8) we deduce that for $(t-s) \le \Delta_1$:

$$|I_3| \le M_4 \|g\|_{0,T,\beta}\Big\{\|x - \tilde{x}\|_{s,t,\beta}(t-s)^{2\beta} + |x_s - \tilde{x}_s|(t-s)^\beta\Big\}, \tag{3.20}$$

with $M_4 = 2c_\beta \|\sigma\|_{C^1}(1 + b_1 T\ (1 + |x_0|)e^{2M_1 b_1 T})$. We use (3.12), (3.13), (3.20), and (3.15), in (3.10) and we may write that for $(t-s) \le \Delta_2 := \min(\Delta_1, \tilde{\Delta}_1)$

$$\begin{aligned} \|x - \tilde{x}\|_{s,t,\beta} \le{}& b_1|x_s - \tilde{x}_s|(t-s)^{1-\beta} + M_4\|g\|_{0,T,\beta}|x_s - \tilde{x}_s| \\ & + \left\|b - \tilde{b}\right\|_\infty (t-s)^{1-\beta} + M_3\|\sigma - \tilde{\sigma}\|_{C^1}\|g\|_{0,T,\beta} + M_3\|\tilde{\sigma}\|_{C^1}\|g - \tilde{g}\|_{0,T,\beta} \\ & + \|x - \tilde{x}\|_{s,t,\beta}\Big[b_1(t-s) + M_4\|g\|_{0,T,\beta}(t-s)^\beta\Big] \\ \le{}& b_1|x_s - \tilde{x}_s|(t-s)^{1-\beta} + M_4\|g\|_{0,T,\beta}|x_s - \tilde{x}_s| + E \\ & + M_5\|x - \tilde{x}\|_{s,t,\beta}\Big(1 + \|g\|_{0,T,\beta}\Big)(t-s)^\beta, \end{aligned} \tag{3.21}$$

where

$$E = \left\|b - \widetilde{b}\right\|_{\infty}(t-s)^{1-\beta} + M_3\|\sigma - \widetilde{\sigma}\|_{C^1}\|g\|_{0,T,\beta} + M_3\|\widetilde{\sigma}\|_{C^1}\|g - \widetilde{g}\|_{0,T,\beta} \tag{3.22}$$

and $M_5 = \max(b_1 T^{1-\beta}, M_4)$. With $\Delta_3 = (2M_5(1 + \|g\|_{0,T,\beta}))^{-1/\beta}$ we may write when $(t-s) \leq \Delta := \min(\Delta_2, \Delta_3)$ that

$$\|x - \widetilde{x}\|_{s,t,\beta} \leq 2b_1|x_s - \widetilde{x}_s|(t-s)^{1-\beta} + 2M_4\|g\|_{0,T,\beta}|x_s - \widetilde{x}_s| + 2E. \tag{3.23}$$

The final arguments are the same as in the proof of (3.1). For $t - s \leq \Delta$ we have

$$\|x - \widetilde{x}\|_{s,t,\infty} \leq |x_s - \widetilde{x}_s|\left(1 + 2b_1\Delta + 2M_4\|g\|_{0,T,\beta}\Delta^{\beta}\right) + 2E \tag{3.24}$$

that implies by induction if we denote $N = T/\Delta$

$$\begin{aligned}
\|x - \widetilde{x}\|_{0,T,\infty} &\leq (|x_0 - \widetilde{x}_0| + 2NE)\left(1 + 2b_1\Delta + 2M_4\|g\|_{0,T,\beta}\Delta^{\beta}\right)^{N} \\
&\leq N(|x_0 - \widetilde{x}_0| + 2E)\left(1 + \frac{2b_1T + 2M_4T\|g\|_{0,T,\beta}\Delta^{\beta-1}}{T/\Delta}\right)^{T/\Delta} \\
&\leq \left(\frac{T}{\Delta}\right)(|x_0 - \widetilde{x}_0| + 2E)\exp\left\{2b_1T + 2M_4T\|g\|_{0,T,\beta}\Delta^{\beta-1}\right\}.
\end{aligned} \tag{3.25}$$

Now we see that we may find a constant $M$ depending only on $T$, $c_\beta$ and all the coefficients of (2.4) (but not on $g$ and $\widetilde{g}$) such that

$$\|x - \widetilde{x}\|_{0,T,\infty} \leq C(|x_0 - \widetilde{x}_0| + E)\left(1 + \|g\|_{0,T,\beta}\right)^{1/\beta}\exp\left\{C\|g\|_{0,T,\beta}^{1/\beta}\right\}. \tag{3.26}$$

Substituting the bound (3.26) in (3.23) yields that for $(t-s) \leq \Delta$:

$$\|x - \widetilde{x}\|_{s,t,\beta} \leq C(|x_0 - \widetilde{x}_0| + E)\left(1 + \|g\|_{0,T,\beta}\right)^{1+1/\beta}\exp\left\{C\|g\|_{0,T,\beta}^{1/\beta}\right\}. \tag{3.27}$$

Thus by Lemma A.2 from [11] we finally obtain

$$\|x - \widetilde{x}\|_{0,T,\beta} \leq C(|x_0 - \widetilde{x}_0| + E)\left(1 + \|g\|_{0,T,\beta}\right)^{2/\beta}\exp\left\{C\|g\|_{0,T,\beta}^{1/\beta}\right\}. \tag{3.28}$$

Proposition 2.3 is now a consequence of (3.26) and (3.28). □

### *3.3. Proof of Theorem 1.1*

*Proof.* The proof is now very simple. We use (2.10) from Proposition 2.3 with $g = B^H$ and $\tilde{g} = B^{H,n}$. Moreover for any $k \geq 1$ we have the following estimate of the moment of the Hölder norm of the fBm (see [16, Lemma 8] for instance)

$$\mathbb{E}\left(\left\|B^H\right\|_{0,T,\beta}^{2k}\right) \leq 32^k (2T)^{2k(H-\beta)} \frac{(2k)!}{k!}. \tag{3.29}$$

Then the estimate on the moments of $\|X^n - X\|_\beta$ are deduced from easy algebra using Lemma 2.2, (3.29), Hölder's inequality and Young's inequality. The convergence (1.9) follows from the stability assumptions on the coefficients. □

## Appendix

## Proof of Lemma 2.1

*Proof.* With $1-\beta < \alpha < 1/2$, we use (2.1) and we obtain for all $0 \leq s,\ t \leq T$ and all $f \in \mathbf{C}^\beta(s,t)$:

$$\left|\int_s^t f_r dg_r\right| \leq \int_s^t \left|D_{s+}^{\alpha} f_r D_{t-}^{1-\alpha} g_{t-}(r)\right| dr. \tag{A.1}$$

We have

$$\begin{aligned} \left|D_{t-}^{1-\alpha} g_{t-}(r)\right| &\leq \frac{\beta}{(\alpha+\beta-1)\Gamma(\alpha)} \|g\|_{0,T,\beta} |t-r|^{\alpha+\beta-1}, \\ \left|D_{s+}^{\alpha} f_r\right| &\leq \frac{1}{\Gamma(1-\alpha)} \|f\|_{s,r,\infty} (r-s)^{-\alpha} + \frac{\alpha}{(\beta-\alpha)\Gamma(1-\alpha)} \|f\|_{s,r,\beta} (r-s)^{\beta-\alpha}. \end{aligned} \tag{A.2}$$

It follows that

$$\begin{aligned} \left|\int_s^t f_r dg_r\right| &\leq \frac{\beta}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(1-\alpha)} \|g\|_{0,T,\beta} \\ &\quad\times \left\{ \|f\|_{s,t,\infty} \int_s^t (r-s)^{-\alpha}(t-r)^{\alpha+\beta-1} dr + \alpha \|f\|_{s,t,\beta} \int_s^t (r-s)^{\beta-\alpha}(t-r)^{\alpha+\beta-1} dr \right\}. \end{aligned} \tag{A.3}$$

We use the change of variables $r = (t-s)\xi + s$ and we recall that the Beta function is defined by $B(a,b) = \int_0^1 (1-\xi)^{a-1}\xi^{b-1}d\xi = \Gamma(a)\Gamma(b)/\Gamma(a+b)$. Then we get

$$\left|\int_s^t f_r dg_r\right| \leq k_{\alpha,\beta}\|g\|_{0,T,\beta}\left[\|f\|_{s,t,\infty}(t-s)^{\beta} + \|f\|_{s,t,\beta}(t-s)^{2\beta}\right] \quad \text{with}$$
$$k_{\alpha,\beta} = \frac{\beta B(\alpha+\beta, 1-\alpha)}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(1-\alpha)} + \frac{\alpha\beta B(\alpha+\beta, 1+\beta-\alpha)}{(\alpha+\beta-1)(\beta-\alpha)\Gamma(\alpha)\Gamma(1-\alpha)}. \tag{A.4}$$

It is proved in [12] that in fact $k_{\alpha,\beta} \leq \kappa/(\beta - 1/2)$ where $\kappa$ is a universal constant. □

## References


[1] D. Nualart, *The Malliavin Calculus and Related Topics*, Probability and Its Applications, Springer, Berlin, Germany, 2nd edition, 2006.

[2] L. C. Young, "An inequality of the Hölder type, connected with Stieltjes integration," *Acta Mathematica*, vol. 67, no. 1, pp. 251–282, 1936.

[3] T. Lyons, "Differential equations driven by rough signals. I. An extension of an inequality of L. C. Young," *Mathematical Research Letters*, vol. 1, no. 4, pp. 451–464, 1994.

[4] L. Coutin and Z. Qian, "Stochastic analysis, rough path analysis and fractional Brownian motions," *Probability Theory and Related Fields*, vol. 122, no. 1, pp. 108–140, 2002.

[5] M. Zähle, "Integration with respect to fractal functions and stochastic calculus. I," *Probability Theory and Related Fields*, vol. 111, no. 3, pp. 333–374, 1998.

[6] D. Nualart and A. Răşcanu, "Differential equations driven by fractional Brownian motion," *Collectanea Mathematica*, vol. 53, no. 1, pp. 55–81, 2002.

[7] F. Baudoin and M. Hairer, "A version of Hörmander's theorem for the fractional Brownian motion," *Probability Theory and Related Fields*, vol. 139, no. 3-4, pp. 373–395, 2007.

[8] Y. Hu and D. Nualart, "Differential equations driven by Hölder continuous functions of order greater than 1/2," in *Stochastic Analysis and Applications*, vol. 2 of *Abel Symposium*, pp. 399–413, Springer, Berlin, Germany, 2007.

[9] I. Nourdin and T. Simon, "On the absolute continuity of one-dimensional SDEs driven by a fractional Brownian motion," *Statistics & Probability Letters*, vol. 76, no. 9, pp. 907–912, 2006.

[10] D. Nualart and B. Saussereau, "Malliavin calculus for stochastic differential equations driven by a fractional Brownian motion," *Stochastic Processes and Their Applications*, vol. 119, no. 2, pp. 391–409, 2009.

[11] M. Hairer and N. S. Pillai, "Ergodicity of hypoelliptic SDEs driven by fractional Brownian motion," *Annales de l'Institut Henri Poincaré Probabilités et Statistiques*, vol. 47, no. 2, pp. 601–628, 2011.

[12] B. Saussereau, "A remark on the mean square distance between the solutions of fractional SDEs and Brownian SDEs," *Stochastics*, vol. 84, no. 1, pp. 1–19, 2012.

[13] P. K. Friz and N. B. Victoir, *Multidimensional Stochastic Processes as Rough Paths Theory and Applications*, vol. 120 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, UK, 2010.

[14] L. Decreusefond and D. Nualart, "Flow properties of differential equations driven by fractional Brownian motion," in *Stochastic Differential Equations: Theory and Applications*, vol. 2 of *Interdisciplinary Math and Science*, pp. 249–262, World Scientific, River Edge, NJ, USA, 2007.

[15] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, Yverdon, Switzerland, 1993.

[16] B. Saussereau, "Transportation inequalities for stochastic differential equations driven by a fractional Brownian motion," *Bernoulli*, vol. 18, no. 1, pp. 1–23, 2012.