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Research Article

The Itô Integral with respect to an Infinite Dimensional Lévy Process: A Series Approach

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We present an alternative construction of the infinite dimensional Itô integral with respect to a Hilbert space valued Lévy process. This approach is based on the well-known theory of real-valued stochastic integration, and the respective Itô integral is given by a series of Itô integrals with respect to standard Lévy processes. We also prove that this stochastic integral coincides with the Itô integral that has been developed in the literature.

1. Introduction

The Itô integral with respect to an infinite dimensional Wiener process has been developed in [1–3], and for the more general case of an infinite dimensional square-integrable martingale, it has been defined in [4, 5]. In these references, one first constructs the Itô integral for elementary processes and then extends it via the Itô isometry to a larger space, in which the space of elementary processes is dense.

For stochastic integrals with respect to a Wiener process, series expansions of the Itô integral have been considered, for example, in [6–8]. Moreover, in [9], series expansions have been used in order to define the Itô integral with respect to a Wiener process for deterministic integrands with values in a Banach space. Later, in [10], this theory has been extended to general integrands with values in UMD Banach spaces.

To the best of the author's knowledge, a series approach for the construction of the Itô integral with respect to an infinite dimensional Lévy process does not exist in the literature so far. The goal of the present paper is to provide such a construction, which is based on the real-valued Itô integral; see, for example, [11–13], and where the Itô integral is given by a series of Itô integrals with respect to real-valued Lévy processes. This approach has the advantage that we can use results from the finite dimensional case, and it might also be beneficial for lecturers teaching students who are already aware of the real-valued Itô integral and have some background in functional analysis. In particular, it avoids the tedious procedure of proving that elementary processes are dense in the space of integrable processes.

In [14], the stochastic integral with respect to an infinite dimensional Lévy process is defined as a limit of Riemannian sums, and a series expansion is provided. A particular feature of [14] is that stochastic integrals are considered as L^2-curves. The connection to the usual Itô integral for a finite dimensional Lévy process has been established in [15]; see also Appendix B in [16]. Furthermore, we point out [17, 18], where the theory of stochastic integration with respect to Lévy processes has been extended to Banach spaces.

The idea to use series expansions for the definition of the stochastic integral has also been utilized in the context of cylindrical processes; see [19] for cylindrical Wiener processes and [20] for cylindrical Lévy processes.

The construction of the Itô integral, which we present in this paper, is divided into the following steps.

(i) For an H-valued process X (with H denoting a separable Hilbert space) and a real-valued square-integrable martingale M, we define the Itô integral

$$X \cdot M := \sum_{k\in\mathbb{N}} \left(\langle X, f_k \rangle_H \cdot M \right) f_k, \tag{1}$$

where $(f_k)_{k\in\mathbb{N}}$ denotes an orthonormal basis of H, and $\langle X, f_k \rangle_H \cdot M$ denotes the real-valued Itô integral.

We will show that this definition does not depend on the choice of the orthonormal basis.

(ii) Based on the just defined integral, for an $\ell^2(H)$-valued process X and a sequence $(M^j)_{j\in\mathbb{N}}$ of standard Lévy processes, we define the Itô integral as

$$\sum_{j\in\mathbb{N}} X^j \cdot M^j. \tag{2}$$

For this, we will ensure convergence of the series.

(iii) In the next step, let L denote an ℓ^2_λ-valued Lévy process, where ℓ^2_λ is a weighted space of sequences (cf. [21]). From the Lévy process L, we can construct a sequence $(M^j)_{j\in\mathbb{N}}$ of standard Lévy processes, and for a $\ell^2(H)$-valued process X, we define the Itô integral

$$X \cdot L := \sum_{j\in\mathbb{N}} X^j \cdot M^j. \tag{3}$$

(iv) Finally, let L be a general Lévy process on some separable Hilbert space U with covariance operator Q. Then, there exist sequences of eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ and eigenvectors, which diagonalize the operator Q. Denoting by $L_2^0(H)$ an appropriate space of Hilbert Schmidt operators from U to H, our idea is to utilize the integral from the previous step and to define the Itô integral for a $L_2^0(H)$-valued process X as

$$X \cdot L := \Psi(X) \cdot \Phi(L), \tag{4}$$

where $\Phi : U \to \ell^2_\lambda$ and $\Psi : L_2^0(H) \to \ell^2(H)$ are isometric isomorphisms such that $\Phi(L)$ is an ℓ^2_λ-valued Lévy process. We will show that this definition does not depend on the choice of the eigenvalues and eigenvectors.

The remainder of this text is organized as follows. In Section 2, we provide the required preliminaries and notation. After that, we start with the construction of the Itô integral as outlined earlier. In Section 3, we define the Itô integral for H-valued processes with respect to a real-valued square-integrable martingale, and in Section 4, we define the Itô integral for $\ell^2(H)$-valued processes with respect to a sequence of standard Lévy processes. Section 5 gives a brief overview about Lévy processes in Hilbert spaces, together with the required results. Then, in Section 6, we define the Itô integral for $\ell^2(H)$-valued processes with respect to an ℓ^2_λ-valued Lévy process, and in Section 7, we define the Itô integral in the general case, where the integrand is an $L_2^0(H)$-valued process and the integrator a general Lévy process on some separable Hilbert space U. We also prove the mentioned series representation of the stochastic integral and show that it coincides with the usual Itô integral, which has been developed in [5].

2. Preliminaries and Notation

In this section, we provide the required preliminary results and some basic notation. Throughout this text, let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. For the upcoming results, let E be a separable Banach space, and let $T > 0$ be a finite time horizon.

Definition 1. Let $p \geq 1$ be arbitrary.

(1) We define the Lebesgue space

$$\mathscr{L}_T^p(E) := \mathscr{L}^p(\Omega, \mathscr{F}_T, \mathbb{P}; \mathbb{D}([0,T]; E)), \tag{5}$$

where $\mathbb{D}([0,T]; E)$ denotes the Skorokhod space consisting of all càdlàg functions from $[0,T]$ to E, equipped with the supremum norm.

(2) We denote by $\mathscr{A}_T^p(E)$ the space of all E-valued adapted processes $X \in \mathscr{L}_T^p(E)$.

(3) We denote by $\mathscr{M}_T^p(E)$ the space of all E-valued martingales $M \in \mathscr{L}_T^p(E)$.

(4) We define the factor spaces

$$M_T^p(E) := \frac{\mathscr{M}_T^p(E)}{N}, \qquad A_T^p(E) := \frac{\mathscr{A}_T^p(E)}{N}, \tag{6}$$

$$L_T^p(E) := \frac{\mathscr{L}_T^p(E)}{N}, \tag{7}$$

where $N \subset \mathscr{M}_T^p(E)$ denotes the subspace consisting of all $M \in \mathscr{M}_T^p(E)$ with $M = 0$ up to indistinguishability.

Remark 2. Let us emphasize the following.

(1) Since the Skorokhod space $\mathbb{D}([0,T]; E)$ equipped with the supremum norm is a Banach space, the Lebesgue space $L_T^p(E)$ equipped with the standard norm

$$\|X\|_{L_T^p(E)} := \mathbb{E}\left[\|X\|_E^p\right]^{1/p} \tag{8}$$

is a Banach space too.

(2) By the completeness of the filtration $(\mathscr{F}_t)_{t\geq 0}$, adaptedness of an element $X \in L_T^p(E)$ does not depend on the choice of the representative. This ensures that the factor space $A_T^p(E)$ of adapted processes is well defined.

(3) The definition of E-valued martingales relies on the existence of conditional expectation in Banach spaces, which has been established in [1, Proposition 1.10].

Note that we have the inclusions

$$M_T^p(E) \subset A_T^p(E) \subset L_T^p(E). \tag{9}$$

The following auxiliary result shows that these inclusions are closed.

Lemma 3. *Let $p \geq 1$ be arbitrary. Then, the following statements are true:*

(1) *$M_T^p(E)$ is closed in $A_T^p(E)$;*

(2) *$A_T^p(E)$ is closed in $L_T^p(E)$.*

Proof. Let $(M^n)_{n\in\mathbb{N}} \subset M_T^p(E)$ be a sequence, and let $M \in A_T^p(E)$ be such that $M^n \to M$ in $L_T^p(E)$. Furthermore, let $\tau \le T$ be a bounded stopping time. Then, we have

$$\mathbb{E}\left[\|M_\tau\|_E^p\right] \le \mathbb{E}\left[\sup_{t\in[0,T]} \|M_t\|_E^p\right] < \infty, \quad (10)$$

showing that $M_\tau \in L^p(\Omega, \mathcal{F}_\tau, \mathbb{P}; E)$. Furthermore, we have

$$\mathbb{E}\left[\|M_\tau^n - M_\tau\|_E^p\right] \le \mathbb{E}\left[\sup_{t\in[0,T]} \|M_t^n - M_t\|_E^p\right] \longrightarrow 0. \quad (11)$$

By Doob's optional stopping theorem (which also holds true for E-valued martingales; see [2, Remark 2.2.5]), it follows that

$$\mathbb{E}\left[M_\tau\right] = \lim_{n\to\infty} \mathbb{E}\left[M_\tau^n\right] = \lim_{n\to\infty} \mathbb{E}\left[M_0^n\right] = \mathbb{E}\left[M_0\right]. \quad (12)$$

Using Doob's optional stopping theorem again, we conclude that $M \in M_T^p(E)$, proving the first statement.

Now, let $(X^n)_{n\in\mathbb{N}} \subset A_T^p(E)$ be a sequence, and let $X \in L_T^p(E)$ be such that $X^n \to X$ in $L_T^p(E)$. Then, for each $t \in [0, T]$, we have

$$\mathbb{E}\left[\|X_t^n - X_t\|_E^p\right] \le \mathbb{E}\left[\sup_{s\in[0,T]} \|X_s^n - X_s\|_E^p\right] \longrightarrow 0, \quad (13)$$

and, hence, $\mathbb{P}$-almost surely $X_t^{n_k} \to X_t$ for some subsequence $(n_k)_{k\in\mathbb{N}}$, showing that X_t is $\mathcal{F}_t$-measurable. This proves that $X \in A_T^p(E)$, providing the second statement. □

Note that, by Doob's martingale inequality [2, Theorem 2.2.7], for $p > 1$, an equivalent norm on $M_T^p(E)$ is given by

$$\|M\|_{M_T^p(E)} := \mathbb{E}\left[\|M_T\|_E^p\right]^{1/p}. \quad (14)$$

Furthermore, if $E = H$ is a separable Hilbert space, then $M_T^2(H)$ is a separable Hilbert space equipped with the inner product

$$\langle M, N\rangle_{M_T^2(H)} := \mathbb{E}\left[\langle M_T, N_T\rangle_H\right]. \quad (15)$$

Finally, we recall the following result about series of pairwise orthogonal vectors in Hilbert spaces.

Lemma 4. *Let H be a separable Hilbert space, and let $(h_n)_{n\in\mathbb{N}} \subset H$ be a sequence with $\langle h_n, h_m\rangle_H = 0$ for $n \ne m$. Then, the following statements are equivalent.*

(1) *The series $\sum_{n=1}^\infty h_n$ converges in H.*

(2) *The series $\sum_{n\in\mathbb{N}} h_n$ converges unconditionally in H.*

(3) *One has $\sum_{n=1}^\infty \|h_n\|_H^2 < \infty$.*

If the previous conditions are satisfied, then one has

$$\left\|\sum_{n=1}^\infty h_n\right\|_H^2 = \sum_{n=1}^\infty \|h_n\|_H^2. \quad (16)$$

Proof. This follows from [22, Theorem 12.6] and [23, Satz V.4.8]. □

3. The Itô Integral with respect to a Real-Valued Square-Integrable Martingale

In this section, we define the Itô integral for Hilbert space valued processes with respect to a real-valued, square-integrable martingale, which is based on the real-valued Itô integral.

In what follows, let H be a separable Hilbert space, and let $T > 0$ be a finite time horizon. Furthermore, let $M \in \mathcal{M}_T^2(\mathbb{R})$ be a square-integrable martingale. Recall that the quadratic variation $\langle M, M\rangle$ is the (up to indistinguishability) unique real-valued, nondecreasing, predictable process with $\langle M, M\rangle_0 = 0$ such that $M^2 - \langle M, M\rangle$ is a martingale.

Proposition 5. *Let X be an H-valued, predictable process with*

$$\mathbb{E}\left[\int_0^T \|X_s\|_H^2 d\langle M, M\rangle_s\right] < \infty. \quad (17)$$

Then, for every orthonormal basis $(f_k)_{k\in\mathbb{N}}$ of H, the series

$$\sum_{k\in\mathbb{N}} \left(\langle X, f_k\rangle_H \cdot M\right) f_k \quad (18)$$

converges unconditionally in $M_T^2(H)$, and its value does not depend on the choice of the orthonormal basis $(f_k)_{k\in\mathbb{N}}$.

Proof. Let $(f_k)_{k\in\mathbb{N}}$ be an orthonormal basis of H. For $j, k \in \mathbb{N}$ with $j \ne k$, we have

$$\begin{aligned}
&\left\langle \left(\langle X, f_j\rangle_H \cdot M\right) f_j, \left(\langle X, f_k\rangle_H \cdot M\right) f_k \right\rangle_{M_T^2(H)} \\
&= \mathbb{E}\left[\left\langle \left(\int_0^T \langle X_s, f_j\rangle_H dM_s\right) f_j, \left(\int_0^T \langle X_s, f_k\rangle_H dM_s\right) f_k \right\rangle_H\right] \\
&= \mathbb{E}\left[\left(\int_0^T \langle X_s, f_j\rangle_H dM_s\right)\left(\int_0^T \langle X_s, f_k\rangle_H dM_s\right)\langle f_j, f_k\rangle_H\right] \\
&= 0.
\end{aligned} \quad (19)$$

Moreover, by the Itô isometry for the real-valued Itô integral and the monotone convergence theorem, we obtain

$$\begin{aligned}
&\sum_{k=1}^\infty \left\|\left(\langle X, f_k\rangle_H \cdot M\right) f_k\right\|_{M_T^2(H)}^2 \\
&= \sum_{k=1}^\infty \mathbb{E}\left[\left\|\left(\int_0^T \langle X_s, f_k\rangle_H dM_s\right) f_k\right\|_H^2\right] \\
&= \sum_{k=1}^\infty \mathbb{E}\left[\left|\int_0^T \langle X_s, f_k\rangle_H dM_s\right|^2\right] \\
&= \sum_{k=1}^\infty \mathbb{E}\left[\int_0^T \left|\langle X_s, f_k\rangle_H\right|^2 d\langle M, M\rangle_s\right] \\
&= \mathbb{E}\left[\int_0^T \sum_{k=1}^\infty \left|\langle X_s, f_k\rangle_H\right|^2 d\langle M, M\rangle_s\right] \\
&= \mathbb{E}\left[\int_0^T \|X_s\|_H^2 d\langle M, M\rangle_s\right].
\end{aligned} \quad (20)$$

Therefore, by (17) and Lemma 4, the series (18) converges unconditionally in $M_T^2(H)$.

Now, let $(g_k)_{k\in\mathbb{N}}$ be another orthonormal basis of H. We define $\mathbb{J}^f, \mathbb{J}^g \in M_T^2(H)$ by

$$\begin{aligned}\mathbb{J}^f &:= \sum_{k=1}^{\infty} \left(\langle X, f_k\rangle_H \cdot M\right) f_k,\\ \mathbb{J}^g &:= \sum_{k=1}^{\infty} \left(\langle X, g_k\rangle_H \cdot M\right) g_k.\end{aligned} \tag{21}$$

Let $h \in H$ be arbitrary. Then, we have

$$\left\langle h, \mathbb{J}^f\right\rangle_H, \langle h, \mathbb{J}^g\rangle_H \in M_T^2(\mathbb{R}) \tag{22}$$

and the identity

$$\begin{aligned}&\left\|\left\langle h, \mathbb{J}^f\right\rangle_H - \langle h, X\rangle_H \cdot M\right\|^2_{M_T^2(\mathbb{R})}\\ &= \left\|\left\langle h, \sum_{k=1}^{\infty}(\langle X, f_k\rangle_H \cdot M) f_k\right\rangle_H - \langle h, X\rangle_H \cdot M\right\|^2_{M_T^2(\mathbb{R})}\\ &= \left\|\sum_{k=1}^{\infty}(\langle h, f_k\rangle_H\langle f_k, X\rangle_H \cdot M) - \langle h, X\rangle_H \cdot M\right\|^2_{M_T^2(\mathbb{R})}\\ &= \lim_{n\to\infty}\left\|\sum_{k=1}^{n}(\langle h, f_k\rangle_H\langle f_k, X\rangle_H \cdot M) - \langle h, X\rangle_H \cdot M\right\|^2_{M_T^2(\mathbb{R})}\\ &= \lim_{n\to\infty}\left\|\left(\sum_{k=1}^{n}\langle h, f_k\rangle_H\langle f_k, X\rangle_H - \langle h, X\rangle_H\right) \cdot M\right\|^2_{M_T^2(\mathbb{R})}.\end{aligned} \tag{23}$$

For all $x \in H$, we have

$$\left|\sum_{k=1}^{n}\langle x, f_k\rangle_H\langle f_k, h\rangle_H - \langle x, h\rangle_H\right|^2 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{24}$$

and, by the Cauchy-Schwarz inequality,

$$\begin{aligned}&\left|\sum_{k=1}^{n}\langle x, f_k\rangle_H\langle f_k, h\rangle_H - \langle x, h\rangle_H\right|^2\\ &\quad= \left|\sum_{k=n+1}^{\infty}\langle x, f_k\rangle_H\langle f_k, h\rangle_H\right|^2\\ &\quad\le \left(\sum_{k=1}^{\infty}\left|\langle x, f_k\rangle_H\right|^2\right)\left(\sum_{k=1}^{\infty}\left|\langle f_k, h\rangle_H\right|^2\right)\\ &\quad= \|x\|_H^2\|h\|_H^2 \quad \text{for each } n \in \mathbb{N}.\end{aligned} \tag{25}$$

Therefore, by the Itô isometry for the real-valued Itô integral and Lebesgue's dominated convergence theorem together with (17), we obtain

$$\begin{aligned}&\left\|\left\langle h, \mathbb{J}^f\right\rangle_H - \langle h, X\rangle_H \cdot M\right\|^2_{M_T^2(\mathbb{R})}\\ &= \lim_{n\to\infty}\mathbb{E}\left[\left|\int_0^T\left(\sum_{k=1}^{n}\langle h, f_k\rangle_H\langle f_k, X_s\rangle_H - \langle h, X_s\rangle_H\right)dM_s\right|^2\right]\\ &= \lim_{n\to\infty}\mathbb{E}\left[\int_0^T\left|\sum_{k=1}^{n}\langle h, f_k\rangle_H\langle f_k, X\rangle_H - \langle h, X\rangle_H\right|^2 d\langle M, M\rangle_s\right]\\ &= 0.\end{aligned} \tag{26}$$

Analogously, we prove that

$$\left\|\langle h, \mathbb{J}^g\rangle_H - \langle h, X\rangle_H \cdot M\right\|^2_{M_T^2(\mathbb{R})} = 0. \tag{27}$$

Therefore, denoting by $\tilde{\mathbb{J}}^f, \tilde{\mathbb{J}}^g \in \mathscr{M}_T^2(H)$ representatives of $\mathbb{J}^f$, $\mathbb{J}^g$, we obtain

$$\left\langle h, \tilde{\mathbb{J}}_T^f\right\rangle_H = \left\langle h, \tilde{\mathbb{J}}_T^g\right\rangle_H \quad \forall h \in H, \ \mathbb{P}\text{-almost surely.} \tag{28}$$

By separability of H, we deduce that

$$\left\langle h, \tilde{\mathbb{J}}_T^f\right\rangle_H = \left\langle h, \tilde{\mathbb{J}}_T^g\right\rangle_H \quad \mathbb{P}\text{-almost surely,} \quad \forall h \in H. \tag{29}$$

Consequently, we have

$$\tilde{\mathbb{J}}_T^f = \tilde{\mathbb{J}}_T^g \quad \mathbb{P}\text{-almost surely,} \tag{30}$$

Implying that $\mathbb{J}^f = \mathbb{J}^g$. This proves that the value of the series (18) does not depend on the choice of the orthonormal basis. □

Now, Proposition 5 gives rise to the following definition.

Definition 6. For every H-valued, predictable process X satisfying (17), we define the Itô integral $X \cdot M = (\int_0^t X_s dM_s)_{t\in[0,T]}$ as

$$X \cdot M := \sum_{k\in\mathbb{N}}\left(\langle X, f_k\rangle_H \cdot M\right) f_k, \tag{31}$$

where $(f_k)_{k\in\mathbb{N}}$ denotes an orthonormal basis of H.

According to Proposition 5, definition (31) of the Itô integral is independent of the choice of the orthonormal basis $(f_k)_{k\in\mathbb{N}}$, and the integral process $X \cdot M$ belongs to $M_T^2(H)$.

Remark 7. As the proof of Proposition 5 shows, the components of the Itô integral $X \cdot M$ are pairwise orthogonal elements of the Hilbert space $M_T^2(H)$.

Proposition 8. *For every H-valued, predictable process X satisfying* (17)*, one has the Itô isometry*

$$\mathbb{E}\left[\left\|\int_0^T X_s\, dM_s\right\|_H^2\right] = \mathbb{E}\left[\int_0^T \|X_s\|_H^2\, d\langle M, M\rangle_s\right]. \tag{32}$$

Proof. Let $(f_k)_{k\in\mathbb{N}}$ be an orthonormal basis of H. According to (19), we have

$$\left\langle \left(\left\langle X, f_j \right\rangle_H \cdot M \right) f_j, \left(\left\langle X, f_k \right\rangle_H \cdot M \right) f_k \right\rangle_{M_T^2(H)} = 0 \quad \text{for } j \neq k. \tag{33}$$

Thus, by Lemma 4 and (20), we obtain

$$\begin{aligned} \mathbb{E}\left[\left\| \int_0^T X_s dM_s \right\|_H^2 \right] &= \|X \cdot M\|_{M_T^2(H)}^2 \\ &= \left\| \sum_{k=1}^{\infty} \left(\left\langle X, f_k \right\rangle_H \cdot M \right) f_k \right\|_{M_T^2(H)}^2 \\ &= \sum_{k=1}^{\infty} \left\| \left(\left\langle X, f_k \right\rangle_H \cdot M \right) f_k \right\|_{M_T^2(H)}^2 \\ &= \mathbb{E}\left[\int_0^T \|X_s\|_H^2 d\langle M, M\rangle_s \right], \end{aligned} \tag{34}$$

finishing the proof. □

Proposition 9. *Let X be a H-valued simple process of the form*

$$X = X_0 \mathbb{1}_{\{0\}} + \sum_{i=1}^{n} X_i \mathbb{1}_{(t_i, t_{i+1}]} \tag{35}$$

with $0 = t_1 < \cdots < t_{n+1} = T$ and $\mathcal{F}_{t_i}$-measurable random variables $X_i : \Omega \to H$ for $i = 0, \ldots, n$. Then, one has

$$X \cdot M = \sum_{i=1}^{n} X_i \left(M^{t_{i+1}} - M^{t_i} \right). \tag{36}$$

Proof. Let $(f_k)_{k\in\mathbb{N}}$ be an orthonormal basis of H. Then, for each $k \in \mathbb{N}$, the process $\langle X, f_k\rangle$ is a real-valued simple process with representation

$$\left\langle X, f_k \right\rangle_H = \left\langle X_0, f_k \right\rangle_H \mathbb{1}_{\{0\}} + \sum_{i=1}^{n} \left\langle X_i, f_k \right\rangle_H \mathbb{1}_{(t_i, t_{i+1}]}. \tag{37}$$

Thus, by the definition of the real-valued Itô integral for simple processes, we obtain

$$\begin{aligned} X \cdot M &= \sum_{k\in\mathbb{N}} \left(\left\langle X, f_k \right\rangle_H \cdot M \right) f_k \\ &= \sum_{k\in\mathbb{N}} \left(\sum_{i=1}^{n} \left\langle X_i, f_k \right\rangle_H \left(M^{t_{i+1}} - M^{t_i} \right) \right) f_k \\ &= \sum_{i=1}^{n} \left(\sum_{k\in\mathbb{N}} \left\langle X_i, f_k \right\rangle_H f_k \right) \left(M^{t_{i+1}} - M^{t_i} \right) \\ &= \sum_{i=1}^{n} X_i \left(M^{t_{i+1}} - M^{t_i} \right), \end{aligned} \tag{38}$$

finishing the proof. □

Lemma 10. *Let X be a H-valued, predictable process satisfying* (17)*. Then, for every orthonormal basis $(f_k)_{k\in\mathbb{N}}$ of H, one has*

$$\sum_{k=1}^{\infty} \left| \left\langle X, f_k \right\rangle_H \right|^2 \cdot \langle M, M\rangle = \|X\|_H^2 \cdot \langle M, M\rangle, \tag{39}$$

where the convergence takes place in $A_T^1(\mathbb{R})$.

Proof. We define the integral process

$$\mathbb{I} := \|X\|_H^2 \cdot \langle M, M\rangle \tag{40}$$

and the sequence $(\mathbb{I}^n)_{n\in\mathbb{N}}$ of partial sums by

$$\mathbb{I}^n := \sum_{k=1}^{n} \left| \left\langle X, f_k \right\rangle_H \right|^2 \cdot \langle M, M\rangle. \tag{41}$$

By (17) we have $\mathbb{I} \in A_T^1(\mathbb{R})$ and $(\mathbb{I}^n)_{n\in\mathbb{N}} \subset A_T^1(\mathbb{R})$. Furthermore, by Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} \|\mathbb{I} - \mathbb{I}^n\|_{L_T^1(\mathbb{R})} &= \mathbb{E}\left[\sup_{t\in[0,T]} \left| \mathbb{I}_t - \mathbb{I}_t^n \right| \right] \\ &= \mathbb{E}\left[\sup_{t\in[0,T]} \left| \int_0^t \sum_{k=n+1}^{\infty} \left| \left\langle X_s, f_k \right\rangle \right|^2 d\langle M, M\rangle_s \right| \right] \\ &= \mathbb{E}\left[\int_0^T \sum_{k=n+1}^{\infty} \left| \left\langle X_s, f_k \right\rangle \right|^2 d\langle M, M\rangle_s \right] \longrightarrow 0 \\ &\qquad \text{for } n \longrightarrow \infty, \end{aligned} \tag{42}$$

which concludes the proof. □

Remark 11. As a consequence of the Doob-Meyer decomposition theorem, for two square-integrable martingales $X, Y \in \mathscr{M}_T^2(H)$, there exists (up to indistinguishability) a unique real-valued, predictable process $\langle X, Y\rangle$ with finite variation paths and $\langle X, Y\rangle_0 = 0$ such that $\langle X, Y\rangle_H - \langle X, Y\rangle$ is a martingale.

Proposition 12. *For every H-valued, predictable process X satisfying* (17)*, one has*

$$\langle X \cdot M, X \cdot M\rangle = \|X\|_H^2 \cdot \langle M, M\rangle. \tag{43}$$

Proof. Let $(f_k)_{k\in\mathbb{N}}$ be an orthonormal basis of H. We define the process $\mathbb{J} := X \cdot M$ and the sequence $(\mathbb{J}^n)_{n\in\mathbb{N}}$ of partial sums by

$$\mathbb{J}^n := \sum_{k=1}^{n} \left(\left\langle X, f_k \right\rangle_H \cdot M \right) f_k. \tag{44}$$

By Proposition 5, we have

$$\mathbb{J}^n \longrightarrow \mathbb{J} \quad \text{in } M_T^2(H). \tag{45}$$

Defining the integral process $\mathbb{I}$ by (40) and the sequence $(\mathbb{I}^n)_{n\in\mathbb{N}}$ of partial sums by (41), using Lemma 10, we have

$$\mathbb{I}^n \longrightarrow \mathbb{I} \quad \text{in } A^1_T(\mathbb{R}). \tag{46}$$

Furthermore, we define the process $M \in A^1_T(\mathbb{R})$ and the sequence $(M^n)_{n\in\mathbb{N}} \subset A^1_T(\mathbb{R})$ as

$$\begin{aligned} M &:= \|\mathbb{J}\|_H^2 - \mathbb{I}, \\ M^n &:= \|\mathbb{J}^n\|_H^2 - \mathbb{I}^n, \quad n \in \mathbb{N}. \end{aligned} \tag{47}$$

Then, we have $(M^n)_{n\in\mathbb{N}} \subset M^1_T(\mathbb{R})$. Indeed, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} M^n &= \left\| \sum_{k=1}^n \left(\langle X, f_k\rangle_H \cdot M\right) f_k \right\|_H^2 - \sum_{k=1}^n \left|\langle X, f_k\rangle_H\right|^2 \cdot \langle M, M\rangle \\ &= \sum_{k=1}^n \left\| \left(\langle X, f_k\rangle_H \cdot M\right) f_k \right\|_H^2 - \sum_{k=1}^n \left|\langle X, f_k\rangle_H\right|^2 \cdot \langle M, M\rangle \\ &= \sum_{k=1}^n \left(\left|\langle X, f_k\rangle_H \cdot M\right|^2 - \left|\langle X, f_k\rangle_H\right|^2 \cdot \langle M, M\rangle \right). \end{aligned} \tag{48}$$

For every $k \in \mathbb{N}$, the quadratic variation of the real-valued process $\langle X, f_k\rangle_H \cdot M$ is given by

$$\left\langle \langle X, f_k\rangle_H \cdot M, \langle X, f_k\rangle_H \cdot M \right\rangle = \left|\langle X, f_k\rangle_H\right|^2 \cdot \langle M, M\rangle \tag{49}$$

see, for example, [12, Theorem I.4.40.d], which shows that M^n is a martingale. Since $M^n \in A^1_T(\mathbb{R})$, we deduce that $M^n \in M^1_T(\mathbb{R})$.

Next, we prove that $M^n \to M$ in $A^1_T(\mathbb{R})$. Indeed, since

$$\left| \, \|\mathbb{J}\|_H^2 - \|\mathbb{J}^n\|_H^2 \, \right| \le \|\mathbb{J} - \mathbb{J}^n\|_H^2 + 2\|\mathbb{J}\|_H \|\mathbb{J} - \mathbb{J}^n\|_H, \tag{50}$$

by the Cauchy-Schwarz inequality and (45) we obtain

$$\begin{aligned} &\left\| \|\mathbb{J}\|_H^2 - \|\mathbb{J}^n\|_H^2 \right\|_{L^1_T(\mathbb{R})} \\ &= \mathbb{E}\left[\sup_{t\in[0,T]} \left| \, \|\mathbb{J}_t\|_H^2 - \|\mathbb{J}_t^n\|_H^2 \right| \right] \\ &\le \mathbb{E}\left[\sup_{t\in[0,T]} \|\mathbb{J}_t - \mathbb{J}_t^n\|_H^2 \right] + 2\mathbb{E}\left[\sup_{t\in[0,T]} \|\mathbb{J}_t\|_H \|\mathbb{J}_t - \mathbb{J}_t^n\|_H \right] \\ &\le \mathbb{E}\left[\sup_{t\in[0,T]} \|\mathbb{J}_t - \mathbb{J}_t^n\|_H^2 \right] \\ &\quad + 2\mathbb{E}\left[\sup_{t\in[0,T]} \|\mathbb{J}_t\|_H^2 \right]^{1/2} \mathbb{E}\left[\sup_{t\in[0,T]} \|\mathbb{J}_t - \mathbb{J}_t^n\|_H^2 \right]^{1/2} \\ &= \|\mathbb{J} - \mathbb{J}^n\|_{L^2_T(H)}^2 + 2\|\mathbb{J}\|_{L^2_T(H)} \\ &\quad \times \|\mathbb{J} - \mathbb{J}^n\|_{L^2_T(H)} \longrightarrow 0. \end{aligned} \tag{51}$$

Therefore, together with (46), we get

$$\begin{aligned} \|M - M^n\|_{L^1_T(\mathbb{R})} \le &\left\| \|\mathbb{J}\|^2 - \|\mathbb{J}^n\|^2 \right\|_{L^1_T(\mathbb{R})} \\ &+ \|\mathbb{I} - \mathbb{I}^n\|_{L^1_T(\mathbb{R})} \longrightarrow 0, \end{aligned} \tag{52}$$

showing that $M^n \to M$ in $A^1_T(\mathbb{R})$. Now, Lemma 3 yields that $M \in M^1_T(\mathbb{R})$, which concludes the proof. □

Theorem 13. *Let $N \in \mathscr{M}^2_T(\mathbb{R})$ be another square-integrable martingale, and let X, Y be two H-valued, predictable processes satisfying* (17) *and*

$$\mathbb{E}\left[\int_0^T \|Y_s\|_H^2 d\langle N, N\rangle_s \right] < \infty. \tag{53}$$

Then, one has

$$\langle X \cdot M, Y \cdot N\rangle = \langle X, Y\rangle_H \cdot \langle M, N\rangle. \tag{54}$$

Proof. Using Proposition 12 and the identities

$$\begin{aligned} \langle x, y\rangle_H &= \frac{1}{4}\left(\|x+y\|_H^2 - \|x-y\|_H^2 \right), \quad x, y \in H, \\ \langle M, N\rangle &= \frac{1}{4}\left(\langle M+N, M+N\rangle - \langle M-N, M-N\rangle \right), \end{aligned} \tag{55}$$

identity (54) follows from a straightforward calculation. □

Proposition 14. *Let $N \in \mathscr{M}^2_T(\mathbb{R})$ be another square-integrable martingale such that $\langle M, N\rangle = 0$, and let X, Y be two H-valued, predictable processes satisfying* (17) *and* (53). *Then, one has*

$$\langle X \cdot M, Y \cdot N\rangle_{M^2_T(H)} = 0. \tag{56}$$

Proof. Using Remark 11, Theorem 13, and the hypothesis $\langle M, N\rangle = 0$, we obtain

$$\begin{aligned} &\langle X \cdot M, Y \cdot N\rangle_{M^2_T(H)} \\ &= \mathbb{E}\left[\left\langle \int_0^T X_s dM_s, \int_0^T Y_s dN_s \right\rangle_H \right] \\ &= \mathbb{E}\left[\left\langle \int_0^T X_s dM_s, \int_0^T Y_s dN_s \right\rangle \right] \\ &= \mathbb{E}\left[\int_0^T \langle X_s, Y_s\rangle_H d\langle M, N\rangle_s \right] = 0, \end{aligned} \tag{57}$$

completing the proof. □

4. The Itô Integral with respect to a Sequence of Standard Lévy Processes

In this section, we introduce the Itô integral for $\ell^2(H)$-valued processes with respect to a sequence of standard Lévy processes, which is based on the Itô integral (31) from the previous section. We define the space of sequences

$$\ell^2(H) := \left\{ \left(h^j\right)_{j\in\mathbb{N}} \subset H : \sum_{j=1}^\infty \left\|h^j\right\|_H^2 < \infty \right\}, \tag{58}$$

which, equipped with the inner product

$$\langle h, g\rangle_{\ell^2(H)} = \sum_{j=1}^{\infty}\left\langle h^j, g^j\right\rangle_H, \tag{59}$$

is a separable Hilbert space.

Definition 15. A sequence $(M^j)_{j\in\mathbb{N}}$ of real-valued Lévy processes is called a sequence of standard Lévy processes if it consists of square-integrable martingales with $\langle M^j, M^k\rangle_t = \delta_{jk}\cdot t$ for all $j,k\in\mathbb{N}$. Here, δ_{jk} denotes the Kronecker delta

$$\delta_{jk} = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j \neq k. \end{cases} \tag{60}$$

For the rest of this section, let $(M^j)_{j\in\mathbb{N}}$ be a sequence of standard Lévy processes.

Proposition 16. *For every $\ell^2(H)$-valued, predictable process X with*

$$\mathbb{E}\left[\int_0^T \|X_s\|^2_{\ell^2(H)} ds\right] < \infty, \tag{61}$$

the series

$$\sum_{j\in\mathbb{N}} X^j \cdot M^j \tag{62}$$

converges unconditionally in $M_T^2(H)$.

Proof. For $j,k\in\mathbb{N}$ with $j\neq k$, we have $\langle M^j, M^k\rangle = 0$, and, hence, by Proposition 14, we obtain

$$\left\langle X^j\cdot M^j, X^k\cdot M^k\right\rangle_{M_T^2(H)} = 0. \tag{63}$$

Moreover, by the Itô isometry (Proposition 8) and the monotone convergence theorem, we have

$$\begin{aligned}\sum_{j=1}^{\infty}\left\|X^j\cdot M^j\right\|^2_{M_T^2(H)} &= \sum_{j=1}^{\infty}\mathbb{E}\left[\left\|\int_0^T X_s^j dM_s^j\right\|_H^2\right] \\ &= \sum_{j=1}^{\infty}\mathbb{E}\left[\int_0^T \left\|X_s^j\right\|_H^2 ds\right] \\ &= \mathbb{E}\left[\int_0^T \sum_{j=1}^{\infty}\left\|X_s^j\right\|_H^2 ds\right] \\ &= \mathbb{E}\left[\int_0^T \|X_s\|^2_{\ell^2(H)} ds\right].\end{aligned} \tag{64}$$

Thus, by (61) and Lemma 4, the series (62) converges unconditionally in $M_T^2(H)$. □

Therefore, for a $\ell^2(H)$-valued, predictable process X satisfying (61) we can define the Itô integral as the series (62).

Remark 17. As the proof of Proposition 16 shows, the components of the Itô integral $\sum_{j\in\mathbb{N}} X^j\cdot M^j$ are pairwise orthogonal elements of the Hilbert space $M_T^2(H)$.

Proposition 18. *For each $\ell^2(H)$-valued, predictable process X satisfying (61), one has the Itô isometry*

$$\mathbb{E}\left[\left\|\sum_{j\in\mathbb{N}}\int_0^T X_s^j dM_s^j\right\|_H^2\right] = \mathbb{E}\left[\int_0^T \|X_s\|^2_{\ell^2(H)} ds\right]. \tag{65}$$

Proof. Using (63), Lemma 4, and identity (64), we obtain

$$\begin{aligned}\mathbb{E}\left[\left\|\sum_{j=1}^{\infty}\int_0^T X_s^j dM_s^j\right\|_H^2\right] &= \left\|\sum_{j=1}^{\infty} X^j\cdot M^j\right\|^2_{M_T^2(H)} \\ &= \sum_{j=1}^{\infty}\left\|X^j\cdot M^j\right\|^2_{M_T^2(H)} \\ &= \mathbb{E}\left[\int_0^T \|X_s\|^2_{\ell^2(H)} ds\right],\end{aligned} \tag{66}$$

completing the proof. □

Proposition 19. *Let X be a $\ell^2(H)$-valued simple process of the form*

$$X = X_0 \mathbb{1}_{\{0\}} + \sum_{i=1}^{n} X_i \mathbb{1}_{(t_i,t_{i+1}]} \tag{67}$$

with $0 = t_1 < \cdots < t_{n+1} = T$ and $\mathcal{F}_{t_i}$-measurable random variables $X_i : \Omega \to \ell^2(H)$ for $i = 0,\ldots,n$. Then, one has

$$X\cdot M = \sum_{i=1}^{n}\sum_{j\in\mathbb{N}} X_i^j\left(\left(M^j\right)^{t_{i+1}} - \left(M^j\right)^{t_i}\right). \tag{68}$$

Proof. For each $j\in\mathbb{N}$, the process X^j is a H-valued simple process having the representation

$$X^j = X_0^j \mathbb{1}_{\{0\}} + \sum_{i=1}^{n} X_i^j \mathbb{1}_{(t_i,t_{i+1}]}. \tag{69}$$

Hence, by Proposition 9, we obtain

$$\begin{aligned}X\cdot M &= \sum_{j\in\mathbb{N}} X^j\cdot M^j \\ &= \sum_{j\in\mathbb{N}}\sum_{i=1}^{n} X_i^j\left(\left(M^j\right)^{t_{i+1}} - \left(M^j\right)^{t_i}\right) \\ &= \sum_{i=1}^{n}\sum_{j\in\mathbb{N}} X_i^j\left(\left(M^j\right)^{t_{i+1}} - \left(M^j\right)^{t_i}\right),\end{aligned} \tag{70}$$

which finishes the proof. □

5. Lévy Processes in Hilbert Spaces

In this section, we provide the required results about Lévy processes in Hilbert spaces. Let U be a separable Hilbert space.

Definition 20. A U-valued càdlàg, adapted process L is called a Lévy process if the following conditions are satisfied.

(1) We have $L_0 = 0$.

(2) $L_t - L_s$ is independent of $\mathscr{F}_s$ for all $s \le t$.

(3) We have $L_t - L_s \overset{\mathrm{d}}{=} L_{t-s}$ for all $s \le t$.

Definition 21. A U-valued Lévy process L with $\mathbb{E}[\|L_t\|_U^2] < \infty$ and $\mathbb{E}[L_t] = 0$ for all $t \ge 0$ is called a square-integrable Lévy martingale.

Note that any square-integrable Lévy martingale L is indeed a martingale; that is,

$$\mathbb{E}\left[X_t \mid \mathscr{F}_s\right] = X_s \quad \forall s \le \mathrm{t} \tag{71}$$

see [5, Proposition 3.25]. According to [5, Theorem 4.44], for each square-integrable Lévy martingale L, there exists a unique self-adjoint, nonnegative definite trace class operator $Q \in L(U)$, called the *covariance operator* of L, such that for all $t, s \in \mathbb{R}_+$ and $u_1, u_2 \in U$, we have

$$\mathbb{E}\left[\langle L_t, u_1\rangle_U \langle L_s, u_2\rangle_U\right] = (t \wedge s)\left\langle Q u_1, u_2\right\rangle_U. \tag{72}$$

Moreover, for all $u_1, u_2 \in U$, the angle bracket process is given by

$$\langle\langle L, u_1\rangle_U, \langle L, u_2\rangle_U\rangle_t = t\langle Q u_1, u_2\rangle_U, \quad t \ge 0; \tag{73}$$

see [5, Theorem 4.49].

Lemma 22. *Let L be a U-valued square-integrable Lévy martingale with covariance operator Q, let V be another separable Hilbert space, and let $\Phi : U \to V$ be an isometric isomorphism. Then, the process $\Phi(L)$ is a V-valued square-integrable Lévy martingale with covariance operator $Q_\Phi := \Phi Q \Phi^{-1}$.*

Proof. The process $\Phi(L)$ is a V-valued càdlàg, adapted process with $\Phi(L_0) = \Phi(0) = 0$. Let $s \le t$ be arbitrary. Then, the random variable $\Phi(L_t) - \Phi(L_s) = \Phi(L_t - L_s)$ is independent of $\mathscr{F}_s$, and we have

$$\Phi\left(L_t\right) - \Phi\left(L_s\right) = \Phi\left(L_t - L_s\right) \overset{\mathrm{d}}{=} \Phi\left(L_{t-s}\right). \tag{74}$$

Moreover, for each $t \in \mathbb{R}_+$, we have

$$\begin{gathered}\mathbb{E}\left[\left\|\Phi\left(L_t\right)\right\|_V^2\right] = \mathbb{E}\left[\left\|L_t\right\|_U^2\right] < \infty, \\ \mathbb{E}\left[\Phi\left(L_t\right)\right] = \Phi\mathbb{E}\left(L_t\right) = 0,\end{gathered} \tag{75}$$

showing that $\Phi(L)$ is a V-valued square-integrable Lévy martingale.

Let $t, s \in \mathbb{R}_+$ and $v_i \in V$, $i = 1, 2$ be arbitrary, and set $u_i := \Phi^{-1} v_i \in U$, $i = 1, 2$. Then, we have

$$\begin{aligned}&\mathbb{E}\left[\langle\Phi(L_t), v_1\rangle_V \langle\Phi(L_s), v_2\rangle_V\right] \\ &\quad = \mathbb{E}\left[\langle\Phi(L_t), \Phi(u_1)\rangle_V \langle\Phi(L_s), \Phi(u_2)\rangle_V\right] \\ &\quad = \mathbb{E}\left[\langle L_t, u_1\rangle_U \langle L_s, u_2\rangle_U\right] = (t \wedge s)\left\langle Q u_1, \mathrm{u}_2\right\rangle_U \\ &\quad = (t \wedge s)\left\langle Q\Phi^{-1} v_1, \Phi^{-1} v_2\right\rangle_U \\ &\quad = (t \wedge s)\left\langle \Phi Q \Phi^{-1} v_1, v_2\right\rangle_V = (t \wedge s)\left\langle Q_\Phi v_1, v_2\right\rangle_V,\end{aligned} \tag{76}$$

showing that the Lévy martingale $\Phi(L)$ has the covariance operator Q_Φ. □

Now, let $Q \in L(U)$ be a self-adjoint, positive definite trace class operator. Then, there exists a sequence $(\lambda_j)_{j\in\mathbb{N}} \subset (0, \infty)$ with $\sum_{j=1}^{\infty} \lambda_j < \infty$ and an orthonormal basis $(e_j^{(\lambda)})_{j\in\mathbb{N}}$ of U such that

$$Q e_j^{(\lambda)} = \lambda_j e_j^{(\lambda)} \quad \forall j \in \mathbb{N}. \tag{77}$$

We define the sequence of pairwise orthogonal vectors $(e_j)_{j\in\mathbb{N}}$ as

$$e_j := \sqrt{\lambda_j} e_j^{(\lambda)}, \quad j \in \mathbb{N}. \tag{78}$$

Proposition 23. *Let L be a U-valued square-integrable Lévy martingale with covariance operator Q. Then, the sequence $(M^j)_{j\in\mathbb{N}}$ given by*

$$M^j := \frac{1}{\sqrt{\lambda_j}}\left\langle L, e_j^{(\lambda)}\right\rangle_U, \quad j \in \mathbb{N}, \tag{79}$$

is a sequence of standard Lévy processes.

Proof. For each $j \in \mathbb{N}$, the process M^j is a real-valued square-integrable Lévy martingale. By (73), for all $j, k \in \mathbb{N}$, we obtain

$$\begin{aligned}\left\langle M^j, M^k\right\rangle_t &= \frac{1}{\sqrt{\lambda_j \lambda_k}}\left\langle\left\langle L, e_j^{(\lambda)}\right\rangle_U, \left\langle L, e_k^{(\lambda)}\right\rangle_U\right\rangle_t \\ &= \frac{t\left\langle Q e_j^{(\lambda)}, e_k^{(\lambda)}\right\rangle_U}{\sqrt{\lambda_j \lambda_k}} = \frac{t\lambda_j\left\langle e_j^{(\lambda)}, e_k^{(\lambda)}\right\rangle_U}{\sqrt{\lambda_j \lambda_k}} = \delta_{jk} \cdot t,\end{aligned} \tag{80}$$

showing that $(M^j)_{j\in\mathbb{N}}$ is a sequence of standard Lévy processes. □

6. The Itô Integral with respect to an ℓ_λ^2-Valued Lévy Process

In this section, we introduce the Itô integral for $\ell^2(H)$-valued processes with respect to an ℓ_λ^2-valued Lévy process, which is based on the Itô integral (62) from Section 4.

Let $(\lambda_j)_{j\in\mathbb{N}} \subset (0,\infty)$ be a sequence with $\sum_{j=1}^{\infty}\lambda_j < \infty$ and denote by ℓ_λ^2 the weighted space of sequences

$$\ell_\lambda^2 := \left\{ \left(v^j\right)_{j\in\mathbb{N}} \subset \mathbb{R} : \sum_{j=1}^{\infty} \lambda_j \left|v^j\right|^2 < \infty \right\}, \tag{81}$$

which, equipped with the inner product

$$\langle v, w\rangle_{\ell_\lambda^2} = \sum_{j=1}^{\infty} \lambda_j v^j w^j, \tag{82}$$

is a separable Hilbert space. Note that we have the strict inclusion $\ell^2 \subsetneq \ell_\lambda^2$, where ℓ^2 denotes the space of sequences

$$\ell^2 = \left\{ \left(v^j\right)_{j\in\mathbb{N}} \subset \mathbb{R} : \sum_{j=1}^{\infty} \left|v^j\right|^2 < \infty \right\}. \tag{83}$$

We denote by $(g_j)_{j\in\mathbb{N}}$ the standard orthonormal basis of ℓ^2, which is given by

$$g_1 = (1,0,\ldots), \quad g_2 = (0,1,0,\ldots), \ldots. \tag{84}$$

Then, the system $(g_j^{(\lambda)})_{j\in\mathbb{N}}$ defined as

$$g_j^{(\lambda)} := \frac{g_j}{\sqrt{\lambda_j}}, \quad j \in \mathbb{N}, \tag{85}$$

is an orthonormal basis of ℓ_λ^2. Let $Q \in L(\ell_\lambda^2)$ be a linear operator such that

$$Q g_j^{(\lambda)} = \lambda_j g_j^{(\lambda)}, \quad \forall j \in \mathbb{N}. \tag{86}$$

Then, Q is a nuclear, self-adjoint, positive definite operator. Let L be an ℓ_λ^2-valued, square-integrable Lévy martingale with covariance operator Q. According to Proposition 23, the sequence $(M^j)_{j\in\mathbb{N}}$ given by

$$M^j := \frac{1}{\sqrt{\lambda_j}} \left\langle L, g_j^{(\lambda)} \right\rangle_{\ell_\lambda^2}, \quad j \in \mathbb{N}, \tag{87}$$

is a sequence of standard Lévy processes.

Definition 24. For every $\ell^2(H)$-valued, predictable process X satisfying (61), we define the Itô integral $X \cdot L := (\int_0^t X_s dL_s)_{t\in[0,T]}$ as

$$X \cdot L := \sum_{j\in\mathbb{N}} X^j \cdot M^j. \tag{88}$$

Remark 25. Note that $L_2^0(H) \cong \ell^2(H)$, where $L_2^0(H)$ denotes the space of Hilbert-Schmidt operators from ℓ^2 to H. In [21], the Itô integral for $L_2^0(H)$-valued processes with respect to an ℓ_λ^2-valued Wiener process has been constructed in the usual fashion (first for elementary and afterwards for general processes), and then the series representation (88) has been proven; see [21, Proposition 2.2.1].

Now, let $(\mu_k)_{k\in\mathbb{N}}$ be another sequence with $\sum_{k=1}^{\infty}\mu_k < \infty$, and let $\Phi : \ell_\lambda^2 \to \ell_\mu^2$ be an isometric isomorphism such that

$$Q_\Phi g_k^{(\mu)} = \mu_k g_k^{(\mu)}, \quad \forall k \in \mathbb{N}. \tag{89}$$

By Lemma 22, the process $\Phi(L)$ is a ℓ_μ^2-valued, square integrable Lévy martingale with covariance operator Q_Φ, and by Proposition 23, the sequence $(N^k)_{k\in\mathbb{N}}$ given by

$$N^k := \frac{1}{\sqrt{\mu_k}} \left\langle \Phi(L), g_k^{(\mu)} \right\rangle_{\ell_\mu^2}, \quad k \in \mathbb{N}, \tag{90}$$

is a sequence of standard Lévy processes.

Theorem 26. *Let $\Psi \in L(\ell^2(H))$ be an isometric isomorphism such that*

$$\langle h, \Psi(w)\rangle_H = \Phi\left(\langle h, w\rangle_H\right) \quad \forall h \in H,\ w \in \ell^2(H). \tag{91}$$

Then, for every $\ell^2(H)$-valued, predictable process X satisfying (61)*, one has*

$$\mathbb{E}\left[\int_0^T \|\Psi(X_s)\|_{\ell^2(H)}^2 ds\right] < \infty \tag{92}$$

and the identity

$$X \cdot L = \Psi(X) \cdot \Phi(L). \tag{93}$$

Proof. Since Ψ is an isometry, by (61), we have

$$\mathbb{E}\left[\int_0^T \|\Psi(X_s)\|_{\ell^2(H)}^2 ds\right] = \mathbb{E}\left[\int_0^T \|X_s\|_{\ell^2(H)}^2 ds\right] < \infty, \tag{94}$$

showing (92). Moreover, by (89), we have

$$\Phi Q \Phi^{-1} g_k^{(\mu)} = Q_\Phi g_k^{(\mu)} = \mu_k g_k^{(\mu)} \quad \forall k \in \mathbb{N}, \tag{95}$$

and, hence, we get

$$Q\left(\Phi^{-1} g_k^{(\mu)}\right) = \mu_k \left(\Phi^{-1} g_k^{(\mu)}\right) \quad \forall k \in \mathbb{N}. \tag{96}$$

By (86) and (96), the vectors $(g_j^{(\lambda)})_{j\in\mathbb{N}}$ and $(\Phi^{-1} g_k^{(\mu)})_{k\in\mathbb{N}}$ are eigenvectors of Q with corresponding eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ and $(\mu_k)_{k\in\mathbb{N}}$. Therefore, and since Φ is an isometry, for $j, k \in \mathbb{N}$ with $\lambda_j \neq \mu_k$, we obtain

$$\left\langle \Phi g_j^{(\lambda)}, g_k^{(\mu)} \right\rangle_{\ell_\mu^2} = \left\langle g_j^{(\lambda)}, \Phi^{-1} g_k^{(\mu)} \right\rangle_{\ell_\lambda^2} = 0. \tag{97}$$

Let $h \in H$ be arbitrary. Then, we have

$$\begin{aligned}
\langle h, X \cdot L \rangle_H &= \left\langle h, \sum_{j=1}^{\infty} X^j \cdot M^j \right\rangle_H \\
&= \sum_{j=1}^{\infty} \left\langle h, X^j \cdot M^j \right\rangle_H \\
&= \sum_{j=1}^{\infty} \left\langle h, X^j \right\rangle_H \cdot M^j \\
&= \sum_{j=1}^{\infty} \frac{1}{\sqrt{\lambda_j}} \left\langle \langle h, X \rangle_H, g_j^{(\lambda)} \right\rangle_{\ell_\lambda^2} \cdot \frac{1}{\sqrt{\lambda_j}} \left\langle L, g_j^{(\lambda)} \right\rangle_{\ell_\lambda^2} \\
&= \sum_{j=1}^{\infty} \frac{1}{\sqrt{\lambda_j}} \left\langle \Phi(\langle h, X \rangle_H), \Phi g_j^{(\lambda)} \right\rangle_{\ell_\mu^2} \\
&\quad \cdot \frac{1}{\sqrt{\lambda_j}} \left\langle \Phi(L), \Phi g_j^{(\lambda)} \right\rangle_{\ell_\mu^2} \\
&= \sum_{j=1}^{\infty} \frac{1}{\sqrt{\lambda_j}} \left\langle \Phi(\langle h, X \rangle_H), \Phi g_j^{(\lambda)} \right\rangle_{\ell_\mu^2} \\
&\quad \cdot \frac{1}{\sqrt{\lambda_j}} \left(\sum_{k=1}^{\infty} \left\langle \Phi(L), g_k^{(\mu)} \right\rangle_{\ell_\mu^2} \left\langle g_k^{(\mu)}, \Phi g_j^{(\lambda)} \right\rangle_{\ell_\mu^2} \right).
\end{aligned} \tag{98}$$

Since $(\lambda_j)_{j\in\mathbb{N}}$ and $(\mu_k)_{k\in\mathbb{N}}$ are eigenvalues of Q, for each $j \in \mathbb{N}$, there are only finitely many $k \in \mathbb{N}$ such that $\lambda_j = \mu_k$. Therefore, by (97), and since $(\Phi(g_j^{(\lambda)}))_{j\in\mathbb{N}}$ is an orthonormal basis of ℓ_μ^2, we obtain

$$\begin{aligned}
&\langle h, X \cdot L \rangle_H \\
&= \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} \frac{1}{\lambda_j} \left\langle \Phi(\langle h, X \rangle_H), \Phi g_j^{(\lambda)} \right\rangle_{\ell_\mu^2} \left\langle \Phi g_j^{(\lambda)}, g_k^{(\mu)} \right\rangle_{\ell_\mu^2} \right) \\
&\quad \cdot \langle \Phi (L), g_k^{(\mu)} \rangle_{\ell_\mu^2} \\
&= \sum_{k=1}^{\infty} \frac{1}{\mu_k} \left(\sum_{j=1}^{\infty} \left\langle \Phi(\langle h, X \rangle_H), \Phi g_j^{(\lambda)} \right\rangle_{\ell_\mu^2} \left\langle \Phi g_j^{(\lambda)}, g_k^{(\mu)} \right\rangle_{\ell_\mu^2} \right) \\
&\quad \cdot \left\langle \Phi(L), g_k^{(\mu)} \right\rangle_{\ell_\mu^2} \\
&= \sum_{k=1}^{\infty} \frac{1}{\sqrt{\mu_k}} \left\langle \Phi(\langle h, X \rangle_H), g_k^{(\mu)} \right\rangle_{\ell_\mu^2} \cdot \frac{1}{\sqrt{\mu_k}} \left\langle \Phi(L), g_k^{(\mu)} \right\rangle_{\ell_\mu^2} \\
&= \sum_{k=1}^{\infty} \Phi(\langle h, X \rangle_H)^k \cdot N^k.
\end{aligned} \tag{99}$$

Thus, taking into account (91) gives us

$$\begin{aligned}
\langle h, X \cdot L \rangle_H &= \sum_{k=1}^{\infty} \left\langle h, \Psi(X)^k \right\rangle_H \cdot N^k \\
&= \sum_{k=1}^{\infty} \left\langle h, \Psi(X)^k \cdot N^k \right\rangle_H \\
&= \left\langle h, \sum_{k=1}^{\infty} \Psi(X)^k \cdot N^k \right\rangle_H \\
&= \langle h, \Psi (X) \cdot \Phi (L) \rangle_H.
\end{aligned} \tag{100}$$

Since $h \in H$ was arbitrary, using the separability of H as in the proof of Proposition 5, we arrive at (93). □

Remark 27. From a geometric point of view, Theorem 26 says that the "angle" measured by the Itô integral is preserved under isometries.

7. The Itô Integral with respect to a General Lévy Process

In this section, we define the Itô integral with respect to a general Lévy process, which is based on the Itô integral (88) from the previous section.

Let U be a separable Hilbert space, and let $Q \in L(U)$ be a nuclear, self-adjoint, positive definite linear operator. Then, there exist a sequence $(\lambda_j)_{j\in\mathbb{N}} \subset (0, \infty)$ with $\sum_{j=1}^{\infty} \lambda_j < \infty$ and an orthonormal basis $(e_j^{(\lambda)})_{j\in\mathbb{N}}$ of U such that

$$Qe_j^{(\lambda)} = \lambda_j e_j^{(\lambda)}, \quad \forall j \in \mathbb{N}; \tag{101}$$

namely, λ_j are the eigenvalues of Q, and each $e_j^{(\lambda)}$ is an eigenvector corresponding to λ_j. The space $U_0 := Q^{1/2}(U)$, equipped with the inner product

$$\langle u, v \rangle_{U_0} := \left\langle Q^{-1/2}u, Q^{-1/2}v \right\rangle_U, \tag{102}$$

is another separable Hilbert space, and the sequence $(e_j)_{j\in\mathbb{N}}$ given by

$$e_j = \sqrt{\lambda_j} e_j^{(\lambda)}, \quad j \in \mathbb{N}, \tag{103}$$

is an orthonormal basis of U_0. We denote by $L_2^0(H) := L_2(U_0, H)$ the space of Hilbert-Schmidt operators from U_0 into H, which, endowed with the Hilbert-Schmidt norm

$$\|S\|_{L_2^0(H)} := \left(\sum_{j=1}^{\infty} \left\| Se_j \right\|_H^2 \right)^{1/2}, \quad S \in L_2^0 (H), \tag{104}$$

itself is a separable Hilbert space. We define the isometric isomorphisms

$$\Phi_\lambda : U \longrightarrow \ell_\lambda^2, \qquad \Phi_\lambda e_j^{(\lambda)} := g_j^{(\lambda)} \quad \text{for } j \in \mathbb{N}, \tag{105}$$

$$\begin{aligned}
&\Psi_\lambda : L_2^0 (H) \longrightarrow \ell^2 (H), \\
&\Psi_\lambda (S) := \left(Se_j \right)_{j\in\mathbb{N}} \quad \text{for } S \in L_2^0 (H).
\end{aligned} \tag{106}$$

Recall that $(g_j^{(\lambda)})_{j\in\mathbb{N}}$ denotes the orthonormal basis of ℓ_λ^2, which we have defined in (85). Let L be a U-valued square-integrable Lévy martingale with covariance operator Q.

Lemma 28. *The following statements are true.*

(1) *The process $\Phi_\lambda(L)$ is an ℓ_λ^2-valued square-integrable Lévy martingale with covariance operator Q_{Φ_λ}.*

(2) *One has*

$$Q_{\Phi_\lambda} g_j^{(\lambda)} = \lambda_j g_j^{(\lambda)}, \quad \forall j \in \mathbb{N}. \tag{107}$$

Proof. By Lemma 22, the process $\Phi_\lambda(L)$ is an ℓ_λ^2-valued square-integrable Lévy martingale with covariance operator Q_{Φ_λ}. Furthermore, by (105) and (101), for all $j \in \mathbb{N}$, we obtain

$$\begin{aligned} Q_{\Phi_\lambda} g_j^{(\lambda)} &= \Phi_\lambda Q \Phi_\lambda^{-1} g_j^{(\lambda)} = \Phi_\lambda Q e_j^{(\lambda)} \\ &= \Phi_\lambda \left(\lambda_j e_j^{(\lambda)}\right) = \lambda_j \Phi_\lambda e_j^{(\lambda)} = \lambda_j g_j^{(\lambda)}, \end{aligned} \tag{108}$$

showing (107). □

Now, our idea is to the define the Itô integral for an $L_2^0(H)$-valued, predictable process X with

$$\mathbb{E}\left[\int_0^T \|X_s\|_{L_2^0(H)}^2 ds\right] < \infty \tag{109}$$

by setting

$$X \cdot L := \Psi_\lambda(X) \cdot \Phi_\lambda(L), \tag{110}$$

where the right-hand side of (110) denotes the Itô integral (88) from Definition 24. One might suspect that this definition depends on the choice of the eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ and eigenvectors $(e_j^{(\lambda)})_{j\in\mathbb{N}}$. In order to prove that this is not the case, let $(\mu_k)_{k\in\mathbb{N}} \subset (0,\infty)$ be another sequence with $\sum_{k=1}^\infty \mu_k < \infty$, and let $(f_k^{(\mu)})_{k\in\mathbb{N}}$ be another orthonormal basis of U such that

$$Q f_k^{(\mu)} = \mu_k f_k^{(\mu)}, \quad \forall k \in \mathbb{N}. \tag{111}$$

Then, the sequence $(f_k)_{k\in\mathbb{N}}$ given by

$$f_k = \sqrt{\mu_k} f_k^{(\mu)}, \quad k \in \mathbb{N}, \tag{112}$$

is an orthonormal basis of U_0. Analogous to (105) and (106), we define the isometric isomorphisms

$$\begin{aligned} &\Phi_\mu : U \longrightarrow \ell_\mu^2, \qquad \Phi_\mu f_k^{(\mu)} := g_k^{(\mu)} \quad \text{for } k \in \mathbb{N}, \\ &\Psi_\mu : L_2^0(H) \longrightarrow \ell^2(H), \\ &\Psi_\mu(S) := (S f_k)_{k\in\mathbb{N}} \quad \text{for } S \in L_2^0(H). \end{aligned} \tag{113}$$

Furthermore, we define the isometric isomorphisms

$$\begin{aligned} \Phi &:= \Phi_\mu \circ \Phi_\lambda^{-1} : \ell_\lambda^2 \longrightarrow \ell_\mu^2, \\ \Psi &:= \Psi_\mu \circ \Psi_\lambda^{-1} : \ell^2(H) \longrightarrow \ell^2(H). \end{aligned} \tag{114}$$

The following diagram illustrates the situation:

$$\begin{tikzcd} (\ell_\lambda^2, \ell^2(H)) \arrow[rr, "(\Phi,\Psi)"] & & (\ell_\mu^2, \ell^2(H)) \\ & (U, L_2^0(H)) \arrow[ul, "(\Phi_\lambda,\Psi_\lambda)"] \arrow[ur, "(\Phi_\mu,\Psi_\mu)"'] & \end{tikzcd} \tag{115}$$

In order to show that the Itô integral (110) is well defined, we have to show that

$$\Psi_\lambda(X) \cdot \Phi_\lambda(L) = \Psi_\mu(X) \cdot \Phi_\mu(L). \tag{116}$$

For this, we prepare the following auxiliary result.

Lemma 29. *For all $h \in H$ and $w \in \ell^2(H)$, one has*

$$\langle h, \Psi(w)\rangle_H = \Phi\left(\langle h, w\rangle_H\right). \tag{117}$$

Proof. By (101) and (111), the vectors $(e_j^{(\lambda)})_{j\in\mathbb{N}}$ and $(f_k^{(\mu)})_{k\in\mathbb{N}}$ are eigenvectors of Q with corresponding eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ and $(\mu_k)_{k\in\mathbb{N}}$. Therefore, for $j,k \in \mathbb{N}$ with $\lambda_j \neq \mu_k$, we have $\langle e_j^{(\lambda)}, f_k^{(\mu)}\rangle_U = 0$. For each $v \in \ell_\lambda^2$, we obtain

$$\begin{aligned} \Phi(v) &= \left(\Phi_\mu \circ \Phi_\lambda^{-1}\right)(v) \\ &= \Phi_\mu\left(\sum_{k=1}^\infty \left\langle \Phi_\lambda^{-1} v, f_k^{(\mu)}\right\rangle_U f_k^{(\mu)}\right) \\ &= \sum_{k=1}^\infty \left\langle \Phi_\lambda^{-1} v, f_k^{(\mu)}\right\rangle_U g_k^{(\mu)} \\ &= \left(\frac{1}{\sqrt{\mu_k}}\left\langle \Phi_\lambda^{-1} v, f_k^{(\mu)}\right\rangle_U\right)_{k\in\mathbb{N}} \\ &= \left(\frac{1}{\sqrt{\mu_k}}\sum_{j=1}^\infty \left\langle \Phi_\lambda^{-1} v, e_j^{(\lambda)}\right\rangle_U \left\langle e_j^{(\lambda)}, f_k^{(\mu)}\right\rangle_U\right)_{k\in\mathbb{N}} \\ &= \left(\frac{1}{\sqrt{\mu_k}}\sum_{j=1}^\infty \left\langle v, g_j^{(\lambda)}\right\rangle_{\ell_\lambda^2} \left\langle e_j^{(\lambda)}, f_k^{(\mu)}\right\rangle_U\right)_{k\in\mathbb{N}} \\ &= \left(\sum_{j=1}^\infty \frac{\sqrt{\lambda_j}}{\sqrt{\mu_k}} \left\langle e_j^{(\lambda)}, f_k^{(\mu)}\right\rangle_U v^j\right)_{k\in\mathbb{N}} \\ &= \left(\sum_{j=1}^\infty \left\langle e_j^{(\lambda)}, f_k^{(\mu)}\right\rangle_U v^j\right)_{k\in\mathbb{N}}. \end{aligned} \tag{118}$$

Let $w \in \ell^2(H)$ be arbitrary. By (106), we have

$$w = \Psi_\lambda\left(\Psi_\lambda^{-1}(w)\right) = \left(\Psi_\lambda^{-1}(w) e_j\right)_{j\in\mathbb{N}}, \tag{119}$$

and, hence,

$$\begin{aligned}
\Psi(w) &= \left(\Psi_\mu \circ \Psi_\lambda^{-1}\right)(w) \\
&= \left(\Psi_\lambda^{-1}(w) f_k\right)_{k\in\mathbb{N}} \\
&= \left(\Psi_\lambda^{-1}(w)\left(\sum_{j=1}^{\infty}\left\langle f_k, e_j\right\rangle_{U_0} e_j\right)\right)_{k\in\mathbb{N}} \\
&= \left(\sum_{j=1}^{\infty}\left\langle f_k, e_j\right\rangle_{U_0} \Psi_\lambda^{-1}(w) e_j\right)_{k\in\mathbb{N}} \\
&= \left(\sum_{j=1}^{\infty}\left\langle f_k, e_j\right\rangle_{U_0} w^j\right)_{k\in\mathbb{N}} \\
&= \left(\sum_{j=1}^{\infty}\left\langle e_j^{(\lambda)}, f_k^{(\mu)}\right\rangle_{U} w^j\right)_{k\in\mathbb{N}}.
\end{aligned} \tag{120}$$

Therefore, for all $h \in H$ and $w \in \ell^2(H)$, we obtain

$$\begin{aligned}
\langle h, \Psi(w)\rangle_H &= \left(\sum_{j=1}^{\infty}\left\langle e_j^{(\lambda)}, f_k^{(\mu)}\right\rangle_{U}\left\langle h, w^j\right\rangle_H\right)_{k\in\mathbb{N}} \\
&= \Phi\left(\langle h, w\rangle_H\right),
\end{aligned} \tag{121}$$

finishing the proof. □

Proposition 30. *The following statements are true.*

(1) $\Phi_\lambda(L)$ *is an* ℓ_λ^2*-valued Lévy process with covariance operator* Q_{Φ_λ}*, and one has*

$$Q_{\Phi_\lambda} g_j^{(\lambda)} = \lambda_j g_j^{(\lambda)}, \quad \forall j \in \mathbb{N}. \tag{122}$$

(2) $\Phi_\mu(L)$ *is an* ℓ_μ^2*-valued Lévy process with covariance operator* Q_{Φ_μ}*, and one has*

$$Q_{\Phi_\mu} g_k^{(\mu)} = \mu_k g_k^{(\mu)} \quad \forall k \in \mathbb{N}. \tag{123}$$

(3) *For every* $L_2^0(H)$*-valued, predictable process* X *with* (109)*, one has*

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \left\|\Psi_\lambda(X_s)\right\|_{\ell^2(H)}^2 ds\right] < \infty, \\
&\mathbb{E}\left[\int_0^T \left\|\Psi_\mu(X_s)\right\|_{\ell^2(H)}^2 ds\right] < \infty
\end{aligned} \tag{124}$$

and the identity (116)*.*

Proof. The first two statements follow from Lemma 28. Since Ψ_λ and Ψ_μ are isometries, we obtain

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \left\|\Psi_\lambda(X_s)\right\|_{\ell^2(H)}^2 ds\right] &= \mathbb{E}\left[\int_0^T \left\|X_s\right\|_{L_2^0(H)}^2 ds\right] \\
&= \mathbb{E}\left[\int_0^T \left\|\Psi_\mu(X_s)\right\|_{\ell^2(H)}^2 ds\right],
\end{aligned} \tag{125}$$

which, together with (109), yields (124). Now, Theorem 26 applies by virtue of Lemma 29 and yields

$$\begin{aligned}
\Psi_\lambda(X)\cdot\Phi_\lambda(L) &= \Psi\left(\Psi_\lambda(X)\right)\cdot\Phi\left(\Phi_\lambda(L)\right) \\
&= \Psi_\mu(X)\cdot\Phi_\mu(L),
\end{aligned} \tag{126}$$

proving (116). □

Definition 31. For every $L_2^0(H)$-valued process X satisfying (109), we define the Itô-Integral $X \cdot L = (\int_0^t X_s dL_s)_{t\in[0,T]}$ by (110).

By virtue of Proposition 30, Definition (110) of the Itô integral neither depends on the choice of the eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ nor on the eigenvectors $(e_j^{(\lambda)})_{j\in\mathbb{N}}$.

Now, we will the prove the announced series representation of the Itô integral. According to Proposition 23, the sequences $(M^j)_{j\in\mathbb{N}}$ and $(N^j)_{j\in\mathbb{N}}$ of real-valued processes given by

$$M^j := \frac{1}{\sqrt{\lambda_j}}\left\langle L, e_j^{(\lambda)}\right\rangle_U, \qquad N^j := \frac{1}{\sqrt{\lambda_j}}\left\langle \Phi_\lambda(L), g_j^{(\lambda)}\right\rangle_{\ell_\lambda^2} \tag{127}$$

are sequences of standard Lévy processes.

Proposition 32. *For every* $L_2^0(H)$*-valued, predictable process* X *satisfying* (109)*, the process* $(\xi^j)_{j\in\mathbb{N}}$ *given by*

$$\xi^j := X e_j, \quad j \in \mathbb{N}, \tag{128}$$

is a $\ell^2(H)$*-valued, predictable process, and, one has*

$$X \cdot L = \sum_{j\in\mathbb{N}} \xi^j \cdot M^j, \tag{129}$$

where the right-hand side of (129) *converges unconditionally in* $M_T^2(H)$*.*

Proof. Since Φ_λ is an isometry, for each $j \in \mathbb{N}$, we obtain

$$\begin{aligned}
M^j &= \frac{1}{\sqrt{\lambda_j}}\left\langle L, e_j^{(\lambda)}\right\rangle_U = \frac{1}{\sqrt{\lambda_j}}\left\langle \Phi_\lambda(L), \Phi_\lambda e_j^{(\lambda)}\right\rangle_{\ell_\lambda^2} \\
&= \frac{1}{\sqrt{\lambda_j}}\left\langle \Phi_\lambda(L), g_j^{(\lambda)}\right\rangle_{\ell_\lambda^2} = N^j.
\end{aligned} \tag{130}$$

Thus, by Definitions 31 and 24, we obtain

$$\begin{aligned}
X \cdot L &= \Psi_\lambda(X)\cdot\Phi_\lambda(L) \\
&= \sum_{j\in\mathbb{N}} \Psi_\lambda(X)^j \cdot N^j \\
&= \sum_{j\in\mathbb{N}} \xi^j \cdot M^j,
\end{aligned} \tag{131}$$

and, by Proposition 16, the series converges unconditionally in $M_T^2(H)$. □

Remark 33. By Remark 17 and the proof of Proposition 32, the components of the Itô integral $\sum_{j\in\mathbb{N}} \xi^j \cdot M^j$ are pairwise orthogonal elements of the Hilbert space $M_T^2(H)$.

Proposition 34. *For every $L_2^0(H)$-valued process X satisfying* (109)*, one has the Itô isometry*

$$\mathbb{E}\left[\left\|\int_0^T X_s dL_s\right\|_H^2\right] = \mathbb{E}\left[\int_0^T \|L_s\|_{L_2^0(H)}^2 ds\right]. \quad (132)$$

Proof. By the Itô isometry (Proposition 18), and since Ψ_λ is an isometry, we obtain

$$\begin{aligned}\mathbb{E}\left[\left\|\int_0^T X_s dL_s\right\|_H^2\right] &= \mathbb{E}\left[\left\|\int_0^T \Psi_\lambda(X_s) d\Phi_\lambda(L)_s\right\|_H^2\right] \\ &= \mathbb{E}\left[\int_0^T \|\Psi_\lambda(X_s)\|_{\ell^2(H)}^2 ds\right] \\ &= \mathbb{E}\left[\int_0^T \|X_s\|_{L_2^0(H)}^2 ds\right],\end{aligned} \quad (133)$$

completing the proof. □

We shall now prove that the stochastic integral, which we have defined so far, coincides with the Itô integral developed in [5]. For this purpose, it suffices to consider elementary processes. Note that for each operator $S \in L(U,H)$, the restriction $S|_{U_0}$ belongs to $L_2^0(H)$, because

$$\begin{aligned}\sum_{j=1}^{\infty} \|Se_j\|_H^2 &\le \sum_{j=1}^{\infty} \|S\|_{L(U,H)}^2 \|e_j\|_U^2 \\ &= \|S\|_{L(U,H)}^2 \sum_{j=1}^{\infty} \left\|\sqrt{\lambda_j} e_j^{(\lambda)}\right\|_{U_0}^2 \\ &= \|S\|_{L(U,H)}^2 \sum_{j=1}^{\infty} \lambda_j < \infty.\end{aligned} \quad (134)$$

Proposition 35. *Let X be a $L(U,H)$-valued simple process of the form*

$$X = X_0 \mathbb{1}_{\{0\}} + \sum_{i=1}^{n} X_i \mathbb{1}_{(t_i, t_{i+1}]} \quad (135)$$

with $0 = t_1 < \cdots < t_{n+1} = T$ and $\mathcal{F}_{t_i}$-measurable random variables $X_i : \Omega \to L(U,H)$ for $i = 0, \ldots, n$. Then, one has

$$X|_{U_0} \cdot L = \sum_{i=1}^{n} X_i \left(L^{t_{i+1}} - L^{t_i}\right). \quad (136)$$

Proof. The process $\Psi_\lambda(X|_{U_0})$ is an $\ell^2(H)$-valued simple process having the representation

$$\Psi_\lambda\left(X|_{U_0}\right) = \Psi_\lambda\left(X_0|_{U_0}\right) \mathbb{1}_{\{0\}} + \sum_{i=1}^{n} \Psi_\lambda\left(X_i|_{U_0}\right) \mathbb{1}_{(t_i, t_{i+1}]}. \quad (137)$$

Thus, by Proposition 19, and since Φ_λ is an isometry, we obtain

$$\begin{aligned}X|_{U_0} \cdot L &= \Psi_\lambda\left(X|_{U_0}\right) \cdot \Phi_\lambda(L) \\ &= \sum_{i=1}^{n} \sum_{j\in\mathbb{N}} \Psi_\lambda\left(X_i|_{U_0}\right)^j \left(\left(N^j\right)^{t_{i+1}} - \left(N^j\right)^{t_i}\right) \\ &= \sum_{i=1}^{n} \sum_{j\in\mathbb{N}} X_i e_j \frac{\left\langle \Phi_\lambda(L^{t_{i+1}} - L^{t_i}), g_j^{(\lambda)} \right\rangle_{\ell_\lambda^2}}{\sqrt{\lambda_j}} \\ &= \sum_{i=1}^{n} \sum_{j\in\mathbb{N}} X_i e_j^{(\lambda)} \left\langle L^{t_{i+1}} - L^{t_i}, \Phi_\lambda^{-1} g_j^{(\lambda)} \right\rangle_U \\ &= \sum_{i=1}^{n} \sum_{j\in\mathbb{N}} X_i e_j^{(\lambda)} \left\langle L^{t_{i+1}} - L^{t_i}, e_j^{(\lambda)} \right\rangle_U \\ &= \sum_{i=1}^{n} X_i \left(\sum_{j\in\mathbb{N}} \left\langle L^{t_{i+1}} - L^{t_i}, e_j^{(\lambda)} \right\rangle_U e_j^{(\lambda)} \right) \\ &= \sum_{i=1}^{n} X_i \left(L^{t_{i+1}} - L^{t_i}\right),\end{aligned} \quad (138)$$

completing the proof. □

Therefore, and since the space of simple processes is dense in the space of all predictable processes satisfying (109); see, for example, [5, Corollary 8.17], the Itô integral (110) coincides with that in [5] for every $L_2^0(H)$-valued, predictable process X satisfying (109). In particular, for a driving Wiener process, it coincides with the Itô integral from [1–3].

By a standard localization argument, we can extend the definition of the Itô integral to all predictable processes X satisfying

$$\mathbb{P}\left(\int_0^T \|X_s\|_{L_2^0(H)}^2 \mathrm{d}s < \infty\right) = 1, \quad \forall T > 0. \quad (139)$$

Since the respective spaces of predictable and adapted, measurable processes are isomorphic (see [24]), proceeding as in [24, Section 3.2], we can further extend the definition of the Itô integral to all adapted, measurable processes X satisfying (139).

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