Hindawi Publishing Corporation
International Journal of Stochastic Analysis
Volume 2013, Article ID 790709, 13 pages
http://dx.doi.org/10.1155/2013/790709

Research Article

Time Reversal of Volterra Processes Driven Stochastic Differential Equations

L. Decreusefond

Institut Telecom-Telecom ParisTech-CNRS LTCI, 75013 Paris, France

Correspondence should be addressed to L. Decreusefond; laurent.decreusefond@telecom-paristech.fr

Received 18 June 2012; Accepted 27 December 2012

Academic Editor: Ciprian A. Tudor

We consider stochastic differential equations driven by some Volterra processes. Under time reversal, these equations are transformed into past-dependent stochastic differential equations driven by a standard Brownian motion. We are then in position to derive existence and uniqueness of solutions of the Volterra driven SDE considered at the beginning.

1. Introduction

Fractional Brownian motion (fBm for short) of Hurst index $H \in [0, 1]$ is the Gaussian process which admits the following representation: for any $t \geq 0$,

$$B^H(t) = \int_0^t K_H(t, s)\, \mathrm{d}B(s), \tag{1}$$

where B is a one-dimensional Brownian motion and K_H is a triangular kernel, that is, $K_H(t, s) = 0$ for $s > t$, the definition of which is given in (46). Fractional Brownian motion is probably the first process which is not a semimartingale and for which it is still interesting to develop a stochastic calculus. That means we want to define a stochastic integral and solve stochastic differential equations driven by such a process. From the very beginning of this program, two approaches do exist. One approach is based on the Hölder continuity or the finite p variation of the fBm sample paths. The other way to proceed relies on the gaussianity of fBm. The former is mainly deterministic and was initiated by Zähle [1], Feyel and de la Pradelle [2], and Russo and Vallois [3, 4]. Then, came the notion of rough paths was introduced by Lyons [5], whose application to fBm relies on the work of Coutin and Qian [6]. These works have been extended in the subsequent works [7–17]. A new way of thinking came with the independent but related works of Feyel, de la Pradelle [18], and Gubinelli [19]. The integral with respect to fBm was shown to exist as the unique process satisfying some characterization (analytic in the case of [18], algebraic in [19]). As a byproduct, this showed that almost all the existing integrals throughout the literature were all the same as they all satisfy these two conditions. Behind each approach, but the last too, is a construction of an integral defined for a regularization of fBm, then the whole work is to show that, under some convenient hypothesis, the approximate integrals converge to a quantity which is called the stochastic integral with respect to fBm. The main tool to prove the convergence is either integration by parts in the sense of fractional deterministic calculus, or enrichment of the fBm by some iterated integrals proved to exist independently or by analytic continuation [20, 21].

In the probabilistic approach [22–30], the idea is also to define an approximate integral and then prove its convergence. It turns out that the key tool is here the integration by parts in the sense of Malliavin calculus.

In dimension greater than one, with the deterministic approach, one knows how to define the stochastic integral and prove existence and uniqueness of fBm-driven SDEs for fBm with Hurst index greater than 1/4. Within the probabilistic framework, one knows how to define a stochastic integral for any value of H but one cannot prove existence and uniqueness of SDEs whatever the value of H. The primary motivation of this work is to circumvent this problem.

In [26, 27], we defined stochastic integrals with respect to fBm as a "damped-Stratonovitch" integral with respect to the underlying standard Brownian motion. This integral is defined as the limit of Riemann-Stratonovitch sums, the convergence of which is proved after an integration by parts in the sense of Malliavin calculus. Unfortunately, this manipulation generates nonadaptiveness: formally the result can be expressed as

$$\int_0^t u(s) \circ \mathrm{d}B^H(s) = \delta\left(\mathscr{K}_t^* u\right) + \operatorname{trace}\left(\mathscr{K}_t^* \nabla u\right), \quad (2)$$

where $\mathscr{K}$ is defined by

$$\mathscr{K} f(t) = \frac{d}{dt} \int_0^t K_H(t,s) f(s) \, \mathrm{d}s \quad (3)$$

and $\mathscr{K}_t^*$ is the adjoint of $\mathscr{K}$ in $\mathscr{L}^2([0,t],\mathbf{R})$. In particular, there exists k such that

$$\mathscr{K}_t^* f(s) = \int_s^t k(t,u) f(u) \, \mathrm{d}u \quad (4)$$

for any $f \in \mathscr{L}^2([0,t],\mathbf{R})$ so that even if u is adapted (with respect to the Brownian filtration), the process $(s \mapsto \mathscr{K}_t^* u(s))$ is anticipative. However, the stochastic integral process $(t \mapsto \int_0^t u(s) \circ \mathrm{d}B^H(s))$ remains adapted; hence, the anticipative aspect is, in some sense, artificial. The motivation of this work is to show that, up to time reversal, we can work with adapted process and Itô integrals. The time-reversal properties of fBm were already studied in [31] in a different context. It was shown there that the time reversal of the solution of an fBm-driven SDE of the form

$$dY(t) = u(Y(t)) \, \mathrm{d}t + \mathrm{d}B^H(t) \quad (5)$$

is still a process of the same form. With a slight adaptation of our method to fBm-driven SDEs with drift, one should recover the main theorem of [31].

In what follows, there is no restriction on the dimension, but we need to assume that any component of B^H is an fBm of Hurst index greater than 1/2. Consider that we want to solve the following equation:

$$X_t = x + \int_0^t \sigma\left(X_s\right) \circ \mathrm{d}B^H(s), \quad 0 \le t \le T, \quad (6)$$

where σ is a deterministic function whose properties will be fixed below. It turns out that it is essential to investigate the more general equations:

$$X_{r,t} = x + \int_r^t \sigma\left(X_{r,s}\right) \circ \mathrm{d}B^H(s), \quad 0 \le r \le t \le T. \quad (\mathrm{A}')$$

The strategy is then as follows. We will first consider the reciprocal problem:

$$Y_{r,t} = x - \int_r^t \sigma\left(Y_{s,t}\right) \circ \mathrm{d}B^H(s), \quad 0 \le r \le t \le T. \quad (\mathrm{B}')$$

The first critical point is that when we consider $\{Z_{r,t} := Y_{t-r,t}, r \in [0,t]\}$, this process solves an adapted, past-dependent, and stochastic differential equation with respect to a standard Brownian motion. Moreover, because K_H is lower-triangular and sufficiently regular, the trace term vanishes in the equation defining Z. We have then reduced the problem to an SDE with coefficients dependent on the past, a problem which can be handled by the usual contraction methods. We do not claim that the results presented are new (for instance, see the brilliant monograph [32] for detailed results obtained via rough paths theory), but it seems interesting to have purely probabilistic methods which show that fBm driven SDEs do have strong solutions which are homeomorphisms. Moreover, the approach given here shows the irreducible difference between the case $H < 1/2$ and $H > 1/2$. The trace term only vanishes in the latter situation, so that such an SDE is merely a usual SDE with past-dependent coefficients. This representation may be fruitful, for instance, to analyze the support and prove the absolute continuity of solutions of (6).

This paper is organized as follows. After some preliminaries on fractional Sobolev spaces, often called Besov-Liouville space, we address, in Section 3, the problem of Malliavin calculus and time reversal. This part is interesting in its own since stochastic calculus of variations is a framework oblivious to time. Constructing such a notion of time is achieved using the notion of resolution of the identity as introduced in [33, 34]. We then introduce the second key ingredient which is the notion of strict causality or quasinilpotence; see [35] for a related application. In Section 4, we show that solving (B') reduces to solve a past-dependent stochastic differential equation with respect to a standard Brownian motion; see (C) below. Then, we prove existence, uniqueness, and some properties of this equation. Technical lemmas are postponed to Section 5.

2. Besov-Liouville Spaces

Let $T > 0$ be fix real number. For a measurable function $f : [0,T] \to \mathbf{R}^n$, we define $\tau_T f$ by

$$\tau_T f(s) = f(T-s) \quad \text{for any } s \in [0,T]. \quad (7)$$

For $t \in [0,T]$, $e_t f$ will represent the restriction of f to $[0,t]$, that is, $e_t f = f \mathbf{1}_{[0,t]}$. For any linear map A, we denote by A_T^*, its adjoint in $\mathscr{L}^2([0,T];\mathbf{R}^n)$. For $\eta \in (0,1]$, the space of η-Hölder continuous functions on $[0,T]$ is equipped with the norm:

$$\|f\|_{\mathrm{Hol}(\eta)} = \sup_{0<s<t<T} \frac{\left|f(t)-f(s)\right|}{|t-s|^\eta} + \|f\|_\infty. \quad (8)$$

Its topological dual is denoted by $\mathrm{Hol}(\eta)^*$. For $f \in \mathscr{L}^1([0,T];\ \mathbf{R}^n;\ \mathrm{d}t)$ (denoted by $\mathscr{L}^1$ for short), the left and right fractional integrals of f are defined by

$$\begin{gathered}\left(I_{0^+}^\gamma f\right)(x) = \frac{1}{\Gamma(\gamma)} \int_0^x f(t)(x-t)^{\gamma-1} \mathrm{d}t, \quad x \ge 0, \\ \left(I_{T^-}^\gamma f\right)(x) = \frac{1}{\Gamma(\gamma)} \int_x^T f(t)(t-x)^{\gamma-1} \mathrm{d}t, \quad x \le T,\end{gathered} \quad (9)$$

where $\gamma > 0$ and $I^0_{0^+} = I^0_{T^-} = \mathrm{Id}$. For any $\gamma \geq 0$, $p, q \geq 1$, any $f \in \mathscr{L}^p$ and $g \in \mathscr{L}^q$ where $p^{-1} + q^{-1} \leq \gamma$, we have

$$\int_0^T f(s) \left(I^\gamma_{0^+} g\right)(s)\, \mathrm{d}s = \int_0^T \left(I^\gamma_{T^-} f\right)(s)\, g(s)\, \mathrm{d}s. \tag{10}$$

The Besov-Liouville space $I^\gamma_{0^+}(\mathscr{L}^p) := \mathscr{I}^+_{\gamma,p}$ is usually equipped with the norm:

$$\left\| I^\gamma_{0^+} f \right\|_{\mathscr{I}^+_{\gamma,p}} = \|f\|_{\mathscr{L}^p}. \tag{11}$$

Analogously, the Besov-Liouville space $I^\gamma_{T^-}(\mathscr{L}^p) := \mathscr{I}^-_{\gamma,p}$ is usually equipped with the norm:

$$\left\| I^{-\gamma}_{T^-} f \right\|_{\mathscr{I}^-_{\gamma,p}} = \|f\|_{\mathscr{L}^p}. \tag{12}$$

We then have the following continuity results (see [2, 36]):

Proposition 1. *Consider the following.*

(i) *If $0 < \gamma < 1$, $1 < p < 1/\gamma$, then $I^\gamma_{0^+}$ is a bounded operator from $\mathscr{L}^p$ into $\mathscr{L}^q$ with $q = p(1 - \gamma p)^{-1}$.*

(ii) *For any $0 < \gamma < 1$ and any $p \geq 1$, $\mathscr{I}^+_{\gamma,p}$ is continuously embedded in $Hol(\gamma - 1/p)$ provided that $\gamma - 1/p > 0$.*

(iii) *For any $0 < \gamma < \beta < 1$, $\mathrm{Hol}(\beta)$ is compactly embedded in $\mathscr{I}_{\gamma,\infty}$.*

(iv) *For $\gamma p < 1$, the spaces $\mathscr{I}^+_{\gamma,p}$ and $\mathscr{I}^-_{\gamma,p}$ are canonically isomorphic. We will thus use the notation $\mathscr{I}_{\gamma,p}$ to denote any of these spaces.*

3. Malliavin Calculus and Time Reversal

Our reference probability space is $\Omega = \mathscr{C}_0([0,T], \mathbf{R}^n)$, the space of $\mathbf{R}^n$-valued, continuous functions, null at time 0. The Cameron-Martin space is denoted by $\mathbf{H}$ and is defined as $\mathbf{H} = I^1_{0^+}(\mathscr{L}^2([0,T]))$. In what follows, the space $\mathscr{L}^2([0,T])$ is identified with its topological dual. We denote by κ the canonical embedding from $\mathbf{H}$ into Ω. The probability measure $\mathbf{P}$ on Ω is such that the canonical map $W : \omega \mapsto (\omega(t),\ t \in [0,T])$ defines a standard n-dimensional Brownian motion. A mapping ϕ from Ω into some separable Hilbert space $\mathfrak{H}$ is called cylindrical if it is of the form $\phi(w) = \sum_{i=1}^d f_i(\langle v_{i,1}, w\rangle, \ldots, \langle v_{i,n}, w\rangle) x_i$, where for each $i, f_i \in \mathscr{C}^\infty_0(\mathbf{R}^n, \mathbf{R})$ and $(v_{i,j},\ j = 1, \ldots, n)$ is a sequence of Ω^*. For such a function we define $\nabla^{\mathrm{W}}\phi$ as

$$\nabla^{\mathrm{W}}\phi(w) = \sum_{i,j=1} \partial_j f_i\left(\langle v_{i,1}, w\rangle, \ldots, \langle v_{i,n}, w\rangle\right) \tilde{v}_{i,j} \otimes x_i, \tag{13}$$

where $\tilde{v}$ is the image of $v \in \Omega^*$ by the map $(I^1_{0^+} \circ \kappa)^*$. From the quasi-invariance of the Wiener measure [37], it follows that ∇^{W} is a closable operator on $L^p(\Omega; \mathfrak{H})$, $p \geq 1$, and we will denote its closure with the same notation. The powers of ∇^{W} are defined by iterating this procedure. For $p > 1$, $k \in \mathbf{N}$, we denote by $\mathbb{D}_{p,k}(\mathfrak{H})$ the completion of $\mathfrak{H}$-valued cylindrical functions under the following norm:

$$\|\phi\|_{p,k} = \sum_{i=0}^k \left\| (\nabla^{\mathrm{W}})^i \phi \right\|_{L^p(\Omega; \mathfrak{H} \otimes \mathscr{L}^p([0,1])^{\otimes i})}. \tag{14}$$

We denote by $\mathbb{L}_{p,1}$ the space $\mathbb{D}_{p,1}(\mathscr{L}^p([0,T];\ \mathbf{R}^n))$. The divergence, denoted as δ^{W}, is the adjoint of ∇^{W}: v belongs to $\mathrm{Dom}_p \delta^{\mathrm{W}}$ whenever, for any cylindrical ϕ,

$$\left| \mathbf{E}\left[\int_0^T v_s \nabla^{\mathrm{W}}_s \phi \mathrm{d}s \right] \right| \leq c \|\phi\|_{L^p} \tag{15}$$

and, for such a process v,

$$\mathbf{E}\left[\int_0^T v_s \nabla^{\mathrm{W}}_s \phi \mathrm{d}s \right] = \mathbf{E}\left[\phi\ \delta^{\mathrm{W}} v \right]. \tag{16}$$

We introduced the temporary notation W for standard Brownian motion to clarify the forthcoming distinction between a standard Brownian motion and its time reversal. Actually, the time reversal of a standard Brownian is also a standard Brownian motion, and thus, both of them "live" in the same Wiener space. We now precise how their respective Malliavin gradient and divergence are linked. Consider $B = (B(t),\ t \in [0,T])$ an n-dimensional standard Brownian motion and $\breve{B}^T = (B(T) - B(T-t), t \in [0,T])$ its time reversal. Consider the following map:

$$\begin{aligned} \Theta_T : \Omega &\longrightarrow \Omega \\ \omega &\longmapsto \breve{\omega} = \omega(T) - \tau_T \omega, \end{aligned} \tag{17}$$

and the commutative diagram:

$$\begin{tikzcd} \mathscr{L}^2 \arrow[r, "\tau_T"] \arrow[d, "I^1_0+"'] & \mathscr{L}^2 \arrow[d, "I^1_0+"] \\ \Omega \supset \mathbf{H} \arrow[r, "\Theta_T"'] & \mathbf{H} \subset \Omega \end{tikzcd} \tag{18}$$

Note that $\Theta_T^{-1} = \Theta_T$ since $\omega(0) = 0$. For a function $f \in \mathscr{C}^\infty_b(\mathbf{R}^{nk})$, we define the following:

$$\begin{aligned} &\nabla_r f\left(\omega(t_1), \ldots, \omega(t_k)\right) \\ &\quad = \sum_{j=1}^k \partial_j f\left(\omega(t_1), \ldots, \omega(t_k)\right) \mathbf{1}_{[0,t_j]}(r), \\ &\breve{\nabla}_r f\left(\breve{\omega}(t_1), \ldots, \breve{\omega}(t_k)\right) \\ &\quad = \sum_{j=1}^k \partial_j f\left(\breve{\omega}(t_1), \ldots, \breve{\omega}(t_k)\right) \mathbf{1}_{[0,t_j]}(r). \end{aligned} \tag{19}$$

The operator $\nabla = \nabla^B$ (resp., $\breve{\nabla} = \nabla^{\breve{B}}$) is the Malliavin gradient associated with a standard Brownian motion (resp., its time reversal). Since

$$\begin{aligned} &f\left(\breve{\omega}(t_1), \ldots, \breve{\omega}(t_k)\right) \\ &\quad = f\left(\omega(T) - \omega(T - t_1), \ldots, \omega(T) - \omega(T - t_k)\right), \end{aligned} \tag{20}$$

we can consider $f(\breve{\omega}(t_1),\dots,\breve{\omega}(t_k))$ as a cylindrical function with respect to the standard Brownian motion. As such its gradient is given by

$$\begin{aligned}\nabla_r f(\breve{\omega}(t_1),\dots,\breve{\omega}(t_k)) \\ = \sum_{j=1}^{k}\partial_j f(\breve{\omega}(t_1),\dots,\breve{\omega}(t_k))\,\mathbf{1}_{[T-t_j,T]}(r).\end{aligned} \tag{21}$$

We thus have, for any cylindrical function F,

$$\nabla F\circ\Theta_T(\omega) = \tau_T\breve{\nabla}F(\breve{\omega}). \tag{22}$$

Since $\Theta_T^*\mathbf{P}=\mathbf{P}$ and τ_T is continuous from $\mathscr{L}^p$ into itself for any p, it is then easily shown that the spaces $\mathbb{D}_{p,k}$ and $\breve{\mathbb{D}}_{p,k}$ (with obvious notations) coincide for any p, k and that (22) holds for any element of one of these spaces. Hence we have proved the following theorem.

Theorem 2. *For any $p\ge 1$ and any integer k, the spaces $\mathbb{D}_{p,k}$ and $\breve{\mathbb{D}}_{p,k}$ coincide. For any $F\in\mathbb{D}_{p,k}$ for some p,k,*

$$\nabla(F\circ\Theta_T) = \tau_T\breve{\nabla}(F\circ\Theta_T),\ \mathbf{P}\ a.s. \tag{23}$$

By duality, an analog result follows for divergences.

Theorem 3. *A process u belongs to the domain of δ if and only if $\tau_T u$ belongs to the domain of $\breve{\delta}$, and, then, the following equality holds:*

$$\breve{\delta}(u(\breve{\omega}))(\breve{\omega}) = \delta(\tau_T u(\breve{\omega}))(\omega) = \delta(\tau_T u\circ\Theta_T)(\omega). \tag{24}$$

Proof. For $u\in\mathscr{L}^2$, for cylindrical F, we have on the one hand:

$$\mathbf{E}\left[F(\breve{\omega})\,\breve{\delta}u(\breve{\omega})\right] = \mathbf{E}\left[\left(\breve{\nabla}F(\breve{\omega}),u\right)_{\mathscr{L}^2}\right], \tag{25}$$

and on the other hand,

$$\begin{aligned}\mathbf{E}\left[\left(\breve{\nabla}F(\breve{\omega}),u\right)_{\mathscr{L}^2}\right] &= \mathbf{E}\left[\left(\tau_T\nabla F\circ\Theta_T(\omega),u\right)_{\mathscr{L}^2}\right] \\ &= \mathbf{E}\left[\left(\nabla F\circ\Theta_T(\omega),\tau_T u\right)_{\mathscr{L}^2}\right] \\ &= \mathbf{E}\left[F\circ\Theta_T(\omega)\,\delta(\tau_T u)(\omega)\right] \\ &= \mathbf{E}\left[F(\breve{\omega})\,\delta(\tau_T u)(\omega)\right].\end{aligned} \tag{26}$$

Since this is valid for any cylindrical F, (24) holds for $u\in\mathscr{L}^2$. Now, for u in the domain of divergence (see [37, 38]),

$$\delta u = \sum_i\left((u,h_i)_{\mathscr{L}^2}\delta h_i - (\nabla u, h_i\otimes h_i)_{\mathscr{L}^2\otimes\mathscr{L}^2}\right), \tag{27}$$

where $(h_i,\ i\in\mathbf{N})$ is an orthonormal basis of $\mathscr{L}^2([0,T];\ \mathbf{R}^n)$. Thus, we have

$$\begin{aligned}\breve{\delta}(u(\breve{\omega}))(\breve{\omega}) &= \sum_i\Big((u(\breve{\omega}),h_i)_{\mathscr{L}^2}\breve{\delta}h_i(\breve{\omega}) \\ &\qquad -\left(\breve{\nabla}u(\breve{\omega}),h_i\otimes h_i\right)_{\mathscr{L}^2\otimes\mathscr{L}^2}\Big) \\ &= \sum_i\Big((u(\breve{\omega}),h_i)_{\mathscr{L}^2}\delta(\tau_T h_i)(\omega) \\ &\qquad -(\nabla u(\breve{\omega}),\tau_T h_i\otimes h_i)_{\mathscr{L}^2\otimes\mathscr{L}^2}\Big) \\ &= \sum_i\Big((\tau_T u(\breve{\omega}),\tau_T h_i)_{\mathscr{L}^2}\delta(\tau_T h_i)(\omega) \\ &\qquad -(\nabla\tau_T u(\breve{\omega}),\tau_T h_i\otimes\tau_T h_i)_{\mathscr{L}^2\otimes\mathscr{L}^2}\Big),\end{aligned} \tag{28}$$

where we have taken into account that τ_T is in an involution. Since $(h_i,\ i\in\mathbf{N})$ is an orthonormal basis of $\mathscr{L}^2([0,T];\ \mathbf{R}^n)$, identity (24) is satisfied for any u in the domain of δ. □

3.1. Causality and Quasinilpotence. In anticipative calculus, the notion of trace of an operator plays a crucial role, We refer to [39] for more details on trace.

Definition 4. Let V be a bounded map from $\mathscr{L}^2([0,T];\ \mathbf{R}^n)$ into itself. The map V is said to be trace class, whenever for one CONB $(h_n,\ n\ge 1)$ of $\mathscr{L}^2([0,T];\ \mathbf{R}^n)$,

$$\sum_{n\ge 1}\left|(Vh_n,h_n)_{\mathscr{L}^2}\right|\ \text{ is finite.} \tag{29}$$

Then, the trace of V is defined by

$$\operatorname{trace}(V) = \sum_{n\ge 1}(Vh_n,h_n)_{\mathscr{L}^2}. \tag{30}$$

It is easily shown that the notion of trace does not depend on the choice of the CONB.

Definition 5. A family E of projections $(E_\lambda,\ \lambda\in[0,1])$ in $\mathscr{L}^2([0,T];\ \mathbf{R}^n)$ is called a resolution of the identity if it satisfies the conditions:

(1) $E_0=0$ and $E_1=\mathrm{Id}$

(2) $E_\lambda E_\mu = E_{\lambda\wedge\mu}$

(3) $\lim_{\mu\downarrow\lambda}E_\mu = E_\lambda$ for any $\lambda\in[0,1)$ and $\lim_{\mu\uparrow 1}E_\mu=\mathrm{Id}$.

For instance, the family $E=(e_{\lambda T},\lambda\in[0,1])$ is a resolution of the identity in $\mathscr{L}^2([0,T];\ \mathbf{R}^n)$.

Definition 6. A partition π of $[0,T]$ is a sequence $\{0=t_0<t_1<\cdots<t_n=T\}$. Its mesh is denoted by $|\pi|$ and defined by $|\pi|=\sup_i|t_{i+1}-t_i|$.

The causality plays a crucial role in what follows. The next definition is just the formalization in terms of operator of the intuitive notion of causality.

Definition 7. A continuous map V from $\mathscr{L}^2([0,T];\ \mathbf{R}^n)$ into itself is said to be E causal if and only if the following condition holds:

$$E_\lambda V E_\lambda = E_\lambda V \quad \text{for any } \lambda \in [0,1]. \tag{31}$$

For instance, an operator V in integral form $Vf(t) = \int_0^T V(t,s)f(s)\mathrm{d}s$ is causal if and only if $V(t,s) = 0$ for $s \geq t$, that is, computing $Vf(t)$ needs only the knowledge of f up to time t and not after. Unfortunately, this notion of causality is insufficient for our purpose, and we are led to introduce the notion of strict causality as in [40].

Definition 8. Let V be a causal operator. It is a strictly causal operator, whenever for any $\varepsilon > 0$, there exists a partition π of $[0,T]$ such that, for any $\pi' = \{0 = t_0 < t_1 < \cdots < t_n = T\} \subset \pi$,

$$\left\|(E_{t_{i+1}} - E_{t_i})V(E_{t_{i+1}} - E_{t_i})\right\|_{\mathscr{L}^2} < \varepsilon, \quad \text{for } i = 0,\ldots,n-1. \tag{32}$$

Note carefully that the identity map is causal but not strictly causal. Indeed, if $V = \mathrm{Id}$, for any $s < t$,

$$\left\|(E_t - E_s)V(E_t - E_s)\right\|_{\mathscr{L}^2} = \left\|E_t - E_s\right\|_{\mathscr{L}^2} = 1 \tag{33}$$

since $E_t - E_s$ is a projection. However, if V is hyper-contractive, we have the following result.

Lemma 9. *Assume the resolution of the identity to be either $E = (e_{\lambda T},\ \lambda \in [0,1])$ or $E = (Id - e_{(1-\lambda)T},\ \lambda \in [0,1])$. If V is an E causal map continuous from $\mathscr{L}^2$ into $\mathscr{L}^p$ for some $p > 2$ then V is strictly E causal.*

Proof. Let π be any partition of $[0,T]$. Assume that $E = (e_{\lambda T}, \lambda \in [0,1])$, and the very same proof works for the other mentioned resolution of the identity. According to Hölder formula, we have for any $0 \leq s < t \leq T$,

$$\begin{aligned}
&\left\|(E_t - E_s)\, V\, (E_t - E_s)\, f\right\|_{\mathscr{L}^2} \\
&\quad = \int_s^t \left|V(f\mathbf{1}_{(s,t]})(u)\right|^2 \mathrm{d}u \\
&\quad \leq (t-s)^{1-2/p} \left\|V(f\mathbf{1}_{(s,t]})\right\|_{\mathscr{L}^{p/2}} \\
&\quad \leq c\ (t-s)^{1-2/p} \|f\|_{\mathscr{L}^2}.
\end{aligned} \tag{34}$$

Then, for any $\varepsilon > 0$, there exists $\eta > 0$ such that $|\pi| < \eta$ implies $\|(E_{t_{i+1}} - E_{t_i})V(E_{t_{i+1}} - E_{t_i})f\|_{\mathscr{L}^2} \leq \varepsilon$ for any $\{0 = t_0 < t_1 < \cdots < t_n = T\} \subset \pi$ and any $i = 0,\ldots,n-1$. □

The importance of strict causality lies in the next theorem we borrow from [40].

Theorem 10. *The set of strictly causal operators coincides with the set of quasinilpotent operators, that is, trace-class operators such that* $\operatorname{trace}(V^n) = 0$ *for any integer $n \geq 1$.*

Moreover, we have the following stability theorem.

Theorem 11. *The set of strictly causal operators is a two-sided ideal in the set of causal operators.*

Definition 12. Let E be a resolution of the identity in $\mathscr{L}^2([0,T];\mathbf{R}^n)$. Consider the filtration $\mathscr{F}^E$ defined as

$$\mathscr{F}_t^E = \sigma\left\{\delta^W\left(E_\lambda h\right), \lambda \leq t, h \in \mathscr{L}^2\right\}. \tag{35}$$

An $\mathscr{L}^2$-valued random variable u is said to be $\mathscr{F}^E$ adapted if, for any $h \in \mathscr{L}^2$, the real valued process $\langle E_\lambda u, h\rangle$ is $\mathscr{F}^E$-adapted. We denote by $\mathbb{D}_{p,k}^E(\mathfrak{H})$ the set of $\mathscr{F}^E$ adapted random variables belonging to $\mathbb{D}_{p,k}(\mathfrak{H})$.

If $E = (e_{\lambda T}, \lambda \in [0,1])$, the notion of $\mathscr{F}^E$ adapted processes coincides with the usual one for the Brownian filtration, and it is well known that a process u is adapted if and only if $\nabla_r^W u(s) = 0$ for $r > s$. This result can be generalized to any resolution of the identity.

Theorem 13 (Proposition 3.1 of [33]). *Let u belongs to $\mathbb{L}_{p,1}$. Then u is $\mathscr{F}^E$ adapted if and only if $\nabla^W u$ is E causal.*

We then have the following key theorem.

Theorem 14. *Assume the resolution of the identity to be $E = (e_{\lambda T}, \lambda \in [0,1])$ either $E = (Id - e_{(1-\lambda)T}, \lambda \in [0,1])$ and that V is an E-strictly causal continuous operator from $\mathscr{L}^2$ into $\mathscr{L}^p$ for some $p > 2$. Let u be an element of $\mathbb{D}_{2,1}^E(\mathscr{L}^2)$. Then, $V\nabla^W u$ is of trace class and we have* $\operatorname{trace}(V\nabla^W u) = 0$.

Proof. Since u is adapted, $\nabla^W u$ is E-causal. According to Theorem 11, $V\nabla^W u$ is strictly causal and the result follows by Theorem 10. □

In what follows, E^0 is the resolution of the identity in the Hilbert space $\mathscr{L}^2$ defined by $e_{\lambda\, T} f = f\mathbf{1}_{[0,\lambda T]}$ and $\check{E}^0$ is the resolution of the identity defined by $\check{e}_{\lambda T} f = f\mathbf{1}_{[(1-\lambda)T,T]}$. The filtrations $\mathscr{F}^{E^0}$ and $\mathscr{F}^{\check{E}^0}$ are defined accordingly. Next lemma is immediate when V is given in the form of $Vf(t) = \int_0^t V(t,s)f(s)\mathrm{d}s$. Unfortunately such a representation as an integral operator is not always available. We give here an algebraic proof to emphasize the importance of causality.

Lemma 15. *Let V be a map from $\mathscr{L}^2([0,T];\ \mathbf{R}^n)$ into itself such that V is E^0-causal. Let V^* be the adjoint of V in $\mathscr{L}^2([0,T];\mathbf{R}^n)$. Then, the map $\tau_T V_T^* \tau_T$ is $\check{E}^0$-causal.*

Proof. This is a purely algebraic lemma once we have noticed that

$$\tau_T e_r = \left(\mathrm{Id} - e_{T-r}\right)\tau_T \quad \text{for any } 0 \leq r \leq T. \tag{36}$$

For, it suffices to write

$$\begin{aligned}
\tau_T e_r f(s) &= f(T-s)\,\mathbf{1}_{[0,r]}(T-s) \\
&= f(T-s)\,\mathbf{1}_{[T-r,T]}(s) \\
&= \left(\mathrm{Id} - e_{T-r}\right)\tau_T f(s), \quad \text{for any } 0 \leq s \leq T.
\end{aligned} \tag{37}$$

We have to show that

$$e_r\tau_T V_T^*\tau_T e_r = e_r\tau_T V_T^*\tau_T \text{ or equivalently} \tag{38}$$

$$e_r\tau_T V\tau_T e_r = \tau_T V\tau_T e_r, \tag{39}$$

since $e_r^* = e_r$ and $\tau_T^* = \tau_T$. Now, (37) yields

$$e_r\tau_T V\tau_T e_r = \tau_T V\tau_T e_r - e_{T-r}V\tau_T e_r. \tag{40}$$

Use (37) again to obtain

$$\begin{aligned} e_{T-r}V\tau_T e_r &= e_{T-r}V\left(\mathrm{Id} - e_{T-r}\right)\tau_T \\ &= \left(e_{T-r}V - e_{T-r}Ve_{T-r}\right)\tau_T = 0, \end{aligned} \tag{41}$$

since V is E-causal. □

3.2. Stratonovitch Integrals. In what follows, η belongs to $(0,1]$ and V is a linear operator. For any $p \geq 2$, we set the following.

Hypothesis 1 (p,η). The linear map V is continuous from $\mathscr{L}^p([0,T];\mathbf{R}^n)$ into the Banach space $\mathrm{Hol}(\eta)$.

Definition 16. Assume that Hypothesis 1 (p,η) holds. The Volterra process associated to V, denoted by W^V, is defined by

$$W^V(t) = \delta^W\left(V\left(\mathbf{1}_{[0,t]}\right)\right), \quad \forall t \in [0,T]. \tag{42}$$

For any subdivision π of $[0,T]$, that is, $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$, of mesh $|\pi|$, we consider the Stratonovitch sums:

$$\begin{aligned} R^\pi(t,u) &= \delta^W\left(\sum_{t_i\in\pi}\frac{1}{\theta_i}\int_{t_i\wedge t}^{t_{i+1}\wedge t} Vu(r)\,\mathrm{d}r\ \mathbf{1}_{[t_i,t_{i+1})}\right) \\ &\quad + \sum_{t_i\in\pi}\frac{1}{\theta_i}\iint_{[t_i\wedge t,t_{i+1}\wedge t]^2} V\left(\nabla_r^W u\right)(s)\,\mathrm{d}s\,\mathrm{d}r. \end{aligned} \tag{43}$$

Definition 17. We say that u is V-Stratonovitch integrable on $[0,t]$ whenever the family $R^\pi(t,u)$, defined in (43), converges in probability as $|\pi|$ goes to 0. In this case the limit will be denoted by $\int_0^t u(s)\circ \mathrm{d}W^V(s)$.

Example 18. The first example is the so-called Lévy fractional Brownian motion of Hurst index $H > 1/2$ defined as

$$\frac{1}{\Gamma(H+1/2)}\int_0^t (t-s)^{H-1/2}\mathrm{d}B_s = \delta\left(I_{T^-}^{H-1/2}\left(\mathbf{1}_{[0,t]}\right)\right). \tag{44}$$

This amounts to say that $V = I_{T^-}^{H-1/2}$. Thus Hypothesis 1 $(p, H - 1/2 - 1/p)$ holds provided that $p(H-1/2) > 1$.

Example 19. The other classical example is the fractional Brownian motion with stationary increments of Hurst index $H > 1/2$, which can be written as

$$\int_0^t K_H(t,s)\,\mathrm{d}B(s), \tag{45}$$

where

$$\begin{aligned} K_H(t,r) &= \frac{(t-r)^{H-(1/2)}}{\Gamma(H+(1/2))}F\left(\frac{1}{2}-H, H-\frac{1}{2}, H+\frac{1}{2}, 1-\frac{t}{r}\right) \\ &\quad \times 1_{[0,t)}(r). \end{aligned} \tag{46}$$

The Gauss hypergeometric function $F(\alpha,\beta,\gamma,z)$ (see [41]) is the analytic continuation on $\mathbb{C}\times\mathbb{C}\times\mathbb{C}\setminus\{-1,-2,\ldots\}\times\{z\in\mathbb{C},\ \mathrm{Arg}|1-z| < \pi\}$ of the power series:

$$\begin{gathered} \sum_{k=0}^{+\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k k!}z^k, \\ (a)_0 = 1, \\ (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\cdots(a+k-1). \end{gathered} \tag{47}$$

We know from [36] that K_H is an isomorphism from $\mathscr{L}^p([0,1])$ onto $\mathscr{I}_{H+1/2,p}^+$ and

$$K_H f = I_{0^+}^1 x^{H-1/2} I_{0^+}^{H-1/2} x^{1/2-H} f. \tag{48}$$

Consider that $\mathscr{K}_H = I_{0^+}^{-1}\circ K_H$. Then it is clear that

$$\int_0^t K_H(t,s)\,\mathrm{d}B(s) = \int_0^t \left(\mathscr{K}_H\right)_T^*\left(\mathbf{1}_{[0,t]}\right)(s)\,\mathrm{d}B(s); \tag{49}$$

hence we are in the framework of Definition 17 provided that we take $V = (\mathscr{K}_H)_T^*$. Hypothesis 1 $(p, H-1/2-1/p)$ is satisfied provided that $p(H-1/2) > 1$.

The next theorem then follows from [26].

Theorem 20. *Assume that Hypothesis 1 (p,η) holds. Assume that u belongs to $\mathbb{L}_{p,1}$. Then u is V-Stratonovitch integrable, and there exists a process which we denote by $D^W u$ such that $D^W u$ belongs to $L^p(\mathbf{P}\otimes ds)$ and*

$$\int_0^T u(s)\circ \mathrm{dW^V}(s) = \delta^W(\mathrm{Vu}) + \int_0^{\mathrm{T}} \mathrm{D}^W\mathrm{u}(s)\,\mathrm{ds}. \tag{50}$$

The so-called "trace-term" satisfies the following estimate:

$$\mathbf{E}\left[\int_0^T \left|D^W u(r)\right|^p \mathrm{d}r\right] \leq c\ T^{p\eta}\|u\|_{\mathbb{L}_{p,1}}^p, \tag{51}$$

for some universal constant c. Moreover, for any $r \leq T$, $e_r u$ is V-Stratonovitch integrable and

$$\begin{aligned} \int_0^r u(s)\circ \mathrm{dW^V}(s) &= \int_0^T (e_r u)(s)\circ \mathrm{dW^V}(s) \\ &= \delta^W(Ve_r u) + \int_0^r D^W u(s)\,\mathrm{d}s, \end{aligned} \tag{52}$$

and we have the maximal inequality:

$$\mathbf{E}\left[\left\| \int_0^{\cdot} u(s) \circ \mathrm{d}W^V(s)\right\|_{\mathrm{Hol}\,(\eta)}^p\right] \le c\,\|u\|_{\mathbb{L}_{p,1}}^p, \tag{53}$$

where c does not depend on u.

The main result of this Section is the following theorem which states that the time reversal of a Stratonovitch integral is an adapted integral with respect to the time-reversed Brownian motion. Due to its length, its proof is postponed to Section 5.1.

Theorem 21. *Assume that Hypothesis 1 (p,η) holds. Let u belong to $\mathbb{L}_{p,1}$ and let $\check{V}_T = \tau_t V \tau_T$. Assume furthermore that V is $\check{E}_0$-causal and that $\check{u} = u \circ \Theta_T^{-1}$ is $\mathcal{F}^{\check{E}_0}$-adapted. Then,*

$$\begin{aligned}&\int_{T-t}^{T-r} \tau_T u(s) \circ \mathrm{d}W^V(s)\\&\quad= \int_r^t \check{V}_T\left(\mathbf{1}_{[r,t]}\check{u}\right)(s)\,\mathrm{d}\check{B}^T(s), \quad 0 \le r \le t \le T,\end{aligned} \tag{54}$$

where the last integral is an Itô integral with respect to the time reversed Brownian motion $\check{B}^T(s) = B(T)-B(T-s) = \Theta_T(B)(s)$.

Remark 22. Note that, at a formal level, we could have an easy proof of this theorem. For instance, consider the Lévy fBm, and a simple computation shows that $\check{V}_T = I_{0^+}^{H-1/2}$ for any T. Thus, we are led to compute trace$(I_{0^+}^{H-1/2}\nabla u)$. If we had sufficient regularity, we could write

$$\operatorname{trace}\left(I_{0^+}^{H-1/2}\nabla u\right) = \int_0^T\int_0^s (s-r)^{H-3/2}\nabla_s u(r)\,\mathrm{d}r\,\mathrm{d}s = 0, \tag{55}$$

since $\nabla_s u(r) = 0$ for $s > r$ for u adapted. Obviously, there are many flaws in these lines of proof. The operator $I_{0^+}^{H-1/2}\nabla u$ is not regular enough for such an expression of the trace to be true. Even more, there is absolutely no reason for $\check{V}_T\nabla u$ to be a kernel operator so we cannot hope such a formula. These are the reasons that we need to work with operators and not with kernels.

4. Volterra-Driven SDEs

Let $\mathfrak{G}$ be the group of homeomorphisms of $\mathbf{R}^n$ equipped with the distance. We introduce a distance d on $\mathfrak{G}$ by

$$d(\varphi,\phi) = \rho(\varphi,\phi) + \rho\left(\varphi^{-1},\phi^{-1}\right), \tag{56}$$

where

$$\rho(\varphi,\phi) = \sum_{N=1}^{\infty} 2^{-N}\frac{\sup_{|x|\le N}|\varphi(x)-\phi(x)|}{1+\sup_{|x|\le N}|\varphi(x)-\phi(x)|}. \tag{57}$$

Then, $\mathfrak{G}$ is a complete topological group. Consider the equations:

$$X_{r,t} = x + \int_r^t \sigma\left(X_{r,s}\right) \circ \mathrm{d}W^V(s), \quad 0 \le r \le t \le T, \tag{A}$$

$$Y_{r,t} = x - \int_r^t \sigma\left(Y_{s,t}\right) \circ \mathrm{d}W^V(s), \quad 0 \le r \le t \le T. \tag{B}$$

As a solution of (A) is to be constructed by "inverting" a solution of (B), we need to add to the definition of a solution of (A) or (B) the requirement of being a flow of homeomorphisms. This is the meaning of the following definition.

Definition 23. By a solution of (A), we mean a measurable map:

$$\begin{aligned}\Omega\times[0,T]\times[0,T] &\longrightarrow \mathfrak{G}\\(\omega,r,t) &\longmapsto \left(x\longmapsto X_{r,t}(\omega,x)\right)\end{aligned} \tag{58}$$

such that the following properties are satisfied.

(1) For any $0 \le r \le t \le T$, for any $x \in \mathbf{R}^n$, $X_{r,t}(\omega,x)$ is $\sigma\{W^V(s), r \le s \le t\}$-measurable.

(2) For any $0 \le r \le T$, for any $x \in \mathbf{R}^n$, the processes $(\omega,t) \mapsto X_{r,t}(\omega,x)$ and $(\omega,t) \mapsto X_{r,t}^{-1}(\omega,x)$ belong to $\mathbb{L}_{p,1}$ for some $p \ge 2$.

(3) For any $0 \le r \le s \le t$, for any $x \in \mathbf{R}^n$, the following identity is satisfied:

$$X_{r,t}(\omega,x) = X_{s,t}\left(\omega, X_{r,s}(\omega,x)\right). \tag{59}$$

(4) Equation (A) is satisfied for any $0 \le r \le t \le T$ $\mathbf{P}$-a.s.

Definition 24. By a solution of (B), we mean a measurable map:

$$\begin{aligned}\Omega\times[0,T]\times[0,T] &\longrightarrow \mathfrak{G}\\(\omega,r,t) &\longmapsto \left(x\longmapsto Y_{r,t}(\omega,x)\right)\end{aligned} \tag{60}$$

such that the following properties are satisfied.

(1) For any $0 \le r \le t \le T$, for any $x \in \mathbf{R}^n$, $Y_{r,t}(\omega,x)$ is $\sigma\{W^V(s), r \le s \le t\}$ measurable.

(2) For any $0 \le r \le T$, for any $x \in \mathbf{R}^n$, the processes $(\omega,r) \mapsto Y_{r,t}(\omega,x)$ and $(\omega,r) \mapsto Y_{r,t}^{-1}(\omega,x)$ belong to $\mathbb{L}_{p,1}$ for some $p \ge 2$.

(3) Equation (B) is satisfied for any $0 \le r \le t \le T$ $\mathbf{P}$-a.s..

(4) For any $0 \le r \le s \le t$, for any $x \in \mathbf{R}^n$, the following identity is satisfied:

$$Y_{r,t}(\omega,x) = Y_{r,s}\left(\omega, Y_{s,t}(\omega,x)\right). \tag{61}$$

At last consider the equation, for any $0 \le r \le t \le T$,

$$Z_{r,t} = x - \int_r^t \check{V}_T\left(\sigma\circ Z_{.,t}\ \mathbf{1}_{[r,t]}\right)(s)\,\mathrm{d}\check{B}^T(s), \tag{C}$$

where B is a standard n-dimensional Brownian motion.

Definition 25. By a solution of (C), we mean a measurable map:

$$\begin{gathered}\Omega \times [0,T] \times [0,T] \longrightarrow \mathfrak{G} \\ (\omega, r, t) \longmapsto \left(x \longmapsto Z_{r,t}(\omega, x)\right)\end{gathered} \tag{62}$$

such that the following properties are satisfied.

(1) For any $0 \le r \le t \le T$, for any $x \in \mathbf{R}^n$, $Z_{r,t}(\omega, x)$ is $\sigma\{\check{B}^T(s),\ s \le r \le t\}$ measurable.

(2) For any $0 \le r \le t \le T$, for any $x \in \mathbf{R}^n$, the processes $(\omega, r) \mapsto Z_{r,t}(\omega, x)$ and $(\omega, r) \mapsto Z_{r,t}^{-1}(\omega, x)$ belong to $\mathbb{L}_{p,1}$ for some $p \ge 2$.

(3) Equation (C) is satisfied for any $0 \le r \le t \le T\mathbf{P}$-a.s..

Theorem 26. *Assume that $\check{V}_T$ is an E^0 causal map continuous from $\mathcal{L}^p$ into $\mathcal{I}_{\alpha,p}$ for $\alpha > 0$ and $p \ge 4$ such that $\alpha p > 1$. Assume that σ is Lipschitz continuous and sublinear; see* (96) *for the definition. Then, there exists a unique solution to* (*C*)*. Let Z denote this solution. For any (r, r'),*

$$\mathbf{E}\left[\left|Z_{r,T} - Z_{r',T}\right|^p\right] \le c\left|r - r'\right|^{p\eta}. \tag{63}$$

Moreover,

$$\begin{gathered}(\omega, r) \longmapsto Z_{r,s}\left(\omega, Z_{s,t}(\omega, x)\right) \in \mathbb{L}_{p,1}, \\ \textit{for any } r \le s \le t \le T.\end{gathered} \tag{64}$$

Since this proof needs several lemmas, we defer it to Section 5.2.

Theorem 27. *Assume that $\check{V}_T$ is an E^0-causal map continuous from $\mathcal{L}^p$ into $\mathcal{I}_{\alpha,p}$ for $\alpha > 0$ and $p \ge 2$ such that $\alpha p > 1$. For fixed T, there exists a bijection between the space of solutions of* (*B*) *on* $[0, T]$ *and the set of solutions of* (*C*).

Proof. Set

$$Z_{r,T}(\check{\omega}, x) = Y_{T-r,T}\left(\Theta_T^{-1}(\check{\omega}), x\right) \tag{65}$$

or equivalently

$$Y_{r,T}(\omega, x) = Z_{T-r,T}\left(\Theta_T(\omega), x\right). \tag{66}$$

According to Theorem 21, Y satisfies (B) if and only if Z satisfies (C). The regularity properties are immediate since $\mathcal{L}^p$ is stable by τ_T. □

The first part of the next result is then immediate.

Corollary 28. *Assume that $\check{V}_T$ is an E^0-causal map continuous from $\mathcal{L}^p$ into $\mathcal{I}_{\alpha,p}$ for $\alpha > 0$ and $p \ge 2$ such that $\alpha p > 1$. Then* (*B*) *has one and only one solution and for any $0 \le r \le s \le t$, for any $x \in \mathbf{R}^n$, the following identity is satisfied:*

$$Y_{r,t}(\omega, x) = Y_{r,s}\left(\omega, Y_{s,t}(\omega, x)\right). \tag{67}$$

Proof. According to Theorems 27 and 26, (B) has at most one solution since (C) has a unique solution. As to the existence, points from (1) to (3) are immediately deduced from the corresponding properties of Z and (66).

According to Theorem 26, $(\omega, r) \mapsto Y_{r,s}(\omega, Y_{s,t}(\omega, x))$ belongs to $\mathbb{L}_{p,1}$; hence, we can apply the substitution formula and

$$\begin{aligned}Y_{r,s}&\left(\omega, Y_{s,t}(\omega, x)\right) \\ &= Y_{s,t}(\omega, x) - \int_r^s \sigma\left(Y_{\tau,s}(\omega, x)\right) \circ \mathrm{d}W^V(\tau)\Big|_{x=Y_{s,t}(\omega,x)} \\ &= x - \int_s^t \sigma\left(Y_{\tau,t}(\omega, x) \circ \mathrm{d}W^V(\tau)\right. \\ &\qquad - \int_r^s \sigma\left(Y_{\tau,s}\left(\omega, Y_{s,t}(\omega, x)\right)\right) \circ \mathrm{d}W^V(\tau).\end{aligned} \tag{68}$$

Set

$$R_{\tau,t} = \begin{cases} Y_{\tau,t}(\omega, x) & \text{for } s \le \tau \le t \\ Y_{\tau,s}\left(\omega, Y_{s,t}(\omega, x)\right) & \text{for } r \le \tau \le s. \end{cases} \tag{69}$$

Then, in view of (68), R appears to be the unique solution (B) and thus $R_{s,t}(\omega, x) = Y_{s,t}(\omega, x)$. Point (4) is thus proved. □

Corollary 29. *For x fixed, the random field $(Y_{r,t}(x), 0 \le r \le t \le T)$ admits a continuous version. Moreover,*

$$\begin{aligned}&\mathbf{E}\left[\left|Y_{r,s}(x) - Y_{r',s'}(x)\right|^p\right] \\ &\quad \le c\left(1 + |x|^p\right)\left(\left|s' - s\right|^{p\eta} + \left|r - r'\right|^{p\eta}\right).\end{aligned} \tag{70}$$

We still denote by Y this continuous version.

Proof. Without loss of generality, assume that $s \le s'$ and remark that $Y_{s,s'(x)}$ thus belongs to $\sigma\{\check{B}_u^T, u \ge s\}$:

$$\begin{aligned}\mathbf{E}&\left[\left|Y_{r,s}(x) - Y_{r',s'}(x)\right|^p\right] \\ &\le c\Big(\mathbf{E}\left[\left|Y_{r,s}(x) - Y_{r',s}(x)\right|^p\right] \\ &\qquad + \mathbf{E}\left[\left|Y_{r',s}(x) - Y_{r',s'}(x)\right|^p\right]\Big) \\ &= c\Big(\mathbf{E}\left[\left|Y_{r,s}(x) - Y_{r',s}(x)\right|^p\right] \\ &\qquad + \mathbf{E}\left[\left|Y_{r',s}(x) - Y_{r',s}\left(Y_{s,s'}(x)\right)\right|^p\right]\Big) \\ &= c\Big(\mathbf{E}\left[\left|Z_{s-r,s}(x) - Z_{s-r',s}(x)\right|^p\right] \\ &\qquad + \mathbf{E}\left[\left|Z_{s-r',s}(x) - Z_{s-r',s}\left(Y_{s,s'}(x)\right)\right|^p\right]\Big).\end{aligned} \tag{71}$$

According to Theorem 37,

$$\mathbf{E}\left[\left|Z_{s-r,s}(x) - Z_{s-r',s}(x)\right|^p\right] \le c\left|r - r'\right|^{p\eta}\left(1 + |x|^p\right). \tag{72}$$

In view of Theorem 21, the stochastic integral which appears in (C) is also a Stratonovitch integral; hence, we can apply the substitution formula and say

$$Z_{s-r',s}\left(Y_{s,s'}(x)\right) = Z_{s-r',s}(y)\big|_{y=Y_{s,s'}(x)}. \tag{73}$$

Thus we can apply Theorem 37 and obtain that

$$\begin{aligned}\mathbf{E}\left[\left|Z_{s-r',s}(x)-Z_{s-r',s}(Y_{s,s'}(x))\right|^p\right]\\ \le c\mathbf{E}\left[\left|x-Y_{s,s'}(x)\right|^p\right].\end{aligned}\tag{74}$$

The right hand side of this equation is in turn equal to $\mathbf{E}[|Z_{0,s'}-Z_{s'-s,s'}(x)|^p]$ thus, we get

$$\begin{aligned}\mathbf{E}\left[\left|Z_{s-r',s}(x)-Z_{s-r',s}\left(Y_{s,s'}(x)\right)\right|^p\right]\\ \le c\left(1+|x|^p\right)\left|s'-s\right|^{p\eta}.\end{aligned}\tag{75}$$

Combining (72) and (75) gives

$$\begin{aligned}\mathbf{E}\left[\left|Y_{r,s}(x)-Y_{r',s'}(x)\right|^p\right]\\ \le c\left(1+|x|^p\right)\left(\left|s'-s\right|^{p\eta}+\left|r-r'\right|^{p\eta}\right),\end{aligned}\tag{76}$$

hence the result comes □

Thus, we have the main result of this paper.

Theorem 30. *Assume that $\check{V}_T$ is an E^0-causal map continuous from $\mathscr{L}^p$ into $\mathscr{I}_{\alpha,p}$ for $\alpha>0$ and $p\ge 4$ such that $\alpha p>1$. Then (A) has one and only one solution.*

Proof. Under the hypothesis, we know that (B) has a unique solution which satisfies (67). By definition a solution of (B), the process $Y^{-1}:(\omega,s)\mapsto Y^{-1}_{st}(\omega,x)$ belongs to $\mathbb{L}_{p,1}$; hence, we can apply the substitution formula. Following the lines of proof of the previous theorem, we see that Y^{-1} is a solution of (A).

In the reverse direction, two distinct solutions of (A) would give rise to two solutions of (B) by the same principles. Since this is definitely impossible in view of Corollary 28 (A) has at most one solution. □

5. Technical Proofs

5.1. Substitution Formula. The proof of Theorem 21 relies on several lemmas including one known in anticipative calculus as the substitution formula, compare [38].

Theorem 31. *Assume that Hypothesis 1 (p,η) holds. Let u belong to $\mathbb{L}_{p,1}$. If $V\nabla^W u$ is of trace class, then*

$$\int_0^T D^W u(s)\,\mathrm{d}s=\operatorname{trace}\left(\mathrm{V}\nabla^W\mathrm{u}\right).\tag{77}$$

Moreover,

$$\mathbf{E}\left[\left|\operatorname{trace}(V\nabla^W u)\right|^p\right]\le c\|u\|^p_{\mathbb{L}_{p,1}}.\tag{78}$$

Proof. For each k, let $(\phi_{k,m},\ m=1,\ldots,2^k)$ be the functions $\phi_{k,m}=2^{k/2}\mathbf{1}_{[(m-1)2^{-k},m2^{-k})}$. Let P_k be the projection onto the span of the $\phi_{k,m}$; since $\nabla^W Vu$ is of trace class, we have (see [42])

$$\operatorname{trace}\left(V\nabla^W p_t u\right)=\lim_{k\to+\infty}\operatorname{trace}\left(P_k V\nabla^W p_t u\ P_k\right).\tag{79}$$

Now,

$$\begin{aligned}&\operatorname{trace}\left(P_k\ V\nabla^W u\ P_k\right)\\ &=\sum_{m=1}^{k}\left(V\nabla^W p_t u,\phi_{k,m}\otimes\phi_{k,m}\right)_{\mathscr{L}^2\otimes\mathscr{L}^2}\\ &=\sum_{m=1}^{k}2^k\int_{(m-1)2^{-k}\wedge t}^{m2^{-k}\wedge t}\int_{(m-1)2^{-k}\wedge t}^{m2^{-k}\wedge t}V\left(\nabla^W_r u\right)(s)\,\mathrm{d}s\,\mathrm{d}r.\end{aligned}\tag{80}$$

According to the proof of Theorem 20, the first part of the theorem follows. The second part is then a rewriting of (51). □

For $p\ge 1$, let Γ_p be the set of random fields:

$$\begin{aligned}u:\mathbf{R}^m&\longrightarrow\mathbb{L}_{p,1}\\ x&\longmapsto((\omega,s)\longmapsto u(\omega,s,x))\end{aligned}\tag{81}$$

equipped with the seminorms,

$$p_K(u)=\sup_{x\in K}\|u(x)\|_{\mathbb{L}_{p,1}}\tag{82}$$

for any compact K of $\mathbf{R}^m$.

Corollary 32 (substitution formula). *Assume that Hypothesis 1 (p,η) holds. Let $\{u(x),x\in\mathbf{R}^m\}$ belong to Γ_p. Let F be a random variable such that $((\omega,s)\mapsto u(\omega,s,F))$ belongs to $\mathbb{L}_{p,1}$. Then,*

$$\int_0^T u(s,F)\circ dW^V(s)=\left.\int_0^T u(s,x)\circ dW^V_s\right|_{x=F}.\tag{83}$$

Proof. Simple random fields of the form:

$$u(\omega,s,x)=\sum_{l=1}^{K}H_l(x)\,u_l(\omega,s)\tag{84}$$

with H_l smooth and u_l in $\mathbb{L}_{p,1}$ are dense in Γ_p. In view of (53), it is sufficient to prove the result for such random fields. By linearity, we can reduce the proof to random fields of the form $H(x)u(\omega,s)$. Now for any partition π,

$$\begin{aligned}&\delta^W\left(\sum_{t_i\in\pi}\frac{1}{\theta_i}\int_{t_i\wedge t}^{t_{i+1}\wedge t}H(F)\,V(u(\omega,.))(r)\,\mathrm{d}r\ \mathbf{1}_{[t_i,t_{i+1})}\right)\\ &=H(F)\,\delta^W\left(\sum_{t_i\in\pi}\frac{1}{\theta_i}\int_{t_i\wedge t}^{t_{i+1}\wedge t}V(u(\omega,.))(r)\,\mathrm{d}r\ \mathbf{1}_{[t_i,t_{i+1})}\right)\\ &\quad-\sum_{t_i\in\pi}\int_{t_i\wedge t}^{t_{i+1}\wedge t}\int_{t_i\wedge t}^{t_{i+1}\wedge t}H'(F)\,\nabla^W_s F\ Vu(r)\,\mathrm{d}s\,\mathrm{d}r.\end{aligned}\tag{85}$$

On the other hand,

$$\nabla^W_s\left(H(F)\,u(\omega,r)\right)=H'(F)\,\nabla^W_s F\ u(r),\tag{86}$$

Hence,

$$\sum_{t_i\in\pi}\frac{1}{\theta_i}\iint_{[t_i\wedge t,t_{i+1}\wedge t]^2} V\left(\nabla_r^{\mathrm{W}} H(F)u\right)(s)\,\mathrm{d}s\,\mathrm{d}r = \sum_{t_i\in\pi}\frac{1}{\theta_i}\iint_{[t_i\wedge t,t_{i+1}\wedge t]^2} H'(F)\nabla_s^{\mathrm{W}} F\, Vu(r)\,\mathrm{d}s\,\mathrm{d}r. \tag{87}$$

According to Theorem 20, (83) is satisfied for simple random fields. □

Definition 33. For any $0\le r\le t\le T$, for u in $\mathbb{L}_{p,1}$, we define $\int_r^t u(s)\circ \mathrm{d}W^V(s)$ as

$$\begin{aligned}\int_r^t u(s)\circ \mathrm{d}W^V(s) &= \int_0^t u(s)\circ \mathrm{d}W^V(s) - \int_0^r u(s)\circ \mathrm{d}W^V(s)\\ &= \int_0^T e_t u(s)\,\mathrm{d}W^V(s) - \int_0^T e_r u(s)\circ \mathrm{d}W^V(s)\\ &= \delta^{\mathrm{W}}\left(V(e_t-e_r)u\right) + \int_r^t D^{\mathrm{W}} u(s)\,\mathrm{d}s.\end{aligned} \tag{88}$$

By the very definition of trace class operators, the next lemma is straightforward.

Lemma 34. *Let A and B be two continuous maps from $\mathscr{L}^2([0,T];\ \mathbf{R}^n)$ into itself. Then, the map $\tau_T A\otimes B$ (resp. $A\tau_T\otimes B$) is of trace class if and only if the map $A\otimes\tau_T B$ (resp. $A\otimes B\tau_T$) is of trace class. Moreover, in such a situation,*

$$\begin{aligned}&\operatorname{trace}\left(\tau_T A\otimes B\right)\\ &\quad = \operatorname{trace}\left(A\otimes\tau_T B\right), \textit{resp.} \operatorname{trace}\left(A\tau_T\otimes B\right)\\ &\quad = \operatorname{trace}\left(A\otimes B\tau_T\right).\end{aligned} \tag{89}$$

The next corollary follows by a classical density argument.

Corollary 35. *Let $u\in\mathbb{L}_{2,1}$ such that $\nabla^W\otimes\tau_T Vu$ and $\nabla^W\otimes V\tau_T u$ are of trace class. Then, $\tau_T\nabla^W\otimes Vu$ and $\nabla^W\tau_T\otimes Vu$ are of trace class. Moreover, we have*

$$\begin{aligned}\operatorname{trace}\left(\nabla^W\otimes\tau_T Vu\right) &= \operatorname{trace}\left(\tau_T\nabla^W\otimes Vu\right),\\ \operatorname{trace}\left(\nabla^W\otimes(V\tau_T)u\right) &= \operatorname{trace}\left(\nabla^W\tau_T\otimes Vu\right).\end{aligned} \tag{90}$$

Proof of Theorem 21. We first study the divergence term. In view of Theorem 3 we have

$$\begin{aligned}&\delta^B\left(V\left(e_{T-r}-e_{T-t}\right)\tau_T\check{u}\circ\Theta_T\right)\\ &\quad=\delta^B\left(V\tau_T\left(e_t-e_r\right)\check{u}\circ\Theta_T\right)\\ &\quad=\delta^B\left(\tau_T\check{V}_T\left(e_t-e_r\right)\check{u}\circ\Theta_T\right)\\ &\quad=\check{\delta}\left(\check{V}_T\left(e_t-e_r\right)\check{u}\right)(\check{\omega})\\ &\quad=\int_r^t \check{V}_T\left(\mathbf{1}_{[r,t]}\check{u}\right)(s)\,\mathrm{d}B^T(s).\end{aligned} \tag{91}$$

According to Lemma 15 $(\check{V}_T)^*$ is $\check{E}_0$ causal, and, according to Lemma 9, it is strictly $\check{E}_0$ causal. Thus, Theorem 14 implies that $\check{\nabla}V(e_t-e_r)\check{u}$ is of trace class and quasinilpotent. Hence Corollary 35 induces that

$$\tau_T\check{V}_T\tau_T\otimes\tau_T\check{\nabla}\tau_T\left(e_t-e_r\right)\check{u} \tag{92}$$

is trace class and quasinilpotent. Now, according to Theorem 2, we have

$$\begin{aligned}&\tau_T\check{V}_T\tau_T\otimes\tau_T\check{\nabla}\tau_T\left(e_t-e_r\right)\check{u}\\ &\quad= V\left(\nabla\tau_T\left(e_{T-r}-e_{T-t}\right)\check{u}\circ\Theta_T\right).\end{aligned} \tag{93}$$

According to Theorem 20, we have proved (54). □

5.2. The Forward Equation

Lemma 36. *Assume that Hypothesis 1 (p,η) holds and that σ is Lipschitz continuous. Then, for any $0\le a\le b\le T$, the map*

$$\begin{aligned}\check{V}_T\circ\sigma : C\left([0,T],\mathbf{R}^n\right) &\longrightarrow C\left([0,T],\mathbf{R}^n\right)\\ \phi &\longmapsto \check{V}_T\left(\sigma\circ\psi\ \mathbf{1}_{[a,b]}\right)\end{aligned} \tag{94}$$

is Lipschitz continuous and Gâteaux differentiable. Its differential is given by

$$d\check{V}_T\circ\sigma(\phi)[\psi] = \check{V}_T\left(\sigma'\circ\phi\ \psi\right). \tag{95}$$

Assume furthermore that σ is sublinear; that is,

$$|\sigma(x)|\le c(1+|x|),\quad \textit{for any } x\in\mathbf{R}^n. \tag{96}$$

Then, for any $\psi\in C([0,T],\mathbf{R}^n)$, for any $t\in[0,T]$,

$$\begin{aligned}\left|\check{V}_T(\sigma\circ\psi)(t)\right| &\le cT^{\eta+1/p}\left(1+\int_0^t|\psi(s)|^p\mathrm{d}s\right)\\ &\le cT^{\eta+1/p}\left(1+\|\psi\|_\infty\right).\end{aligned} \tag{97}$$

Proof. Let ψ and ϕ be two continuous functions, since $C([0,T],\mathbf{R}^n)$ is continuously embedded in $\mathscr{L}^p$ and $\check{V}_T(\sigma\circ\psi-\sigma\circ\phi)$ belongs to $\mathrm{Hol}(\eta)$. Moreover,

$$\begin{aligned}&\sup_{t\le T}\left|\check{V}_T\left(\sigma\circ\psi\mathbf{1}_{[a,b]}\right)(t)-\check{V}_T\left(\sigma\circ\phi\ \mathbf{1}_{[a,b]}\right)(t)\right|\\ &\quad\le c\left\|\check{V}_T((\sigma\circ\psi-\sigma\circ\phi)\ \mathbf{1}_{[a,b]})\right\|_{\mathrm{Hol}(\eta)}\\ &\quad\le c\left\|(\sigma\circ\psi-\sigma\circ\phi)\ \mathbf{1}_{[a,b]}\right\|_{\mathscr{L}^p}\\ &\quad\le c\left\|\phi-\psi\right\|_{\mathscr{L}^p([a,b])}\\ &\quad\le c\sup_{t\le T}\left|\psi(t)-\phi(t)\right|,\end{aligned} \tag{98}$$

since σ is Lipschitz continuous.

Let ϕ and ψ be two continuous functions on $[0,T]$. Since σ is Lipschitz continuous, we have

$$\begin{aligned}&\sigma\left(\psi(t)+\varepsilon\phi(t)\right)\\ &\quad=\sigma\left(\psi(t)\right)+\varepsilon\int_0^1\sigma'\left(u\psi(t)+(1-u)\phi(t)\right)\mathrm{d}u.\end{aligned} \tag{99}$$

Moreover, since σ is Lipschitz, σ' is bounded and

$$\int_0^T \left| \int_0^1 \sigma'(u\psi(t) + (1-u)\phi(t))\mathrm{d}u \right|^p \mathrm{d}t \le cT. \tag{100}$$

This means that $(t \mapsto \int_0^1 \sigma'(u\psi(t) + (1-u)\phi(t))\mathrm{d}u)$ belongs to $\mathscr{L}^p$. Hence, according to Hypothesis 1,

$$\left\| \check{V}_T \left(\int_0^1 \sigma' \left(u\psi(.) + (1-u)\phi(.)\right) \mathrm{d} \right) \right\|_C \le cT. \tag{101}$$

Thus,

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \left(\check{V}_T \left(\sigma \circ (\psi + \varepsilon\phi)\right) - \check{V}_T \left(\sigma \circ \psi\right) \right) \text{exists,} \tag{102}$$

and $\check{V}_T \circ \sigma$ is Gâteaux differentiable and its differential is given by (95).

Since $\sigma \circ \psi$ belongs to $C([0,T], \mathbf{R}^n)$, according to Hypothesis 1, we have

$$\begin{aligned} \left| \check{V}_T \left(\sigma \circ \psi\right)(t) \right| &\le c \left(\int_0^t s^{\eta p} |\sigma(\psi(s))|^p \mathrm{d}s \right)^{1/p} \\ &\le cT^\eta \left(\int_0^t (1 + |\psi(s)|^p) \mathrm{d}s \right)^{1/p} \\ &\le cT^{\eta + 1/p} \left(1 + \|\psi\|_\infty^p\right)^{1/p} \\ &\le cT^{\eta + 1/p} \left(1 + \|\psi\|_\infty\right). \end{aligned} \tag{103}$$

The proof is thus complete. □

Following [43], we then have the following nontrivial result.

Theorem 37. *Assume that Hypothesis 1 (p, η) holds and that σ is Lipschitz continuous. Then, there exists one and only one measurable map from $\Omega \times [0,T] \times [0,T]$ into $\mathfrak{G}$ which satisfies the first two points of Definition (C). Moreover,*

$$\begin{aligned} &\mathbf{E}\left[\left| Z_{r,t}(x) - Z_{r',t}(x') \right|^p \right] \\ &\quad \le c \left(1 + |x|^p \vee |x'|^p\right) \\ &\quad \times \left(|r - r'|^{p\eta} + |x - x'|^p \right), \end{aligned} \tag{104}$$

and, for any $x \in \mathbf{R}^n$, for any $0 \le r \le t \le T$, we have

$$\mathbf{E}\left[|Z_{r,t}(x)|^p \right] \le c \left(1 + |x|^p\right) e^{cT^{\eta p + 1}}. \tag{105}$$

Note even if x and x' are replaced by $\sigma\{\check{B}^T(u), t \le u\}$ measurable random variables, the last estimates still hold.

Proof. Existence, uniqueness, and homeomorphy of a solution of (C) follow from [43]. The regularity with respect to r and x is obtained as usual by BDG inequality and Gronwall Lemma. For x or x' random, use the independence of $\sigma\{\check{B}^T(u), t \le u\}$ and $\sigma\{\check{B}^T(u), r \wedge r' \le u \le t\}$. □

Theorem 38. *Assume that Hypothesis 1 (p, η) holds and that σ is Lipschitz continuous and sublinear. Then, for any $x \in \mathbf{R}^n$, for any $0 \le r \le s \le t \le T$, $(\omega, r) \mapsto Z_{r,s}(\omega, Z_{s,t}(x))$ and $(\omega, r) \mapsto Z_{r,t}^{-1}(\omega, x)$ belong to $\mathbb{L}_{p,1}$.*

Proof. According to ([44], Theorem 3.1), the differentiability of $\omega \mapsto Z_{r,t}(\omega, x)$ is ensured. Furthermore,

$$\begin{aligned} \nabla_u Z_{r,t} = &- \check{V}_T \left(\sigma \circ Z_{.,t} \mathbf{1}_{[r,t]}\right)(u) \\ &- \int_r^t \check{V}_T \left(\sigma'\left(Z_{.,t}\right) . \nabla_u Z_{.,t} \mathbf{1}_{[r,t]}\right)(s) \,\mathrm{d}\check{B}(s), \end{aligned} \tag{106}$$

where σ' is the differential of σ. For $M > 0$, let

$$\xi_M = \inf\left\{\tau, \left|\nabla_u Z_{\tau,t}\right|^p \ge M\right\}, \qquad Z_{\tau,t}^M = Z_{\tau \vee \xi_M, t}. \tag{107}$$

Since $\check{V}_T$ is continuous from $\mathscr{L}^p$ into itself and σ is Lipschitz, according to BDG inequality, for $r \le u$,

$$\begin{aligned} &\mathbf{E}\left[\left| \nabla_u Z_{r,t}^M \right|^p \right] \\ &\quad \le c\mathbf{E}\left[\left| \check{V}_T \left(\sigma \circ Z_{.,t}^M \mathbf{1}_{[r,t]}\right)(u) \right|^p \right] \\ &\qquad + c\mathbf{E}\left[\int_r^t \left| \check{V}_T \left(\sigma'\left(Z_{.,t}^M\right) \nabla_u Z_{.,t}^M \mathbf{1}_{[r,t]}\right)(s) \right|^p \mathrm{d}s \right] \\ &\quad \le c \left(1 + \mathbf{E}\left[\int_r^t u^{p\eta} \int_r^u \left|Z_{\tau,t}\right|^p \mathrm{d}\tau \,\mathrm{d}u \right] \right. \\ &\qquad \left. + \mathbf{E}\left[\int_r^t s^{p\eta} \int_r^s \left|\nabla_u Z_{\tau,t}^M\right|^p \mathrm{d}\tau \,\mathrm{d}s \right] \right) \\ &\quad \le c \left(1 + \mathbf{E}\left[\int_r^t \left|Z_{\tau,t}\right|^p \left(t^{p\eta+1} - \tau^{p\eta+1}\right) \mathrm{d}\tau \right] \right. \\ &\qquad \left. + \mathbf{E}\left[\int_r^t \left|\nabla_u Z_{\tau,t}^M\right|^p \left(t^{p\eta+1} - \tau^{p\eta+1}\right) \mathrm{d}\tau \right] \right) \\ &\quad \le ct^{p\eta+1} \left(1 + \mathbf{E}\left[\int_r^t \left|Z_{\tau,t}\right|^p \mathrm{d}\tau \right] + \mathbf{E}\left[\int_r^t \left|\nabla_u Z_{\tau,t}^M\right|^p \mathrm{d}\tau \right] \right). \end{aligned} \tag{108}$$

Then, Gronwall Lemma entails that

$$\mathbf{E}\left[\left| \nabla_u Z_{r,t}^M \right|^p \right] \le c \left(1 + \mathbf{E}\left[\int_r^t \left|Z_{\tau,t}\right|^p \mathrm{d}\tau \right] \right), \tag{109}$$

hence by Fatou lemma,

$$\mathbf{E}\left[\left| \nabla_u Z_{r,t} \right|^p \right] \le c \left(1 + \mathbf{E}\left[\int_r^t \left|Z_{\tau,t}\right|^p \mathrm{d}\tau \right] \right). \tag{110}$$

The integrability of $\mathbf{E}[|\nabla_u Z_{r,t}|^p]$ with respect to u follows.

Now, since $0 \le r \le s \le t \le T$, $Z_{s,t}(x)$ is independent of $Z_{r,s}(x)$, thus the previous computations still hold and $(\omega, r) \mapsto Z_{r,s}(\omega, Z_{s,t}(x))$ belongs to $\mathbb{L}_{p,1}$.

According to [45], to prove that $Z_{r,t}^{-1}(x)$ belongs to $\mathbb{D}_{p,1}$, we need to prove that

(1) for every $h \in \mathscr{L}^2$, there exists an absolutely continuous version of the process $(t \mapsto Z_{r,t}^{-1}(\omega + th, x))$,

(2) there exists $DZ_{r,t}^{-1}$, an $\mathscr{L}^2$-valued random variable such that, for every $h \in \mathscr{L}^2$,

$$\frac{1}{t}\left(Z_{r,t}^{-1}(\omega + th, x) - Z_{r,t}^{-1}(\omega, x)\right) \xrightarrow{t \to 0} \int_0^T DZ_{r,t}^{-1}(s)\, h(s)\, \mathrm{d}s, \tag{111}$$

where the convergence holds in probability,

(3) $DZ_{r,t}^{-1}$ belongs to $\mathscr{L}^2(\Omega, \mathscr{L}^2)$.

We first show that

$$\mathbf{E}\left[\left|\frac{\partial Z_{r,t}}{\partial x}(\omega, Z_{r,t}^{-1}(x))\right|^{-p}\right] \text{ is finite.} \tag{112}$$

Since

$$\begin{aligned}\frac{\partial Z_{r,t}}{\partial x}(\omega, x) &= \mathrm{Id} + \int_r^t \check{V}_T\left(\sigma'\left(Z_{.,t}(x)\right)\frac{\partial Z_{.,t}(\omega, x)}{\partial x}\right)(s)\, \mathrm{d}\breve{B}(s),\end{aligned} \tag{113}$$

let $\Theta_v = \sup_{u \le v} |\partial_x Z_{u,t}(x)|$. The same kind of computations as above entails that (for the sake of brevity, we do not detail the localisation procedure as it is similar to the previous one)

$$\begin{aligned}\mathbf{E}\left[\Theta_v^{2q}\right] &\le c + c\,\mathbf{E}\left[\int_u^t \Theta_s^{2(q-1)}\left(\int_u^s \left|\partial_x Z_{\tau,t}(x)\right|^p \mid \mathrm{d}\tau\right)^{2/p} \mathrm{d}s\right] \\ &\quad + c\,\mathbf{E}\left[\left(\int_u^t \Theta_s^{q-2}\left(\int_u^s \left|\partial_x Z_{\tau,t}(x)\right|^p \mid \mathrm{d}\tau\right)^{2/p}\right)^2 \mathrm{d}s\right].\end{aligned} \tag{114}$$

Hence,

$$\mathbf{E}\left[\Theta_v^{2q}\right] \le c\left(1 + \int_v^t \mathbf{E}\left[\Theta_s^{2q}\right] \mathrm{d}s\right), \tag{115}$$

and (112) follows by Fatou and Gronwall lemmas. Since $Z_{r,t}(\omega, Z_{r,t}^{-1}(\omega, x)) = x$, the implicit function theorem implies that $Z_{r,t}^{-1}(x)$ satisfies the first two properties and that

$$\nabla Z_{r,t}\left(\omega, Z_{r,t}^{-1}(x)\right) + \frac{\partial Z_{r,t}}{\partial x}\left(\omega, Z_{r,t}^{-1}(x)\right) \tilde{\nabla} Z_{r,t}^{-1}(\omega, x). \tag{116}$$

It follows by Hölder inequality and (112) that

$$\left\|DZ_{r,t}^{-1}(x))\right\|_{p,1} \le c\left\|Z_{r,t}(x))\right\|_{2p,1}\left\|(\partial_x Z_{r,t}(x))^{-1}\right\|_{2p}; \tag{117}$$

hence $Z_{r,t}^{-1}$ belongs to $\mathbb{L}_{p,1}$. □

Acknowledgment

The author would like to thank the anonymous referees for their thorough reviews, comments, and suggestions significantly contributed to improve the quality of this contribution.

References

[1] M. Zähle, "Integration with respect to fractal functions and stochastic calculus. I," *Probability Theory and Related Fields*, vol. 111, no. 3, pp. 333–374, 1998.

[2] D. Feyel and A. de La Pradelle, "On fractional Brownian processes," *Potential Analysis*, vol. 10, no. 3, pp. 273–288, 1999.

[3] F. Russo and P. Vallois, "The generalized covariation process and Itô formula," *Stochastic Processes and their Applications*, vol. 59, no. 1, pp. 81–104, 1995.

[4] F. Russo and P. Vallois, "Itô formula for C^1-functions of semimartingales," *Probability Theory and Related Fields*, vol. 104, no. 1, pp. 27–41, 1996.

[5] T. J. Lyons, "Differential equations driven by rough signals," *Revista Matemática Iberoamericana*, vol. 14, no. 2, pp. 215–310, 1998.

[6] L. Coutin and Z. Qian, "Stochastic analysis, rough path analysis and fractional Brownian motions," *Probability Theory and Related Fields*, vol. 122, no. 1, pp. 108–140, 2002.

[7] L. Coutin, P. Friz, and N. Victoir, "Good rough path sequences and applications to anticipating stochastic calculus," *The Annals of Probability*, vol. 35, no. 3, pp. 1172–1193, 2007.

[8] L. Decreusefond and D. Nualart, "Flow properties of differential equations driven by fractional Brownian motion," in *Stochastic Differential Equations: Theory and Applications*, vol. 2, pp. 249–262, 2007.

[9] P. Friz and N. Victoir, "Approximations of the Brownian rough path with applications to stochastic analysis," *Annales de l'Institut Henri Poincaré. Probabilités et Statistiques*, vol. 41, no. 4, pp. 703–724, 2005.

[10] P. Friz and N. Victoir, "A note on the notion of geometric rough paths," *Probability Theory and Related Fields*, vol. 136, no. 3, pp. 395–416, 2006.

[11] M. Gradinaru, I. Nourdin, F. Russo, and P. Vallois, "m-order integrals and generalized Itô's formula: the case of a fractional Brownian motion with any Hurst index," *Annales de l'Institut Henri Poincaré. Probabilités et Statistiques*, vol. 41, no. 4, pp. 781–806, 2005.

[12] A. Lejay and N. Victoir, "On (p, q)-rough paths," *Journal of Differential Equations*, vol. 225, no. 1, pp. 103–133, 2006.

[13] T. Lyons and N. Victoir, "An extension theorem to rough paths," *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire*, vol. 24, no. 5, pp. 835–847, 2007.

[14] A. Neuenkirch and I. Nourdin, "Exact rate of convergence of some approximation schemes associated to SDEs driven by a fractional Brownian motion," *Journal of Theoretical Probability*, vol. 20, no. 4, pp. 871–899, 2007.

[15] I. Nourdin, "Schémas d'approximation associés à une équation différentielle dirigée par une fonction höldérienne; cas du mouvement brownien fractionnaire," *Comptes Rendus Mathématique. Académie des Sciences. Paris*, vol. 340, no. 8, pp. 611–614, 2005.

[16] I. Nourdin and T. Simon, "On the absolute continuity of one-dimensional SDEs driven by a fractional Brownian motion," *Statistics & Probability Letters*, vol. 76, no. 9, pp. 907–912, 2006.

[17] I. Nourdin and C. A. Tudor, "Some linear fractional stochastic equations," *Stochastics*, vol. 78, no. 2, pp. 51–65, 2006.

[18] D. Feyel and A. de La Pradelle, "Curvilinear integrals along enriched paths," *Electronic Journal of Probability*, vol. 11, no. 34, pp. 860–892, 2006.

[19] M. Gubinelli, "Controlling rough paths," *Journal of Functional Analysis*, vol. 216, no. 1, pp. 86–140, 2004.

[20] J. Unterberger, "A rough path over multidimensional fractional Brownian motion with arbitrary Hurst index by Fourier normal ordering," *Stochastic Processes and their Applications*, vol. 120, no. 8, pp. 1444–1472, 2010.

[21] A. Neuenkirch, S. Tindel, and J. Unterberger, "Discretizing the fractional Lévy area," *Stochastic Processes and their Applications*, vol. 120, no. 2, pp. 223–254, 2010.

[22] E. Alòs, O. Mazet, and D. Nualart, "Stochastic calculus with respect to fractional Brownian motion with Hurst parameter lesser than 1/2," *Stochastic Processes and their Applications*, vol. 86, no. 1, pp. 121–139, 2000.

[23] E. Alòs, O. Mazet, and D. Nualart, "Stochastic calculus with respect to Gaussian processes," *The Annals of Probability*, vol. 29, no. 2, pp. 766–801, 2001.

[24] L. Decreusefond, "Stochastic integration with respect to Gaussian processes," *Comptes Rendus Mathématique. Académie des Sciences. Paris*, vol. 334, no. 10, pp. 903–908, 2002.

[25] L. Decreusefond, "Stochastic integration with respect to fractional Brownian motion," in *Theory and Applications of Long-Range Dependence*, pp. 203–226, Birkhäuser, Boston, Mass, USA, 2003.

[26] L. Decreusefond, "Stochastic integration with respect to Volterra processes," *Annales de l'Institut Henri Poincaré. Probabilités et Statistiques*, vol. 41, no. 2, pp. 123–149, 2005.

[27] L. Decreusefond and A. S. Üstünel, "Stochastic analysis of the fractional Brownian motion," *Potential Analysis*, vol. 10, no. 2, pp. 177–214, 1999.

[28] Y. Hu and D. Nualart, "Differential equations driven by Hölder continuous functions of order greater than 1/2," in *Stochastic Analysis and Applications*, vol. 2 of *The Abel Symposium*, pp. 399–413, Springer, Berlin, Germany, 2007.

[29] D. Nualart and A. Răşcanu, "Differential equations driven by fractional Brownian motion," *Universitat de Barcelona. Collectanea Mathematica*, vol. 53, no. 1, pp. 55–81, 2002.

[30] D. Nualart and B. Saussereau, "Malliavin calculus for stochastic differential equations driven by a fractional Brownian motion," *Stochastic Processes and their Applications*, vol. 119, no. 2, pp. 391–409, 2009.

[31] S. Darses and B. Saussereau, "Time reversal for drifted fractional Brownian motion with Hurst index $Hx3e; 1/2$," *Electronic Journal of Probability*, vol. 12, no. 43, pp. 1181–1211, 2007.

[32] P. K. Friz and N. B. Victoir, *Multidimensional Stochastic Processes as Rough Paths*, Cambridge University Press, Cambridge, UK, 2010.

[33] A. S. Üstünel and M. Zakai, "The construction of filtrations on abstract Wiener space," *Journal of Functional Analysis*, vol. 143, no. 1, pp. 10–32, 1997.

[34] L. M. Wu, "Un traitement unifié de la représentation des fonctionnelles de Wiener," in *Séminaire de Probabilités, XXIV*, vol. 1426 of *Lecture Notes in Mathematics*, pp. 166–187, Springer, Berlin, Germany, 1990.

[35] M. Zakai and O. Zeitouni, "When does the Ramer formula look like the Girsanov formula?" *The Annals of Probability*, vol. 20, no. 3, pp. 1436–1441, 1992.

[36] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives*, Gordon and Breach Science, Yverdon, Switzerland, 1993.

[37] A. S. Üstünel, *An Introduction to Analysis on Wiener Space*, vol. 1610 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1995.

[38] D. Nualart, *The Malliavin Calculus and Related Topics*, Springer, 1995.

[39] N. Dunford and J. T. Schwartz, *Linear Operators. Part II*, Wiley Classics Library, 1988.

[40] A. Feintuch and R. Saeks, *System Theory*, vol. 102 of *Pure and Applied Mathematics*, Academic Press, New York, NY, USA, 1982, A Hilbert Space Approach.

[41] A. F. Nikiforov and V. B. Uvarov, *Special Functions of Mathematical Physics*, Birkhäuser, 1988.

[42] B. Simon, *Trace Ideals and their Applications*, vol. 120, American Mathematical Society, Providence, RI, USA, 2nd edition, 2005.

[43] A. Uppman, "Sur le flot d'une équation différentielle stochastique," in *Seminar on Probability, XVI*, vol. 920, pp. 268–284, Springer, Berlin, Germany, 1982.

[44] F. Hirsch, "Propriété d'absolue continuité pour les équations différentielles stochastiques dépendant du passé," *Journal of Functional Analysis*, vol. 76, no. 1, pp. 193–216, 1988.

[45] H. Sugita, "On a characterization of the Sobolev spaces over an abstract Wiener space," *Journal of Mathematics of Kyoto University*, vol. 25, no. 4, pp. 717–725, 1985.