Hindawi Publishing Corporation
International Journal of Stochastic Analysis
Volume 2014, Article ID 793275, 22 pages
http://dx.doi.org/10.1155/2014/793275

Research Article

SPDEs with α-Stable Lévy Noise: A Random Field Approach

Raluca M. Balan

Department of Mathematics and Statistics, University of Ottawa, 585 King Edward Avenue, Ottawa, ON, Canada K1N 6N5

Correspondence should be addressed to Raluca M. Balan; rbalan@uottawa.ca

Received 17 August 2013; Accepted 25 November 2013; Published 4 February 2014

Academic Editor: H. Srivastava

This paper is dedicated to the study of a nonlinear SPDE on a bounded domain in R^d, with zero initial conditions and Dirichlet boundary, driven by an α-stable Lévy noise Z with $\alpha \in (0,2)$, $\alpha \neq 1$, and possibly nonsymmetric tails. To give a meaning to the concept of solution, we develop a theory of stochastic integration with respect to this noise. The idea is to first solve the equation with "truncated" noise (obtained by removing from Z the jumps which exceed a fixed value K), yielding a solution u_K, and then show that the solutions $u_L, L > K$ coincide on the event $t \leq \tau_K$, for some stopping times τ_K converging to infinity. A similar idea was used in the setting of Hilbert-space valued processes. A major step is to show that the stochastic integral with respect to Z_K satisfies a pth moment inequality. This inequality plays the same role as the Burkholder-Davis-Gundy inequality in the theory of integration with respect to continuous martingales.

1. Introduction

Modeling phenomena which evolve in time or space-time and are subject to random perturbations are a fundamental problem in stochastic analysis. When these perturbations are known to exhibit an extreme behavior, as seen frequently in finance or environmental studies, a model relying on the Gaussian distribution is not appropriate. A suitable alternative could be a model based on a heavy-tailed distribution, like the stable distribution. In such a model, these perturbations are allowed to have extreme values with a probability which is significantly higher than in a Gaussian-based model.

In the present paper, we introduce precisely such a model, given rigorously by a stochastic partial differential equation (SPDE) driven by a noise term which has a stable distribution over any space-time region and has independent values over disjoint space-time regions (i.e., it is a Lévy noise). More precisely, we consider the SPDE:

$$Lu(t,x) = \sigma(u(t,x))\dot{Z}(t,x), \quad t > 0,\ x \in \mathcal{O} \tag{1}$$

with zero initial conditions and Dirichlet boundary conditions, where σ is a Lipschitz function, L is a second-order pseudo-differential operator on a bounded domain $\mathcal{O} \subset \mathbb{R}^d$, and $\dot{Z}(t,x) = \partial^{d+1}Z/\partial t\partial x_1,\dots,\partial x_d$ is the formal derivative of an α-stable Lévy noise with $\alpha \in (0,2)$, $\alpha \neq 1$. The goal is to find sufficient conditions on the fundamental solution $G(t,x,y)$ of the equation $Lu = 0$ on $\mathbb{R}_+ \times \mathcal{O}$, which will ensure the existence of a mild solution of (1). We say that a predictable process $u = \{u(t,x); t \geq 0, x \in \mathcal{O}\}$ is a *mild solution* of (1) if for any $t > 0$, $x \in \mathcal{O}$,

$$u(t,x) = \int_0^t \int_{\mathcal{O}} G(t-s,x,y)\,\sigma(u(s,y))\,Z(ds,dy) \quad \text{a.s.} \tag{2}$$

We assume that $G(t,x,y)$ is a function in t, which excludes from our analysis the case of the wave equation with $d \geq 3$.

To explain the connections with other works, we describe briefly the construction of the noise (the details are given in Section 2). This construction is similar to that of a classical α-stable Lévy process and is based on a Poisson random measure (PRM) N on $\mathbb{R}_+ \times \mathbb{R}^d \times (\mathbb{R} \setminus \{0\})$ of intensity $dtdx\nu_\alpha(dz)$, where

$$\nu_\alpha(dz) = \left[p\alpha z^{-\alpha-1}1_{(0,\infty)}(z) + q\alpha(-z)^{-\alpha-1}1_{(-\infty,0)}(z)\right]dz \tag{3}$$

for some $p, q \geq 0$ with $p+q=1$. More precisely, for any set $B \in \mathscr{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$,

$$Z(B) = \int_{B\times\{|z|\leq 1\}} z\widehat{N}(ds,dx,dz) + \int_{B\times\{|z|>1\}} zN(ds,dx,dz) - \mu|B|, \tag{4}$$

where $\widehat{N}(B\times\cdot) = N(B\times\cdot) - |B|\nu_\alpha(\cdot)$ is the compensated process and μ is a constant (specified by Lemma 3). Here, $\mathscr{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$ is the class of bounded Borel sets in $\mathbb{R}_+ \times \mathbb{R}^d$ and $|B|$ is the Lebesgue measure of B.

As the term on the right-hand side of (2) is a stochastic integral with respect to Z, such an integral should be constructed first. Our construction of the integral is an extension to random fields of the construction provided by Giné and Marcus in [1] in the case of an α-stable Lévy process $\{Z(t)\}_{t\in[0,1]}$. Unlike these authors, we do not assume that the measure ν_α is symmetric.

Since any Lévy noise is related to a PRM, in a broad sense, one could say that this problem originates in Itô's papers [2, 3] regarding the stochastic integral with respect to a Poisson noise. SPDEs driven by a compensated PRM were considered for the first time in [4], using the approach based on Hilbert-space-valued solutions. This study was motivated by an application to neurophysiology leading to the cable equation. In the case of the heat equation, a similar problem was considered in [5–7] using the approach based on random-field solutions. One of the results of [6] shows that the heat equation:

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(t,x) + \int_U f(t,x,u(t,x);z)\widehat{N}(t,x,dz) + g(t,x,u(t,x)) \tag{5}$$

has a unique solution in the space of predictable processes u satisfying $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E|u(t,x)|^p < \infty$, for any $p \in (1+2/d, 2]$. In this equation, $\widehat{N}$ is the compensated process corresponding to a PRM N on $\mathbb{R}_+\times\mathbb{R}^d\times U$ of intensity $dt\,dx\,\nu(dz)$, for an arbitrary σ-finite measure space $(U, \mathscr{B}(U), \nu)$ with measure ν satisfying $\int_U |z|^p \nu(dz) < \infty$. Because of this later condition, this result cannot be used in our case with $U = \mathbb{R}\setminus\{0\}$ and $\nu = \nu_\alpha$. For similar reasons, the results of [7] also do not cover the case of an α-stable noise. However, in the case $\alpha > 1$, we will be able to exploit successfully some ideas of [6] for treating the equation with "truncated" noise Z_K, obtained by removing from Z the jumps exceeding a value K (see Section 5.2).

The heat equation with the same type of noise as in the present paper was examined in [8, 9] in the cases $\alpha < 1$ and $\alpha > 1$, respectively, assuming that the noise has only positive jumps (i.e., $q = 0$). The methods used by these authors are different from those presented here, since they investigate the more difficult case of a non-Lipschitz function $\sigma(u) = u^\delta$ with $\delta > 0$. In [8], Mueller removes the atoms of Z of mass smaller than 2^{-n} and solves the equation driven by the noise obtained in this way; here we remove the atoms of Z of mass larger than K and solve the resulting equation. In [9], Mytnik uses a martingale problem approach and gives the existence of a pair (u, Z) which satisfies the equation (the so-called "weak solution"), whereas in the present paper we obtain the existence of a solution u for a *given* noise Z (the so-called "strong solution"). In particular, when $\alpha > 1$ and $\delta = 1/\alpha$, the existence of a "weak solution" of the heat equation with α-stable Lévy noise is obtained in [9] under the condition

$$\alpha < 1 + \frac{2}{d} \tag{6}$$

which we encounter here as well. It is interesting to note that (6) is the necessary and sufficient condition for the existence of the density of the super-Brownian motion with "$\alpha - 1$"-stable branching (see [10]). Reference [11] examines the heat equation with multiplicative noise (i.e., $\sigma(u) = u$), driven by an α-stable Lévy noise Z which does not depend on time.

To conclude the literature review, we should point out that there are many references related to stochastic differential equations with α-stable Lévy noise, using the approach based on Hilbert-space valued solutions. We refer the reader to Section 12.5 of the monograph [12] and to [13–16] for a sample of relevant references. See also the survey article [17] for an approach based on the white noise theory for Lévy processes.

This paper is organized as follows.

(i) In Section 2, we review the construction of the α-stable Lévy noise Z, and we show that this can be viewed as an independently scattered random measure with jointly α-stable distributions.

(ii) In Section 3, we consider the linear equation (1) (with $\sigma(u) = 1$) and we identify the necessary and sufficient condition for the existence of the solution. This condition is verified in the case of some examples.

(iii) Section 4 contains the construction of the stochastic integral with respect to the α-stable noise Z, for $\alpha \in (0, 2)$. The main effort is dedicated to proving a maximal inequality for the tail of the integral process, when the integrand is a simple process. This extends the construction of [1] to the case random fields and nonsymmetric measure ν_α.

(iv) In Section 5, we introduce the process Z_K obtained by removing from Z the jumps exceeding a fixed value K, and we develop a theory of integration with respect to this process. For this, we need to treat separately the cases $\alpha < 1$ and $\alpha > 1$. In both cases, we obtain a pth moment inequality for the integral process for $p \in (\alpha, 1)$ if $\alpha < 1$ and $p \in (\alpha, 2)$ if $\alpha > 1$. This inequality plays the same role as the Burkholder-Davis-Gundy inequality in the theory of integration with respect to continuous martingales.

(v) In Section 6 we prove the main result about the existence of the mild solution of (1). For this, we first solve the equation with "truncated" noise Z_K using a Picard iteration scheme, yielding a solution u_K.

We then introduce a sequence $(\tau_K)_{K\geq 1}$ of stopping times with $\tau_K \uparrow \infty$ a.s. and we show that the solutions u_L, $L > K$ coincide on the event $t \leq \tau_K$. For the definition of the stopping times τ_K, we need again to consider separately the cases $\alpha < 1$ and $\alpha > 1$.

(vi) Appendix A contains some results about the tail of a nonsymmetric stable random variable and the tail of an infinite sum of random variables. Appendix B gives an estimate for the Green function associated with the fractional power of the Laplacian. Appendix C gives a local property of the stochastic integral with respect to Z (or Z_K).

2. Definition of the Noise

In this section we review the construction of the α-stable Lévy noise on $\mathbb{R}_+ \times \mathbb{R}^d$ and investigate some of its properties.

Let $N = \sum_{i\geq 1} \delta_{(T_i,X_i,Z_i)}$ be a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d \times (\mathbb{R} \setminus \{0\})$, defined on a probability space $(\Omega, \mathscr{F}, P)$, with intensity measure $dtdx\nu_\alpha(dz)$, where ν_α is given by (3). Let $(\varepsilon_j)_{j\geq 0}$ be a sequence of positive real numbers such that $\varepsilon_j \to 0$ as $j \to \infty$ and $1 = \varepsilon_0 > \varepsilon_1 > \varepsilon_2 > \cdots$. Let

$$\Gamma_j = \left\{z \in \mathbb{R}; \varepsilon_j < |z| \leq \varepsilon_{j-1}\right\}, \quad j \geq 1, \qquad \Gamma_0 = \{z \in \mathbb{R}; |z| > 1\}. \tag{7}$$

For any set $B \in \mathscr{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$, we define

$$L_j(B) = \int_{B\times\Gamma_j} zN(dt,dx,dz) = \sum_{(T_i,X_i)\in B} Z_i 1_{\{Z_i\in\Gamma_j\}}, \quad j \geq 0. \tag{8}$$

Remark 1. The variable $L_0(B)$ is finite since the sum above contains finitely many terms. To see this, we note that $E[N(B \times \Gamma_0)] = |B|\nu_\alpha(\Gamma_0) < \infty$, and hence $N(B \times \Gamma_0) = \text{card}\{i \geq 1; (T_i, X_i, Z_i) \in B \times \Gamma_0\} < \infty$.

For any $j \geq 0$, the variable $L_j(B)$ has a compound Poisson distribution with jump intensity measure $|B| \cdot \nu_\alpha|_{\Gamma_j}$; that is,

$$E\left[e^{iuL_j(B)}\right] = \exp\left\{|B| \int_{\Gamma_j} \left(e^{iuz} - 1\right) \nu_\alpha(dz)\right\}, \quad u \in \mathbb{R}. \tag{9}$$

It follows that $E(L_j(B)) = |B| \int_{\Gamma_j} z\nu_\alpha(dz)$ and $\text{Var}(L_j(B)) = |B| \int_{\Gamma_j} z^2\nu_\alpha(dz)$ for any $j \geq 0$. Hence, $\text{Var}(L_j(B)) < \infty$ for any $j \geq 1$ and $\text{Var}(L_0(B)) = \infty$. If $\alpha > 1$, then $E(L_0(B))$ is finite. Define

$$Y(B) = \sum_{j\geq 1} \left[L_j(B) - E\left(L_j(B)\right)\right] + L_0(B). \tag{10}$$

This sum converges a.s. by Kolmogorov's criterion since $\{L_j(B) - E(L_j(B))\}_{j\geq 1}$ are independent zero-mean random variables with $\sum_{j\geq 1} \text{Var}(L_j(B)) < \infty$.

From (9) and (10), it follows that $Y(B)$ is an infinitely divisible random variable with characteristic function:

$$E\left(e^{iuY(B)}\right) = \exp\left\{|B| \int_{\mathbb{R}} \left(e^{iuz} - 1 - iuz1_{\{|z|\leq 1\}}\right) \nu_\alpha(dz)\right\}, \quad u \in \mathbb{R}. \tag{11}$$

Hence, $E(Y(B)) = |B| \int_{\mathbb{R}} z1_{\{|z|>1\}}\nu_\alpha(dz)$ and $\text{Var}(Y(B)) = |B| \int_{\mathbb{R}} z^2\nu_\alpha(dz)$.

Lemma 2. *The family $\{Y(B); B \in \mathscr{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)\}$ defined by (10) is an independently scattered random measure; that is,*

(a) *for any disjoint sets $B_1, \dots, B_n$ in $\mathscr{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$, $Y(B_1), \dots, Y(B_n)$ are independent;*

(b) *for any sequence $(B_n)_{n\geq 1}$ of disjoint sets in $\mathscr{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$ such that $\bigcup_{n\geq 1} B_n$ is bounded, $Y(\bigcup_{n\geq 1} B_n) = \sum_{n\geq 1} Y(B_n)$ a.s.*

Proof. (a) Note that for any function $\varphi \in L^2(\mathbb{R}_+ \times \mathbb{R}^d)$ with compact support K, we can define the random variable $Y(\varphi) = \sum_{j\geq 1}[L_j(\varphi) - E(L_j(\varphi))] + L_0(\varphi)$ where $L_j(\varphi) = \int_{K\times\Gamma_j} \varphi(t,x)z\,N(dt,dx,dz)$. For any $u \in \mathbb{R}$, we have

$$E\left(e^{iuY(\varphi)}\right) = \exp\left\{\int_{\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}} \left(e^{iuz\varphi(t,x)} - 1 - iuz\varphi(t,x)\,1_{\{|z|\leq 1\}}\right) dt\,dx\,\nu_\alpha(dz)\right\}. \tag{12}$$

For any disjoint sets $B_1, \dots, B_n$ and for any $u_1, \dots, u_n \in \mathbb{R}$, we have

$$\begin{aligned} E\left[\exp\left(i\sum_{k=1}^n u_k Y(B_k)\right)\right] &= E\left[\exp\left(iY\left(\sum_{k=1}^n u_k 1_{B_k}\right)\right)\right] \\ &= \exp\left\{\int_{\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}} \left(e^{iz\sum_{k=1}^n u_k 1_{B_k}(t,x)} - 1 - iz1_{\{|z|\leq 1\}} \times \sum_{k=1}^n u_k 1_{B_k}(t,x)\right) dt\,dx\,\nu_\alpha(dz)\right\} \\ &= \exp\left\{\sum_{k=1}^n |B_k| \int_{\mathbb{R}} \left(e^{iu_kz} - 1 - iu_kz1_{\{|z|\leq 1\}}\right) \nu_\alpha(dz)\right\} \\ &= \prod_{k=1}^n E\left[\exp\left(iu_kY(B_k)\right)\right], \end{aligned} \tag{13}$$

using (12) with $\varphi = \sum_{k=1}^{n} u_k 1_{B_k}$ for the second equality and (9) for the last equality. This proves that $Y(B_1), \ldots, Y(B_n)$ are independent.

(b) Let $S_n = \sum_{k=1}^{n} Y(B_k)$ and $S = Y(B)$, where $B = \bigcup_{n\geq 1} B_n$. By Lévy's equivalence theorem, $(S_n)_{n\geq 1}$ converges a.s. if and only if it converges in distribution. By (13), with $u_i = u$ for all $i = 1, \ldots, k$, we have

$$E\left(e^{iuS_n}\right) = \exp\left\{\left|\bigcup_{k=1}^{n} B_k\right| \int_{\mathbb{R}} \left(e^{iuz} - 1 - iuz1_{\{|z|\leq 1\}}\right)\nu_\alpha(dz)\right\}. \tag{14}$$

This clearly converges to $E(e^{iuS}) = \exp\{|B| \int_{\mathbb{R}} (e^{iuz} - 1 - iuz1_{\{|z|\leq 1\}})\nu_\alpha(dz)\}$, and hence $(S_n)_{n\geq 1}$ converges in distribution to S. □

Recall that a random variable X has an *α-stable distribution* with parameters $\alpha \in (0, 2)$, $\sigma \in [0, \infty)$, $\beta \in [-1, 1]$, and $\mu \in \mathbb{R}$ if, for any $u \in \mathbb{R}$,

$$E\left(e^{iuX}\right) = \exp\left\{-|u|^\alpha \sigma^\alpha \left(1 - i \operatorname{sgn}(u) \beta \tan\frac{\pi\alpha}{2}\right) + iu\mu\right\}, \quad \text{if } \alpha \neq 1, \tag{15}$$

or

$$E\left(e^{iuX}\right) = \exp\left\{-|u|\sigma\left(1 + i \operatorname{sgn}(u) \beta \frac{2}{\pi} \ln|u|\right) + iu\mu\right\}, \quad \text{if } \alpha = 1 \tag{16}$$

(see Definition 1.1.6 of [18]). We denote this distribution by $S_\alpha(\sigma, \beta, \mu)$.

Lemma 3. *$Y(B)$ has a $S_\alpha(\sigma|B|^{1/\alpha}, \beta, \mu|B|)$ distribution with $\beta = p - q$,*

$$\sigma^\alpha = \int_0^\infty \frac{\sin x}{x^\alpha} dx = \begin{cases} \dfrac{\Gamma(2-\alpha)}{1-\alpha} \cos\dfrac{\pi\alpha}{2}, & \text{if } \alpha \neq 1, \\ \dfrac{\pi}{2}, & \text{if } \alpha = 1, \end{cases} \qquad \mu = \begin{cases} \beta\dfrac{\alpha}{\alpha - 1}, & \text{if } \alpha \neq 1, \\ \beta c_0, & \text{if } \alpha = 1, \end{cases} \tag{17}$$

and $c_0 = \int_0^\infty (\sin z - z1_{\{z\leq 1\}}) z^{-2} dz$. If $\alpha > 1$, then $E(Y(B)) = \mu|B|$.

Proof. We first express the characteristic function (11) of $Y(B)$ in Feller's canonical form (see Section XVII.2 of [19]):

$$E\left(e^{iuY(B)}\right) = \exp\left\{iub|B| + |B| \int_{\mathbb{R}} \frac{e^{iuz} - 1 - iu\sin z}{z^2} M_\alpha(dz)\right\} \tag{18}$$

with $M_\alpha(dz) = z^2\nu_\alpha(dz)$ and $b = \int_{\mathbb{R}} (\sin z - z1_{\{|z|\leq 1\}})\nu_\alpha(dz)$. Then the result follows from the calculations done in Example XVII.3.(g) of [19]. □

From Lemmas 2 and 3, it follows that

$$Z = \left\{Z(B) = Y(B) - \mu|B|; B \in \mathscr{B}_b\left(\mathbb{R}_+ \times \mathbb{R}^d\right)\right\} \tag{19}$$

is an α-stable random measure, in the sense of Definition 3.3.1 of [18], with control measure $m(B) = \sigma^\alpha|B|$ and constant skewness intensity β. In particular, $Z(B)$ has a $S_\alpha(\sigma|B|^{1/\alpha}, \beta, 0)$ distribution.

We say that Z is an *α-stable Lévy noise*. Coming back to the original construction (10) of $Y(B)$ and noticing that

$$\begin{aligned} \mu|B| &= -|B| \int_{\mathbb{R}} z1_{\{|z|\leq 1\}}\nu_\alpha(dz) = -\sum_{j\geq 1} E\left(L_j(B)\right), \quad \text{if } \alpha < 1, \\ \mu|B| &= |B| \int_{\mathbb{R}} z1_{\{|z|>1\}}\nu_\alpha(dz) = E\left(L_0(B)\right), \quad \text{if } \alpha > 1, \end{aligned} \tag{20}$$

it follows that $Z(B)$ can be represented as

$$Z(B) = \sum_{j\geq 0} L_j(B) =: \int_{B\times(\mathbb{R}\setminus\{0\})} zN(dt, dx, dz), \quad \text{if } \alpha < 1, \tag{21}$$

$$\begin{aligned} Z(B) &= \sum_{j\geq 0} \left[L_j(B) - E\left(L_j(B)\right)\right] \\ &=: \int_{B\times(\mathbb{R}\setminus\{0\})} z\widehat{N}(dt, dx, dz), \quad \text{if } \alpha > 1. \end{aligned} \tag{22}$$

Here $\widehat{N}$ is the compensated Poisson measure associated with N; that is, $\widehat{N}(A) = N(A) - E(N(A))$ for any relatively compact set A in $\mathbb{R}_+ \times \mathbb{R}^d \times (\overline{\mathbb{R}} \setminus \{0\})$.

In the case $\alpha = 1$, we will assume that $p = q$ so that ν_α is symmetric around 0, $E(L_j(B)) = 0$ for all $j \geq 1$, and $Z(B)$ admits the same representation as in the case $\alpha < 1$.

3. The Linear Equation

As a preliminary investigation, we consider first equation (1) with $\sigma = 1$:

$$Lu(t, x) = \dot{Z}(t, x), \quad t > 0,\ x \in \mathcal{O} \tag{23}$$

with zero initial conditions and Dirichlet boundary conditions. In this section $\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$ or $\mathcal{O} = \mathbb{R}^d$.

By definition, the process $\{u(t, x); t \geq 0, x \in \mathcal{O}\}$ given by

$$u(t, x) = \int_0^t \int_{\mathcal{O}} G(t - s, x, y) Z(ds, dy) \tag{24}$$

is a mild solution of (23), provided that the stochastic integral on the right-hand side of (24) is well defined.

We define now the stochastic integral of a deterministic function φ:

$$Z(\varphi) = \int_0^\infty \int_{\mathbb{R}^d} \varphi(t, x) Z(dt, dx). \tag{25}$$

If $\varphi \in L^{\alpha}(\mathbb{R}_+ \times \mathbb{R}^d)$, this can be defined by approximation with simple functions, as explained in Section 3.4 of [18]. The process $\{Z(\varphi); \varphi \in L^{\alpha}(\mathbb{R}_+ \times \mathbb{R}^d)\}$ has jointly α-stable finite dimensional distributions. In particular, each $Z(\varphi)$ has a $S_{\alpha}(\sigma_{\varphi}, \beta, 0)$-distribution with scale parameter:

$$\sigma_{\varphi} = \sigma\left(\int_0^{\infty} \int_{\mathbb{R}^d} |\varphi(t,x)|^{\alpha} dx\, dt\right)^{1/\alpha}. \tag{26}$$

More generally, a measurable function $\varphi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ is integrable with respect to Z if there exists a sequence $(\varphi_n)_{n\geq 1}$ of simple functions such that $\varphi_n \to \varphi$ a.e., and, for any $B \in \mathscr{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$, the sequence $\{Z(\varphi_n 1_B)\}_n$ converges in probability (see [20]).

The next results show that condition $\varphi \in L^{\alpha}(\mathbb{R}_+ \times \mathbb{R}^d)$ is also necessary for the integrability of φ with respect to Z. Due to Lemma 2, this follows immediately from the general theory of stochastic integration with respect to independently scattered random measures developed in [20].

Lemma 4. *A deterministic function φ is integrable with respect to Z if and only if $\varphi \in L^{\alpha}(\mathbb{R}_+ \times \mathbb{R}^d)$.*

Proof. We write the characteristic function of $Z(B)$ in the form used in [20]:

$$\begin{aligned} &E\left(e^{iuZ(B)}\right) \\ &\quad = \exp\left\{\int_B \left[iua + \int_{\mathbb{R}} \left(e^{iuz} - 1 - iu\tau(z)\right)\nu_{\alpha}(dz)\right] dt dx\right\} \end{aligned} \tag{27}$$

with $a = \beta - \mu$, $\tau(z) = z$ if $|z| \leq 1$ and $\tau(z) = \text{sgn}(z)$ if $|z| > 1$. By Theorem 2.7 of [20], φ is integrable with respect to Z if and only if

$$\begin{gathered} \int_{\mathbb{R}_+ \times \mathbb{R}^d} |U(\varphi(t,x))|\, dt\, dx < \infty, \\ \int_{\mathbb{R}_+ \times \mathbb{R}^d} V(\varphi(t,x))\, dt\, dx < \infty, \end{gathered} \tag{28}$$

where $U(y) = ay + \int_{\mathbb{R}}(\tau(yz) - y\tau(z))\nu_{\alpha}(dz)$ and $V(y) = \int_{\mathbb{R}}(1 \wedge |yz|^2)\nu_{\alpha}(dz)$. Direct calculations show that, in our case, $U(y) = -(\beta/(\alpha-1))y^{\alpha}$ if $\alpha \neq 1$, $U(y) = 0$ if $\alpha = 1$, and $V(y) = (2/(2-\alpha))y^{\alpha}$. □

The following result follows immediately from (24) and Lemma 4.

Proposition 5. *Equation* (23) *has a mild solution if and only if for any $t > 0$, $x \in \mathcal{O}$*

$$I_{\alpha}(t) = \int_0^t \int_{\mathcal{O}} G(s,x,y)^{\alpha} dy\, ds < \infty. \tag{29}$$

In this case, $\{u(t,x); t \geq 0, x \in \mathcal{O}\}$ has jointly α-stable finite-dimensional distributions. In particular, $u(t,x)$ has a $S_{\alpha}(\sigma I_{\alpha}(t)^{1/\alpha}, \beta, 0)$ distribution.

Condition (29) can be easily verified in the case of several examples.

Example 6 (heat equation). Let $L = \partial/\partial t - (1/2)\Delta$. Assume first that $\mathcal{O} = \mathbb{R}^d$. Then $G(t,x,y) = \overline{G}(t, x-y)$, where

$$\overline{G}(t,x) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x|^2}{2t}\right), \tag{30}$$

and condition (29) is equivalent to (6). In this case, $I_{\alpha}(t) = c_{\alpha,d} t^{d(1-\alpha)/2+1}$. If $\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$, then $G(t,x,y) \leq \overline{G}(t, x-y)$ (see page 74 of [11]) and condition (29) is implied by (6).

Example 7 (parabolic equation). Let $L = \partial/\partial t - \mathscr{L}$ where

$$\mathscr{L} f(x) = \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^d b_i(x) \frac{\partial f}{\partial x_i}(x) \tag{31}$$

is the generator of a Markov process with values in $\mathbb{R}^d$, without jumps (a diffusion). Assume that $\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$ or $\mathcal{O} = \mathbb{R}^d$. By Aronson estimate (see, e.g., Theorem 2.6 of [12]), under some assumptions on the coefficients a_{ij}, b_i, there exist some constants $c_1, c_2 > 0$ such that

$$G(t,x,y) \leq c_1 t^{-d/2} \exp\left(-\frac{|x-y|^2}{c_2 t}\right) \tag{32}$$

for all $t > 0$ and $x, y \in \mathcal{O}$. In this case, condition (29) is implied by (6).

Example 8 (heat equation with fractional power of the Laplacian). Let $L = \partial/\partial t + (-\Delta)^{\gamma}$ for some $\gamma > 0$. Assume that $\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$ or $\mathcal{O} = \mathbb{R}^d$. Then (see, e.g., Appendix B.5 of [12])

$$\begin{aligned} G(t,x,y) &= \int_0^{\infty} \mathscr{G}(s,x,y)\, g_{t,\gamma}(s)\, ds \\ &= \int_0^{\infty} \mathscr{G}\left(t^{1/\gamma}s, x, y\right) g_{1,\gamma}(s)\, ds, \end{aligned} \tag{33}$$

where $\mathscr{G}(t,x,y)$ is the fundamental solution of $\partial u/\partial t - \Delta u = 0$ on $\mathcal{O}$ and $g_{t,\gamma}$ is the density of the measure $\mu_{t,\gamma}$, $(\mu_{t,\gamma})_{t\geq 0}$ being a convolution semigroup of measures on $[0,\infty)$ whose Laplace transform is given by

$$\int_0^{\infty} e^{-us} g_{t,\gamma}(s)\, ds = \exp\left(-tu^{\gamma}\right), \quad \forall u > 0. \tag{34}$$

Note that if $\gamma < 1$, $g_{t,\gamma}$ is the density of S_t, where $(S_t)_{t\geq 0}$ is a γ-stable subordinator with Lévy measure $\rho_{\gamma}(dx) = (\gamma/\Gamma(1-\gamma))x^{-\gamma-1}1_{(0,\infty)}(x)dx$.

Assume first that $\mathcal{O} = \mathbb{R}^d$. Then $G(t,x,y) = \overline{G}(t,x-y)$, where

$$\overline{G}(t,x) = \int_{\mathbb{R}^d} e^{i\xi\cdot x} e^{-t|\xi|^{2\gamma}} d\xi. \tag{35}$$

If $\gamma < 1$, then $\overline{G}(t,\cdot)$ is the density of X_t, with $(X_t)_{t\geq 0}$ being a symmetric (2γ)-stable Lévy process with values in $\mathbb{R}^d$ defined by $X_t = W_{S_t}$, with $(W_t)_{t\geq 0}$ a Brownian motion in $\mathbb{R}^d$ with variance 2. By Lemma B.1 (Appendix B), if $\alpha > 1$, then (29) holds if and only if

$$\alpha < 1 + \frac{2\gamma}{d}. \tag{36}$$

If $\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$, then $G(t,x,y) \leq \overline{G}(t,x-y)$ (by Lemma 2.1 of [8]). In this case, if $\alpha > 1$, then (29) is implied by (36).

Example 9 (cable equation in $\mathbb{R}$). Let $Lu = \partial u/\partial t - \partial^2 u/\partial x^2 + u$ and $\mathcal{O} = \mathbb{R}$. Then $G(t,x,y) = \overline{G}(t,x-y)$, where

$$\overline{G}(t,x) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{|x|^2}{4t} - t\right), \tag{37}$$

and condition (29) holds for any $\alpha \in (0,2)$.

Example 10 (wave equation in $\mathbb{R}^d$ with $d = 1,2$). Let $L = \partial^2/\partial t^2 - \Delta$ and $\mathcal{O} = \mathbb{R}^d$ with $d = 1$ or $d = 2$. Then $G(t,x,y) = \overline{G}(t,x-y)$, where

$$\begin{aligned} \overline{G}(t,x) &= \frac{1}{2} 1_{\{|x|<t\}}, \quad \text{if } d = 1, \\ \overline{G}(t,x) &= \frac{1}{2\pi} \cdot \frac{1}{\sqrt{t^2 - |x|^2}} 1_{\{|x|<t\}}, \quad \text{if } d = 2. \end{aligned} \tag{38}$$

Condition (29) holds for any $\alpha \in (0,2)$. In this case, $I_\alpha(t) = 2^{-\alpha} t^2$ if $d = 1$ and $I_\alpha(t) = ((2\pi)^{1-\alpha}/(2-\alpha)(3-\alpha)) t^{3-\alpha}$ if $d = 2$.

4. Stochastic Integration

In this section we construct a stochastic integral with respect to Z by generalizing the ideas of [1] to the case of random fields. Unlike these authors, we do not assume that $Z(B)$ has a symmetric distribution, unless $\alpha = 1$.

Let $\mathcal{F}_t = \mathcal{F}_t^N \vee \mathcal{N}$ where $\mathcal{N}$ is the σ-field of negligible sets in $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_t^N$ is the σ-field generated by $N([0,s] \times A \times \Gamma)$ for all $s \in [0,t]$, $A \in \mathcal{B}_b(\mathbb{R}^d)$ and for all Borel sets $\Gamma \subset \mathbb{R} \setminus \{0\}$ bounded away from 0. Note that $\mathcal{F}_t^Z \subset \mathcal{F}_t^N$ where $\mathcal{F}_t^Z$ is the σ-field generated by $Z([0,s] \times A)$, $s \in [0,t]$, and $A \in \mathcal{B}_b(\mathbb{R}^d)$.

A process $X = \{X(t,x)\}_{t\geq 0, x\in\mathbb{R}^d}$ is called *elementary* if it is of the form

$$X(t,x) = 1_{(a,b]}(t)\, 1_A(x)\, Y, \tag{39}$$

where $0 \leq a < b$, $A \in \mathcal{B}_b(\mathbb{R}^d)$, and Y is $\mathcal{F}_a$-measurable and bounded. A *simple process* is a linear combination of elementary processes. Note that any simple process X can be written as

$$X(t,x) = 1_{\{0\}}(t)\, Y_0(x) + \sum_{i=0}^{N-1} 1_{(t_i, t_{i+1}]}(t)\, Y_i(x) \tag{40}$$

with $0 = t_0 < t_1 < \cdots < t_N < \infty$ and $Y_i(x) = \sum_{j=1}^{m_i} 1_{A_{ij}}(x) Y_{ij}$, where $(Y_{ij})_{j=1,\ldots,m_i}$ are $\mathcal{F}_{t_i}$-measurable and $(A_{ij})_{j=1,\ldots,m_j}$ are disjoint sets in $\mathcal{B}_b(\mathbb{R}^d)$. Without loss of generality, we assume that $Y_0 = 0$.

We denote by $\mathcal{P}$ the *predictable σ-field* on $\Omega \times \mathbb{R}_+ \times \mathbb{R}^d$, that is, the σ-field generated by all simple processes. We say that a process $X = \{X(t,x)\}_{t\geq 0, x\in\mathbb{R}^d}$ is *predictable* if the map $(\omega,t,x) \mapsto X(\omega,t,x)$ is $\mathcal{P}$-measurable.

Remark 11. One can show that the predictable σ-field $\mathcal{P}$ is the σ-field generated by the class $\mathcal{C}$ of processes X such that $t \mapsto X(\omega,t,x)$ is left continuous for any $\omega \in \Omega$, $x \in \mathbb{R}^d$ and $(\omega,x) \mapsto X(\omega,t,x)$ is $\mathcal{F}_t \times \mathcal{B}(\mathbb{R}^d)$-measurable for any $t > 0$.

Let $\mathcal{L}_\alpha$ be the class of all predictable processes X such that

$$\|X\|_{\alpha,T,B}^\alpha := E \int_0^T \int_B |X(t,x)|^\alpha dx\, dt < \infty, \tag{41}$$

for all $T > 0$ and $B \in \mathcal{B}_b(\mathbb{R}^d)$. Note that $\mathcal{L}_\alpha$ is a linear space.

Let $(E_k)_{k\geq 1}$ be an increasing sequence of sets in $\mathcal{B}_b(\mathbb{R}^d)$ such that $\bigcup_k E_k = \mathbb{R}^d$. We define

$$\begin{aligned} \|X\|_\alpha &= \sum_{k\geq 1} \frac{1 \wedge \|X\|_{\alpha,k,E_k}}{2^k}, \quad \text{if } \alpha > 1, \\ \|X\|_\alpha^\alpha &= \sum_{k\geq 1} \frac{1 \wedge \|X\|_{\alpha,k,E_k}^\alpha}{2^k}, \quad \text{if } \alpha \leq 1. \end{aligned} \tag{42}$$

We identify two processes X and Y for which $\|X - Y\|_\alpha = 0$; that is, $X = Y\nu$ a.e., where $\nu = Pdtdx$. In particular, we identify two processes X and Y if X is a modification of Y; that is, $X(t,x) = Y(t,x)$ a.s. for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$.

The space $\mathcal{L}_\alpha$ becomes a metric space endowed with the metric d_α:

$$\begin{aligned} d_\alpha(X,Y) &= \|X - Y\|_\alpha, \quad \text{if } \alpha > 1, \\ d_\alpha(X,Y) &= \|X - Y\|_\alpha^\alpha, \quad \text{if } \alpha \leq 1. \end{aligned} \tag{43}$$

This follows using Minkowski's inequality if $\alpha > 1$ and the inequality $|a+b|^\alpha \leq |a|^\alpha + |b|^\alpha$ if $\alpha \leq 1$.

The following result can be proved similarly to Proposition 2.3 of [21].

Proposition 12. *For any $X \in \mathcal{L}_\alpha$ there exists a sequence $(X_n)_{n\geq 1}$ of bounded simple processes such that $\|X_n - X\|_\alpha \to 0$ as $n \to \infty$.*

By Proposition 5.7 of [22], the α-stable Lévy process $\{Z(t,B) = Z([0,t] \times B); t \geq 0\}$ has a càdlàg modification, for any $B \in \mathcal{B}_b(\mathbb{R}^d)$. We work with these modifications. If X is a simple process given by (40), we define

$$I(X)(t,B) = \sum_{i=0}^{N-1} \sum_{j=1}^{m_i} Y_{ij} Z\left((t_i \wedge t, t_{i+1} \wedge t] \times \left(A_{ij} \cap B\right)\right). \tag{44}$$

Note that, for any $B \in \mathscr{B}_b(\mathbb{R}^d)$, $I(X)(t,B)$ is $\mathscr{F}_t$-measurable for any $t \geq 0$, and $\{I(X)(t,B)\}_{t\geq 0}$ is càdlàg. We write

$$I(X)(t,B) = \int_0^t \int_B X(s,x) Z(ds,dx). \tag{45}$$

The following result will be used for the construction of the integral. This result generalizes Lemma 3.3 of [1] to the case of random fields and nonsymmetric measures ν_α.

Theorem 13. *If X is a bounded simple process then*

$$\sup_{\lambda>0} \lambda^\alpha P\left(\sup_{t\in[0,T]} |I(X)(t,B)| > \lambda\right) \leq c_\alpha E \int_0^T \int_B |X(t,x)|^\alpha dx\, dt, \tag{46}$$

for any $T > 0$ and $B \in \mathscr{B}_b(\mathbb{R}^d)$, where c_α is a constant depending only on α.

Proof. Suppose that X is of the form (40). Since $\{I(X)(t,B)\}_{t\in[0,T]}$ is càdlàg, it is separable. Without loss of generality, we assume that its separating set D can be written as $D = \cup_n F_n$ where $(F_n)_n$ is an increasing sequence of finite sets containing the points $(t_k)_{k=0,\ldots,N}$. Hence,

$$P\left(\sup_{t\in[0,T]} |I(X)(t,B)| > \lambda\right) = \lim_{n\to\infty} P\left(\max_{t\in F_n} |I(X)(t,B)| > \lambda\right). \tag{47}$$

Fix $n \geq 1$. Denote by $0 = s_0 < s_1 < \cdots < s_m = T$ the points of the set F_n. Say $t_k = s_{i_k}$ for some $0 = i_0 < i_1 < \cdots < i_N$. Then each interval $(t_k, t_{k+1}]$ can be written as the union of some intervals of the form $(s_i, s_{i+1}]$:

$$(t_k, t_{k+1}] = \bigcup_{i\in I_k} (s_i, s_{i+1}], \tag{48}$$

where $I_k = \{i; i_k \leq i < i_{k+1}\}$. By (44), for any $k = 0, \ldots, N-1$ and $i \in I_k$,

$$I(X)(s_{i+1},B) - I(X)(s_i,B) = \sum_{j=1}^{m_k} Y_{kj} Z\left((s_i, s_{i+1}] \times (A_{kj} \cap B)\right). \tag{49}$$

For any $i \in I_k$, let $N_i = m_k$, and, for any $j = 1, \ldots, N_i$, define $\beta_{ij} = Y_{kj}$, $H_{ij} = A_{kj}$, and $Z_{ij} = Z((s_i, s_{i+1}] \times (H_{ij} \cap B))$. With this notation, we have

$$I(X)(s_{i+1},B) - I(X)(s_i,B) = \sum_{j=1}^{N_i} \beta_{ij} Z_{ij}, \quad \forall i = 0, \ldots, m. \tag{50}$$

Consequently, for any $l = 1, \ldots, m$

$$I(X)(s_l,B) = \sum_{i=0}^{l-1} \left(I(X)(s_{i+1},B) - I(X)(s_i,B)\right) = \sum_{i=0}^{l-1} \sum_{j=1}^{N_i} \beta_{ij} Z_{ij}. \tag{51}$$

Using (47) and (51), it is enough to prove that for any $\lambda > 0$,

$$P\left(\max_{l=0,\ldots,m-1} \left|\sum_{i=0}^{l} \sum_{j=1}^{N_i} \beta_{ij} Z_{ij}\right| > \lambda\right) \leq c_\alpha \lambda^{-\alpha} E \int_0^T \int_B |X(s,x)|^\alpha dx\, ds. \tag{52}$$

First, note that

$$E \int_0^T \int_B |X(s,x)|^\alpha dx\, ds = \sum_{i=0}^{m-1} (s_{i+1} - s_i) \sum_{j=1}^{N_i} E\left|\beta_{ij}\right|^\alpha \left|H_{ij} \cap B\right|. \tag{53}$$

This follows from the definition (40) of X and (48), since $X(t,x) = \sum_{i=0}^{N-1} \sum_{i\in I_k} 1_{(s_i,s_{i+1}]}(t) \sum_{j=1}^{N_i} \beta_{ij} 1_{H_{ij}}(x)$.

We now prove (52). Let $W_i = \sum_{j=1}^{N_i} \beta_{ij} Z_{ij}$. For the event on the left-hand side, we consider its intersection with the event $\{\max_{0\leq i\leq m-1} |W_i| > \lambda\}$ and its complement. Hence, the probability of this event can be bounded by

$$\sum_{i=0}^{m-1} P(|W_i| > \lambda) + P\left(\max_{0\leq l\leq m-1} \left|\sum_{i=0}^{l} W_i 1_{\{|W_i|\leq\lambda\}}\right| > \lambda\right) =: I + II. \tag{54}$$

We treat separately the two terms.

For the first term, we note that $\overline{\beta}_i = (\beta_{ij})_{1\leq j\leq N_i}$ is $\mathscr{F}_{s_i}$-measurable and $\overline{Z}_i = (Z_{ij})_{1\leq j\leq N_i}$ is independent of $\mathscr{F}_{s_i}$. By Fubini's theorem

$$I = \sum_{i=0}^{m-1} \int_{\mathbb{R}^{N_i}} P\left(\left|\sum_{j=1}^{N_i} x_j Z_{ij}\right| > \lambda\right) P_{\overline{\beta}_i}(d\overline{x}), \tag{55}$$

where $\overline{x} = (x_j)_{1\leq j\leq N_i}$ and $P_{\overline{\beta}_i}$ is the law of $\overline{\beta}_i$.

We examine the tail of $U_i = \sum_{j=1}^{N_i} x_j Z_{ij}$ for a fixed $\overline{x} \in \mathbb{R}^{N_i}$. By Lemma 3, Z_{ij} has a $S_\alpha(\sigma(s_{i+1} - s_i)^{1/\alpha}|H_{ij} \cap B|^{1/\alpha}, \beta, 0)$ distribution. Since the sets $(H_{ij})_{1\leq j\leq N_i}$ are disjoint, the variables $(Z_{ij})_{1\leq j\leq N_i}$ are independent. Using elementary properties of the stable distribution (Properties 1.2.1 and 1.2.3 of [18]), it follows that U_i has a $S_\alpha(\sigma_i, \beta_i^*, 0)$ distribution with parameters:

$$\sigma_i^\alpha = \sigma^\alpha (s_{i+1} - s_i) \sum_{j=1}^{N_i} |x_j|^\alpha |H_{ij} \cap B|,$$
$$\beta_i^* = \frac{\beta}{\sum_{j=1}^{N_i} |x_j|^\alpha |H_{ij} \cap B|} \sum_{j=1}^{N_i} \operatorname{sgn}(x_j) |x_j|^\alpha |H_{ij} \cap B|. \tag{56}$$

By Lemma A.1 (Appendix A), there exists a constant $c_\alpha^* > 0$ such that

$$P(|U_i| > \lambda) \le c_\alpha^* \lambda^{-\alpha} \sigma^\alpha (s_{i+1} - s_i) \sum_{j=1}^{N_i} |x_j|^\alpha |H_{ij} \cap B| \tag{57}$$

for any $\lambda > 0$. Hence,

$$I \le c_\alpha^* \lambda^{-\alpha} \sigma^\alpha \sum_{i=0}^{m-1} (s_{i+1} - s_i) \sum_{j=1}^{N_i} E|\beta_{ij}|^\alpha |H_{ij} \cap B|$$
$$= c_\alpha^* \lambda^{-\alpha} \sigma^\alpha E \int_0^T \int_B |X(s,x)|^\alpha dx\, ds. \tag{58}$$

We now treat II. We consider three cases. For the first two cases we deviate from the original argument of [1] since we do not require that $\beta = 0$.

Case 1 ($\alpha < 1$). Note that

$$II \le P\left(\max_{0 \le l \le m-1} M_l > \lambda\right), \tag{59}$$

where $\{M_l = \sum_{i=0}^{l} |W_i| 1_{\{|W_i| \le \lambda\}}, \mathcal{F}_{s_{l+1}}; 0 \le l \le m-1\}$ is a submartingale. By the submartingale maximal inequality (Theorem 35.3 of [23]),

$$P\left(\max_{0 \le l \le m-1} M_l > \lambda\right) \le \frac{1}{\lambda} E(M_{m-1})$$
$$= \frac{1}{\lambda} \sum_{i=0}^{m-1} E\left(|W_i| 1_{|W_i| \le \lambda}\right). \tag{60}$$

Using the independence between $\overline{\beta}_i$ and $\overline{Z}_i$ it follows that

$$E\left[|W_i| 1_{|W_i| \le \lambda}\right]$$
$$= \int_{\mathbb{R}^{N_i}} E\left[\left|\sum_{j=1}^{N_i} x_j Z_{ij}\right| 1_{\{|\sum_{j=1}^{N_i} x_j Z_{ij}| \le \lambda\}}\right] P_{\overline{\beta}_i}(d\overline{x}). \tag{61}$$

Let $U_i = \sum_{j=1}^{N_i} x_j Z_{ij}$. Using (57) and Remark A.2 (Appendix A), we get

$$E\left[|U_i| 1_{\{|U_i| \le \lambda\}}\right] \le c_\alpha^* \sigma^\alpha \frac{1}{1-\alpha} \lambda^{1-\alpha} (s_{i+1} - s_i)$$
$$\times \sum_{j=1}^{N_i} |x_j|^\alpha |H_{ij} \cap B|. \tag{62}$$

Hence,

$$E\left[|W_i| 1_{|W_i| \le \lambda}\right] \le c_\alpha^* \sigma^\alpha \frac{1}{1-\alpha} \lambda^{1-\alpha} (s_{i+1} - s_i)$$
$$\times \sum_{j=1}^{N_i} E|\beta_{ij}|^\alpha |H_{ij} \cap B|. \tag{63}$$

From (59), (60), and (63), it follows that

$$II \le c_\alpha^* \sigma^\alpha \frac{1}{1-\alpha} \lambda^{-\alpha} E \int_0^T \int_B |X(s,x)|^\alpha dx\, ds. \tag{64}$$

Case 2 ($\alpha > 1$). We have

$$II \le P\left(\max_{0 \le l \le m-1} \left|\sum_{i=0}^{l} X_i\right| > \frac{\lambda}{2}\right) + P\left(\max_{0 \le l \le m-1} Y_i > \frac{\lambda}{2}\right)$$
$$=: II' + II'', \tag{65}$$

where $X_i = W_i 1_{\{|W_i| \le \lambda\}} - E[W_i 1_{\{|W_i| \le \lambda\}} \mid \mathcal{F}_{s_i}]$ and $Y_i = |E[W_i 1_{\{|W_i| \le \lambda\}} \mid \mathcal{F}_{s_i}]|$.

We first treat the term II'. Note that $\{M_l = \sum_{i=0}^{l} X_i, \mathcal{F}_{s_{l+1}}; 0 \le l \le m-1\}$ is a zero-mean square integrable martingale, and

$$II' = P\left(\max_{0 \le l \le m-1} |M_l| > \frac{\lambda}{2}\right) \le \frac{4}{\lambda^2} \sum_{i=0}^{m-1} E\left(X_i^2\right)$$
$$\le \frac{4}{\lambda^2} \sum_{i=0}^{m-1} E\left[W_i^2 1_{\{|W_i| \le \lambda\}}\right]. \tag{66}$$

Let $U_i = \sum_{j=1}^{N_i} x_j Z_{ij}$. Using (57) and Remark A.2 (Appendix A), we get

$$E\left[U_i^2 1_{\{|U_i| \le \lambda\}}\right] \le 2 c_\alpha^* \sigma^\alpha \frac{1}{2-\alpha} \lambda^{2-\alpha} (s_{i+1} - s_i)$$
$$\times \sum_{j=1}^{N_i} |x_j|^\alpha |H_{ij} \cap B|. \tag{67}$$

As in Case 1, we obtain that

$$E\left[W_i^2 1_{\{|W_i| \le \lambda\}}\right] \le c_\alpha^* \sigma^\alpha \frac{2}{2-\alpha} \lambda^{2-\alpha} (s_{i+1} - s_i)$$
$$\times \sum_{j=1}^{N_i} E|\beta_{ij}|^\alpha |H_{ij} \cap B|, \tag{68}$$

and hence

$$II' \le 8 c_\alpha^* \sigma^\alpha \frac{1}{2-\alpha} \lambda^{-\alpha} E \int_0^T \int_B |X(s,x)|^\alpha dx\, ds. \tag{69}$$

We now treat II''. Note that $\{N_l = \sum_{i=0}^{l} Y_i, \mathcal{F}_{s_{l+1}}; 0 \le l \le m-1\}$ is a semimartingale and hence, by the submartingale inequality,

$$II'' \le \frac{2}{\lambda} E(N_{m-1}) = \frac{2}{\lambda} \sum_{i=0}^{m-1} E(Y_i). \tag{70}$$

To evaluate $E(Y_i)$, we note that, for almost all $\omega \in \Omega$,

$$E\left[W_i 1_{\{|W_i|\le\lambda\}} \mid \mathcal{F}_{s_i}\right](\omega) = E\left[\sum_{j=1}^{N_i} \beta_{ij}(\omega) Z_{ij} 1_{\{|\sum_{j=1}^{N_i}\beta_{ij}(\omega)Z_{ij}|\le\lambda\}}\right], \tag{71}$$

due to the independence between $\overline{\beta}_i$ and $\overline{Z}_i$. We let $U_i = \sum_{j=1}^{N_i} x_j Z_{ij}$ with $x_j = \beta_{ij}(\omega)$. Since $\alpha > 1$, $E(U_i) = 0$. Using (57) and Remark A.2, we obtain

$$\begin{aligned}\left|E\left[U_i 1_{\{|U_i|\le\lambda\}}\right]\right| &= \left|E\left[U_i 1_{\{|U_i|>\lambda\}}\right]\right| \le E\left[|U_i| 1_{\{|U_i|>\lambda\}}\right] \\ &\le c_\alpha^* \sigma^\alpha \frac{\alpha}{\alpha-1}\lambda^{1-\alpha}(s_{i+1}-s_i) \\ &\quad\times \sum_{j=1}^{N_i} |x_j|^\alpha |H_{ij}\cap B|.\end{aligned} \tag{72}$$

Hence, $E(Y_i) \le c_\alpha^* \sigma^\alpha (\alpha/(\alpha-1))\lambda^{1-\alpha}(s_{i+1} - s_i)\sum_{j=1}^{N_i} E|\beta_{ij}|^\alpha |H_{ij}\cap B|$ and

$$II'' \le c_\alpha^* \sigma^\alpha \frac{2\alpha}{\alpha-1}\lambda^{-\alpha} E\int_0^T\int_B |X(t,x)|^\alpha dx\, dt. \tag{73}$$

Case 3 ($\alpha = 1$). In this case we assume that $\beta = 0$. Hence, $U_i = \sum_{j=1}^{N_i} x_j Z_{ij}$ has a symmetric distribution for any $\overline{x} \in \mathbb{R}^{N_i}$. Using (71), it follows that $E[W_i 1_{\{|W_i|\le\lambda\}} \mid \mathcal{F}_{s_i}] = 0$ a.s. for all $i = 0, \dots, m-1$. Hence, $\{M_l = \sum_{i=0}^{l} W_i 1_{\{|W_i|\le\lambda\}}, \mathcal{F}_{s_{l+1}}; 0 \le l \le m-1\}$ is a zero-mean square integrable martingale. By the martingale maximal inequality,

$$II \le \frac{1}{\lambda^2} E\left[M_{m-1}^2\right] = \frac{1}{\lambda^2}\sum_{i=0}^{m-1} E\left[W_i^2 1_{\{|W_i|\le\lambda\}}\right]. \tag{74}$$

The result follows using (68). □

We now proceed to the construction of the stochastic integral. If $Y = \{Y(t)\}_{t\ge0}$ is a jointly measurable random process, we define

$$\|Y\|_{\alpha,T}^\alpha = \sup_{\lambda>0} \lambda^\alpha P\left(\sup_{t\in[0,T]} |Y(t)| > \lambda\right). \tag{75}$$

Let $X \in \mathcal{L}_\alpha$ be arbitrary. By Proposition 12, there exists a sequence $(X_n)_{n\ge1}$ of simple functions such that $\|X_n - X\|_\alpha \to 0$ as $n \to \infty$. Let $T > 0$ and $B \in \mathcal{B}_b(\mathbb{R}^d)$ be fixed. By linearity of the integral and Theorem 13,

$$\left\|I(X_n)(\cdot,B) - I(X_m)(\cdot,B)\right\|_{\alpha,T}^\alpha \le c_\alpha \left\|X_n - X_m\right\|_{\alpha,T,B}^\alpha \longrightarrow 0, \tag{76}$$

as $n, m \to \infty$. In particular, the sequence $\{I(X_n)(\cdot,B)\}_n$ is Cauchy in probability in the space $D[0,T]$ equipped with the sup-norm. Therefore, there exists a random element $Y(\cdot,B)$ in $D[0,T]$ such that, for any $\lambda > 0$,

$$P\left(\sup_{t\in[0,T]} \left|I(X_n)(t,B) - Y(t,B)\right| > \lambda\right) \longrightarrow 0. \tag{77}$$

Moreover, there exists a subsequence $(n_k)_k$ such that

$$\sup_{t\in[0,T]} \left|I\left(X_{n_k}\right)(t,B) - Y(t,B)\right| \longrightarrow 0 \quad \text{a.s.} \tag{78}$$

as $k \to \infty$. Hence, $Y(t,B)$ is $\mathcal{F}_t$-measurable for any $t \in [0,T]$. The process $Y(\cdot,B)$ does not depend on the sequence $(X_n)_n$ and can be extended to a càdlàg process on $[0,\infty)$, which is unique up to indistinguishability. We denote this extension by $I(X)(\cdot,B)$ and we write

$$I(X)(t,B) = \int_0^t\int_B X(s,x) Z(ds,dx). \tag{79}$$

If A and B are disjoint sets in $\mathcal{B}_b(\mathbb{R}^d)$, then

$$I(X)(t, A\cup B) = I(X)(t,A) + I(X)(t,B) \quad \text{a.s.} \tag{80}$$

Lemma 14. *Inequality* (46) *holds for any* $X \in \mathcal{L}_\alpha$.

Proof. Let $(X_n)_n$ be a sequence of simple functions such that $\|X_n - X\|_\alpha \to 0$. For fixed B, we denote $I(X) = I(X)(\cdot,B)$. We let $\|\cdot\|_\infty$ be the sup-norm on $D[0,T]$. For any $\varepsilon > 0$, we have

$$\begin{aligned}P\left(\|I(X)\|_\infty > \lambda\right) &\le P\left(\left\|I(X) - I(X_n)\right\|_\infty > \lambda\varepsilon\right) \\ &\quad + P\left(\left\|I(X_n)\right\|_\infty > \lambda(1-\varepsilon)\right).\end{aligned} \tag{81}$$

Multiplying by λ^α and using Theorem 13, we obtain

$$\begin{aligned}&\sup_{\lambda>0}\lambda^\alpha P\left(\|I(X)\|_\infty > \lambda\right) \\ &\quad\le \varepsilon^{-\alpha}\sup_{\lambda>0}\lambda^\alpha P\left(\left\|I(X) - I(X_n)\right\|_\infty > \lambda\right) \\ &\qquad + (1-\varepsilon)^{-\alpha} c_\alpha \left\|X_n\right\|_{\alpha,T,B}^\alpha.\end{aligned} \tag{82}$$

Let $n \to \infty$. Using (76) one can prove that $\sup_{\lambda>0}\lambda^\alpha P(\|I(X_n) - I(X)\|_\infty > \lambda) \to 0$. We obtain that $\sup_{\lambda>0}\lambda^\alpha P(\|I(X)\|_\infty > \lambda) \le (1-\varepsilon)^{-\alpha} c_\alpha \|X\|_{\alpha,T,B}^\alpha$. The conclusion follows letting $\varepsilon \to 0$. □

For an arbitrary Borel set $\mathcal{O} \subset \mathbb{R}^d$ (possibly $\mathcal{O} = \mathbb{R}^d$), we assume, in addition, that $X \in \mathcal{L}_\alpha$ satisfies the condition:

$$E\int_0^T\int_{\mathcal{O}} |X(t,x)|^\alpha dx\, dt < \infty, \quad \forall T > 0. \tag{83}$$

Then we can define $I(X)(\cdot,\mathcal{O})$ as follows. Let $\mathcal{O}_k = \mathcal{O}\cap E_k$ where $(E_k)_k$ is an increasing sequence of sets in $\mathcal{B}_b(\mathbb{R}^d)$ such that $\bigcup_k E_k = \mathbb{R}^d$. By (80), Lemma 14, and (83),

$$\begin{aligned}&\sup_{\lambda>0}\lambda^\alpha P\left(\sup_{t\le T}\left|I(X)(t,\mathcal{O}_k) - I(X)(t,\mathcal{O}_l)\right| > \lambda\right) \\ &\quad\le c_\alpha E\int_0^T\int_{\mathcal{O}_k\setminus\mathcal{O}_l} |X(t,x)|^\alpha dx\, dt \longrightarrow 0,\end{aligned} \tag{84}$$

as $k, l \to \infty$. This shows that $\{I(X)(\cdot,\mathcal{O}_k)\}_k$ is a Cauchy sequence in probability in the space $D[0,T]$ equipped with the sup-norm. We denote by $I(X)(\cdot,\mathcal{O})$ its limit. As above, this process can be extended to $[0,\infty)$ and $I(X)(t,\mathcal{O})$ is $\mathcal{F}_t$-measurable for any $t>0$. We denote

$$I(X)(t,\mathcal{O}) = \int_0^t \int_{\mathcal{O}} X(s,x)\, Z(ds,dx). \tag{85}$$

Similarly, to Lemma 14, one can prove that, for any $X \in \mathcal{L}_\alpha$ satisfying (83),

$$\sup_{\lambda>0} \lambda^\alpha P\left(\sup_{t\le T} |I(X)(t,\mathcal{O})| > \lambda\right) \le c_\alpha E \int_0^T \int_{\mathcal{O}} |X(t,x)|^\alpha dx\, dt. \tag{86}$$

5. The Truncated Noise

For the study of nonlinear equations, we need to develop a theory of stochastic integration with respect to another process Z_K which is defined by removing from Z the jumps whose modulus exceeds a fixed value $K>0$. More precisely, for any $B \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$, we define

$$Z_K(B) = \int_{B\times\{0<|z|\le K\}} zN(ds,dx,dz), \quad \text{if } \alpha \le 1, \tag{87}$$

$$Z_K(B) = \int_{B\times\{0<|z|\le K\}} z\widehat{N}(ds,dx,dz), \quad \text{if } \alpha > 1. \tag{88}$$

We treat separately the cases $\alpha \le 1$ and $\alpha > 1$.

5.1. The Case $\alpha \le 1$. Note that $\{Z_K(B); B \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)\}$ is an independently scattered random measure on $\mathbb{R}_+ \times \mathbb{R}^d$ with characteristic function given by

$$E\left(e^{iuZ_K(B)}\right) = \exp\left\{|B| \int_{|z|\le K} \left(e^{iuz} - 1\right) \nu_\alpha(dz)\right\}, \quad \forall u \in \mathbb{R}. \tag{89}$$

We first examine the tail of $Z_K(B)$.

Lemma 15. *For any set $B \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$,*

$$\sup_{\lambda>0} \lambda^\alpha P\left(|Z_K(B)| > \lambda\right) \le r_\alpha |B|, \tag{90}$$

where $r_\alpha > 0$ is a constant depending only on α (given by Lemma A.3).

Proof. This follows from Example 3.7 of [1]. We denote by $\nu_{\alpha,K}$ the restriction of ν_α to $\{z \in \mathbb{R}; 0 < |z| \le K\}$. Note that

$$\nu_{\alpha,K}(\{z \in \mathbb{R}; |z| > t\}) = \begin{cases} t^{-\alpha} - K^{-\alpha}, & \text{if } 0 < t \le K, \\ 0, & \text{if } t > K, \end{cases} \tag{91}$$

and hence $\sup_{t>0} t^\alpha \nu_{\alpha,K}(\{z \in \mathbb{R}; |z| > t\}) = 1$. Next we observe that we do not need to assume that the measure $\nu_{\alpha,K}$ is symmetric since we use a modified version of Lemma 2.1 of [24] given by Lemma A.3 (Appendix A). □

In fact, since the tail of $\nu_{\alpha,K}$ vanishes if $t > K$, we can obtain another estimate for the tail of $Z_K(B)$ which, together with (90), will allow us to control its pth moment for $p \in (\alpha, 1)$. This new estimate is given below.

Lemma 16. *If $\alpha < 1$, then*

$$P\left(|Z_K(B)| > u\right) \le \frac{\alpha}{1-\alpha} K^{1-\alpha} |B| u^{-1}, \quad \forall u > K. \tag{92}$$

If $\alpha = 1$, then $P(|Z_K(B)| > u) \le K|B|u^{-2}$ for all $u > K$.

Proof. We use the same idea as in Example 3.7 of [1]. For each $k \ge 1$, let $Z_{k,K}(B)$ be a random variable with characteristic function:

$$E\left(e^{iuZ_{k,K}(B)}\right) = \exp\left\{|B| \int_{\{k^{-1}<|z|\le K\}} \left(e^{iuz} - 1\right) \nu_\alpha(dz)\right\}. \tag{93}$$

Since $\{Z_{k,K}(B)\}_k$ converges in distribution to $Z_K(B)$, it suffices to prove the lemma for $Z_{k,K}(B)$. Let μ_k be the restriction of ν_α to $\{z; k^{-1} < |z| \le K\}$. Since μ_k is finite, $Z_{k,K}(B)$ has a compound Poisson distribution with

$$P\left(|Z_{k,K}(B)| > u\right) = e^{-|B|\mu_k(\mathbb{R})} \sum_{n\ge0} \frac{|B|^n}{n!} \mu_k^{*n}(\{z; |z| > u\}), \tag{94}$$

where μ_k^{*n} denotes the n-fold convolution. Note that

$$\mu_k^{*n}(\{z; |z| > u\}) = \left[\mu_k(\mathbb{R})\right]^n P\left(\left|\sum_{i=1}^n \eta_i\right| > u\right), \tag{95}$$

where $(\eta_i)_{i\ge1}$ are i.i.d. random variables with law $\mu_k/\mu_k(\mathbb{R})$.

Assume first that $\alpha < 1$. To compute $P(|\sum_{i=1}^n \eta_i| > u)$ we consider the intersection with the event $\{\max_{1\le i\le n}|\eta_i| > u\}$ and its complement. Note that $P(|\eta_i| > u) = 0$ for any $u > K$. Using this fact and Markov's inequality, we obtain that, for any $u > K$,

$$P\left(\left|\sum_{i=1}^n \eta_i\right| > u\right) \le P\left(\left|\sum_{i=1}^n \eta_i 1_{\{|\eta_i|\le u\}}\right| > u\right) \le \frac{1}{u} \sum_{i=1}^n E\left(|\eta_i| 1_{\{|\eta_i|\le u\}}\right). \tag{96}$$

Note that $P(|\eta_i| > s) \le (s^{-\alpha} - K^{-\alpha})/\mu_k(\mathbb{R})$ if $s \le K$. Hence, for any $u > K$

$$E\left(|\eta_i| 1_{\{|\eta_i|\le u\}}\right) \le \int_0^u P(|\eta_i| > s)\, ds = \int_0^K P(|\eta_i| > s)\, ds \le \frac{1}{\mu_k(\mathbb{R})} \frac{\alpha}{1-\alpha} K^{1-\alpha}. \tag{97}$$

Combining all these facts, we get that for any $u > K$

$$\mu_k^{*n}(\{z; |z| > u\}) \le \left[\mu_k(\mathbb{R})\right]^{n-1} \frac{\alpha}{1-\alpha} K^{1-\alpha} n u^{-1}, \tag{98}$$

and the conclusion follows from (94).

Assume now that $\alpha = 1$. In this case, $E(\eta_i 1_{\{|\eta_i|\le u\}}) = 0$ since η_i has a symmetric distribution. Using Chebyshev's inequality this time, we obtain

$$P\left(\left|\sum_{i=1}^{n}\eta_i\right| > u\right) \le P\left(\left|\sum_{i=1}^{n}\eta_i 1_{\{|\eta_i|\le u\}}\right| > u\right) \le \frac{1}{u^2}\sum_{i=1}^{n}E\left(\eta_i^2 1_{\{|\eta_i|\le u\}}\right). \quad (99)$$

The result follows as above using the fact that, for any $u > K$,

$$E\left(\eta_i^2 1_{\{|\eta_i|\le u\}}\right) \le 2\int_0^u sP\left(|\eta_i| > s\right) ds = 2\int_0^K sP\left(|\eta_i| > s\right) ds \le \frac{1}{\mu_k(\mathbb{R})}K. \quad (100)$$

□

Lemma 17. *If $\alpha < 1$ then*

$$E|Z_K(B)|^p \le C_{\alpha,p}K^{p-\alpha}|B| \quad \text{for any } p \in (\alpha, 1), \quad (101)$$

where $C_{\alpha,p}$ is a constant depending on α and p. If $\alpha = 1$, then

$$E|Z_K(B)|^p \le C_p K^{p-1}|B| \quad \text{for any } p \in (1,2), \quad (102)$$

where C_p is a constant depending on p.

Proof. Note that

$$E|Z_K(B)|^p = \int_0^\infty P\left(|Z_K(B)|^p > t\right) dt = p\int_0^\infty P\left(|Z_K(B)| > u\right) u^{p-1} du. \quad (103)$$

We consider separately the integrals for $u \le K$ and $u > K$. For the first integral we use (90):

$$\int_0^K P\left(|Z_K(B)| > u\right) u^{p-1} du \le r_\alpha |B| \int_0^K u^{-\alpha+p-1} du = r_\alpha |B| \frac{1}{p-\alpha}K^{p-\alpha}. \quad (104)$$

For the second one we use Lemma 16: if $\alpha < 1$ then

$$\int_K^\infty P\left(|Z_K(B)| > u\right) u^{p-1} du \le \frac{\alpha}{1-\alpha}K^{1-\alpha}|B| \int_K^\infty u^{p-2} du = \frac{\alpha}{(1-\alpha)(1-p)}|B| K^{p-\alpha}, \quad (105)$$

and if $\alpha = 1$, then

$$\int_K^\infty P\left(|Z_K(B)| > u\right) u^{p-1} du \le K|B| \int_K^\infty u^{p-3} du = |B| \frac{1}{2-p}K^{p-1}. \quad (106)$$

□

We now proceed to the construction of the stochastic integral with respect to Z_K. For this, we use the same method as for Z. Note that $\mathcal{F}_t^{Z_K} \subset \mathcal{F}_t$, where $\mathcal{F}_t^{Z_K}$ is the σ-field generated by $Z_K([0,s]\times A)$ for all $s \in [0,t]$ and $A \in \mathcal{B}_b(\mathbb{R}^d)$. For any $B \in \mathcal{B}_b(\mathbb{R}^d)$, we will work with a càdlàg modification of the Lévy process $\{Z_K(t,B) = Z_K([0,t]\times B); t \ge 0\}$.

If X is a simple process given by (40), we define

$$I_K(X)(t,B) = \int_0^t \int_B X(s,x) Z_K(ds,dx) \quad (107)$$

by the same formula (44) with Z replaced by Z_K. The following result shows that $I_K(X)(t,B)$ has the same tail behavior as $I(X)(t,B)$.

Proposition 18. *If X is a bounded simple process then*

$$\sup_{\lambda>0}\lambda^\alpha P\left(\sup_{t\in[0,T]}\left|I_K(X)(t,B)\right| > \lambda\right) \le d_\alpha E\int_0^T \int_B |X(t,x)|^\alpha dx\, dt, \quad (108)$$

for any $T > 0$ and $B \in \mathcal{B}_b(\mathbb{R}^d)$, where d_α is a constant depending only on α.

Proof. As in the proof of Theorem 13, it is enough to prove that

$$P\left(\max_{l=0,\dots,m-1}\left|\sum_{i=0}^{l}\sum_{j=1}^{N_i}\beta_{ij}Z_{ij}^*\right| > \lambda\right) \le d_\alpha \lambda^{-\alpha}\sum_{i=0}^{m-1}\left(s_{i+1}-s_i\right)\sum_{j=1}^{N_i}E\left|\beta_{ij}\right|^\alpha \left|H_{ij}\cap B\right|, \quad (109)$$

where $Z_{ij}^* = Z_K((s_i, s_{i+1}]\times(H_{ij}\cap B))$. This reduces to showing that $U_i^* = \sum_{j=1}^{N_i} x_j Z_{ij}^*$ satisfies an inequality similar to (57) for any $\overline{x} \in \mathbb{R}^{N_i}$; that is,

$$P\left(\left|U_i^*\right| > \lambda\right) \le d_\alpha^* \lambda^{-\alpha}\left(s_{i+1}-s_i\right)\sum_{j=1}^{N_i}\left|x_j\right|^\alpha \left|H_{ij}\cap B\right|, \quad (110)$$

for any $\lambda > 0$, for some $d_\alpha^* > 0$. We first examine the tail of Z_{ij}^*. By (90),

$$P\left(\left|Z_{ij}^*\right| > \lambda\right) \le r_\alpha\left(s_{i+1}-s_i\right)K_{ij}\lambda^{-\alpha}, \quad (111)$$

where $K_{ij} = |H_{ij}\cap B|$. Letting $\eta_{ij} = K_{ij}^{-1/\alpha}Z_{ij}^*$, we obtain that, for any $u > 0$,

$$P\left(\left|\eta_{ij}\right| > u\right) \le r_\alpha\left(s_{i+1}-s_i\right)u^{-\alpha}, \quad \forall j = 1,\dots,N_i. \quad (112)$$

By Lemma A.3 (Appendix A), it follows that, for any $\lambda > 0$,

$$P\left(\left|\sum_{j=1}^{N_i}b_j\eta_{ij}\right| > \lambda\right) \le r_\alpha^2\left(s_{i+1}-s_i\right)\sum_{j=1}^{N_i}\left|b_j\right|^\alpha \lambda^{-\alpha}, \quad (113)$$

for any sequence $(b_j)_{j=1,\dots,N_i}$ of real numbers. Inequality (110) (with $d_\alpha^* = r_\alpha^2$) follows by applying this to $b_j = x_j K_{ij}^{1/\alpha}$. □

In view of the previous result and Proposition 12, for any process $X \in \mathscr{L}_\alpha$, we can construct the integral

$$I_K(X)(t,B) = \int_0^t \int_B X(s,x) Z_K(ds,dx) \tag{114}$$

in the same manner as $I(X)(t,B)$, and this integral satisfies (108). If in addition the process $X \in \mathscr{L}_\alpha$ satisfies (83), then we can define the integral $I_K(X)(t,\mathcal{O})$ for an arbitrary Borel set $\mathcal{O} \subset \mathbb{R}^d$ (possibly $\mathcal{O} = \mathbb{R}^d$). This integral will satisfy an inequality similar to (108) with B replaced by $\mathcal{O}$.

The appealing feature of $I_K(X)(t,B)$ is that we can control its moments, as shown by the next result.

Theorem 19. *If $\alpha < 1$, then for any $p \in (\alpha, 1)$ and for any $X \in \mathscr{L}_p$,*

$$E|I_K(X)(t,B)|^p \le C_{\alpha,p} K^{p-\alpha} E \int_0^t \int_B |X(s,x)|^p dx\, ds, \tag{115}$$

for any $t > 0$ and $B \in \mathscr{B}_b(\mathbb{R}^d)$, where $C_{\alpha,p}$ is a constant depending on α, p. If $\mathcal{O} \subset \mathbb{R}^d$ is an arbitrary Borel set and we assume, in addition, that the process $X \in \mathscr{L}_p$ satisfies

$$E\int_0^T \int_{\mathcal{O}} |X(s,x)|^p dx\, ds < \infty, \quad \forall T > 0, \tag{116}$$

then inequality (115) *holds with B replaced by $\mathcal{O}$.*

Proof. Consider the following steps.

Step 1. Suppose that X is an elementary process of the form (39). Then $I_K(X)(t,B) = YZ_K(H)$ where $H = (t \wedge a, t \wedge b] \times (A \cap B)$. Note that $Z_K(H)$ is independent of $\mathscr{F}_a$. Hence, $Z_K(H)$ is independent of Y. Let P_Y denote the law of Y. By Fubini's theorem,

$$\begin{aligned} E|YZ_K(H)|^p &= p\int_0^\infty P(|YZ_K(H)| > u)\, u^{p-1} du \\ &= p\int_{\mathbb{R}} \left(\int_0^\infty P(|yZ_K(H)| > u)\, u^{p-1} du \right) P_Y(dy). \end{aligned} \tag{117}$$

We evaluate the inner integral. We split this integral into two parts, for $u \le K|y|$ and $u > K|y|$, respectively. For the first integral, we use (90). For the second one, we use Lemma 16. Therefore, the inner integral is bounded by

$$\begin{aligned} r_\alpha |y|^\alpha |H| \int_0^{K|y|} u^{-\alpha+p-1} du & \\ + \frac{\alpha}{1-\alpha} |y| K^{1-\alpha} |H| & \\ \times \int_{K|y|}^\infty u^{p-2} du = C'_{\alpha,p} K^{p-\alpha} |y|^p |H|, & \end{aligned}$$

$$\begin{aligned} E|YZ_K(H)|^p &\le pC'_{\alpha,p} K^{p-\alpha} |H|\, E|Y|^p \\ &= C_{\alpha,p} K^{p-\alpha} E\int_0^t \int_B |X(s,x)|^p dx\, ds. \end{aligned} \tag{118}$$

Step 2. Suppose now that X is a simple process of the form (40). Then $X(t,x) = \sum_{i=0}^{N-1} \sum_{j=1}^{m_i} X_{ij}(t,x)$ where $X_{ij}(t,x) = 1_{(t_i,t_{i+1}]}(t) 1_{A_{ij}}(x) Y_{ij}$.

Using the linearity of the integral, the inequality $|a+b|^p \le |a|^p + |b|^p$, and the result obtained in Step 1 for the elementary processes X_{ij}, we get

$$\begin{aligned} E|I_K(X)(t,B)|^p &\le E\sum_{i=0}^{N-1}\sum_{j=1}^{m_i} \left|I_K(X_{ij})(t,B)\right|^p \\ &\le C_{\alpha,p} K^{p-\alpha} E\sum_{i=0}^{N-1}\sum_{j=1}^{m_i} \int_0^t \int_B \left|X_{ij}(s,x)\right|^p dx\, ds \\ &= C_{\alpha,p} K^{p-\alpha} E\int_0^t \int_B |X(s,x)|^p dx\, ds. \end{aligned} \tag{119}$$

Step 3. Let $X \in \mathscr{L}_p$ be arbitrary. By Proposition 12, there exists a sequence $(X_n)_n$ of bounded simple processes such that $\|X_n - X\|_p \to 0$. Since $\alpha < p$, it follows that $\|X_n - X\|_\alpha \to 0$. By the definition of $I_K(X)(t,B)$ there exists a subsequence $\{n_k\}_k$ such that $\{I_K(X_{n_k})(t,B)\}_k$ converges to $I_K(X)(t,B)$ a.s. Using Fatou's lemma and the result obtained in Step 2 (for the simple processes X_{n_k}), we get

$$\begin{aligned} E|I_K(X)(t,B)|^p &\le \liminf_{k\to\infty} E\left|I_K(X_{n_k})(t,B)\right|^p \\ &\le C_{\alpha,p} K^{p-\alpha} \liminf_{k\to\infty} E\int_0^t \int_B \left|X_{n_k}(s,x)\right|^p dx\, ds \\ &= C_{\alpha,p} K^{p-\alpha} E\int_0^t \int_B |X(s,x)|^p dx\, ds. \end{aligned} \tag{120}$$

Step 4. Suppose that $X \in \mathscr{L}_p$ satisfies (116). Let $\mathcal{O}_k = \mathcal{O} \cap E_k$ where $(E_k)_k$ is an increasing sequence of sets in $\mathscr{B}_b(\mathbb{R}^d)$ such that $\bigcup_{k\ge1} E_k = \mathbb{R}^d$. By the definition of $I_K(X)(t,\mathcal{O})$, there exists a subsequence $(k_i)_i$ such that $\{I_K(X)(t,\mathcal{O}_{k_i})\}_i$ converges to $I_K(X)(t,\mathcal{O})$ a.s. Using Fatou's lemma, the result obtained in Step 3 (for $B = \mathcal{O}_{k_i}$) and the monotone convergence theorem, we get

$$\begin{aligned} E|I_K(X)(t,\mathcal{O})|^p &\le \liminf_{i\to\infty} E\left|I_K(X)(t,\mathcal{O}_{k_i})\right|^p \\ &\le C_{\alpha,p} K^{p-\alpha} \liminf_{i\to\infty} E\int_0^t \int_{\mathcal{O}_{k_i}} |X(s,x)|^p dx\, ds \\ &= C_{\alpha,p} K^{p-\alpha} E\int_0^t \int_{\mathcal{O}} |X(s,x)|^p dx\, ds. \end{aligned} \tag{121}$$

□

Remark 20. Finding a similar moment inequality for the cases $\alpha = 1$ and $p \in (1,2)$ remains an open problem. The argument used in Step 2 above relies on the fact that $p < 1$. Unfortunately, we could not find another argument to cover the case $p > 1$.

5.2. The Case $\alpha>1$. In this case, the construction of the integral with respect to Z_K relies on an integral with respect to $\widehat{N}$ which exists in the literature. We recall briefly the definition of this integral. For more details, see Section 1.2.2 of [6], Section 24.2 of [25], or Section 8.7 of [12].

Let $\mathbb{E} = \mathbb{R}^d \times (\mathbb{R} \setminus \{0\})$ endowed with the measure $\mu(dx,dz) = dx\nu_\alpha(dz)$ and let $\mathscr{B}_b(\mathbb{E})$ be the class of bounded Borel sets in $\mathbb{E}$. For a simple process $Y = \{Y(t,x,z); t \geq 0, (x,z) \in \mathbb{E}\}$, the integral $I^{\widehat{N}}(Y)(t,B)$ is defined in the usual way, for any $t > 0$, $B \in \mathscr{B}_b(\mathbb{E})$. The process $I^{\widehat{N}}(Y)(\cdot,B)$ is a (càdlàg) zero-mean square-integrable martingale with quadratic variation

$$\left[I^{\widehat{N}}(Y)(\cdot,B)\right]_t = \int_0^t \int_B |Y(s,x,z)|^2 N(ds,dx,dz) \quad (122)$$

and predictable quadratic variation

$$\left\langle I^{\widehat{N}}(Y)(\cdot,B)\right\rangle_t = \int_0^t \int_B |Y(s,x,z)|^2 \nu_\alpha(dz)\,dx\,ds. \quad (123)$$

By approximation, this integral can be extended to the class of all $\mathscr{P} \times \mathscr{B}(\mathbb{R}\setminus\{0\})$-measurable processes Y such that for any $T > 0$ and $B \in \mathscr{B}_b(\mathbb{E})$

$$\|Y\|_{2,T,B}^2 := E\int_0^T \int_B |Y(s,x,z)|^2 \nu_\alpha(dz)\,dx\,ds < \infty. \quad (124)$$

The integral is a martingale with the same quadratic variations as above and has the isometry property: $E|I^{\widehat{N}}(Y)(t,B)|^2 = \|Y\|_{2,T,B}^2$. If, in addition, $\|Y\|_{2,T,\mathbb{E}} < \infty$, then the integral can be extended to $\mathbb{E}$. By the Burkholder-Davis-Gundy inequality for discontinuous martingales, for any $p \geq 1$,

$$E\sup_{t\leq T}\left|I^{\widehat{N}}(Y)(t,\mathbb{E})\right|^p \leq C_p E\left[I^{\widehat{N}}(Y)(\cdot,\mathbb{E})\right]_T^{p/2}. \quad (125)$$

The previous inequality is not suitable for our purposes. A more convenient inequality can be obtained for *another* stochastic integral, constructed for $p \in [1,2]$ fixed, as suggested on page 293 of [6]. More precisely, one can show that, for any bounded simple process Y,

$$\begin{aligned} &E\sup_{t\leq T}\left|I^{\widehat{N}}(Y)(t,\mathbb{E})\right|^p \\ &\leq C_p E\int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}\setminus\{0\}} |Y(t,x,z)|^p \nu_\alpha(dz)\,dx\,dt \\ &=: |Y|_{p,T,\mathbb{E}}^p, \end{aligned} \quad (126)$$

where C_p is the constant appearing in (125) (see Lemma 8.22 of [12]).

By the usual procedure, the integral can be extended to the class of all $\mathscr{P} \times \mathscr{B}(\mathbb{R}\setminus\{0\})$-measurable processes Y such that $[Y]_{p,T,\mathbb{E}} < \infty$. The integral is defined as an element in the space $L^p(\Omega; D[0,T])$ and will be denoted by

$$I^{\widehat{N},p}(Y)(t,\mathbb{E}) = \int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}\setminus\{0\}} Y(s,x,z)\widehat{N}(ds,dx,dz). \quad (127)$$

Its appealing feature is that it satisfies inequality (126).

From now on, we fix $p \in [1,2]$. Based on (88), for any $B \in \mathscr{B}_b(\mathbb{R}^d)$, we let

$$\begin{aligned} I_K(X)(t,B) &= \int_0^t \int_B X(s,x) Z_K(ds,dx) \\ &= \int_0^t \int_B \int_{\{|z|\leq K\}} X(s,x) z\widehat{N}(ds,dx,dz), \end{aligned} \quad (128)$$

for any predictable process $X = \{X(t,x); t \geq 0, x \in \mathbb{R}^d\}$ for which the rightmost integral is well defined. Letting $Y(t,x,z) = X(t,x)z1_{\{0<|z|\leq K\}}$, we see that this is equivalent to saying that $p > \alpha$ and $X \in \mathscr{L}_p$. By (126),

$$E\sup_{t\leq T}\left|I_K(X)(t,B)\right|^p \leq C_{\alpha,p}K^{p-\alpha}E\int_0^T \int_B |X(s,x)|^p dx\,ds, \quad (129)$$

where $C_{\alpha,p} = C_p\alpha/(p-\alpha)$. If, in addition, the process $X \in \mathscr{L}_p$ satisfies (116) then (129) holds with B replaced by $\mathcal{O}$, for an arbitrary Borel set $\mathcal{O} \subset \mathbb{R}^d$.

Note that (129) is the counterpart of (115) for the case $\alpha > 1$. Together, these two inequalities will play a crucial role in Section 6.

Table 1 summarizes all the conditions.

6. The Main Result

In this section, we state and prove the main result regarding the existence of a mild solution of (1). For this result, $\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$. For any $t > 0$, we denote

$$J_p(t) = \sup_{x\in\mathcal{O}} \int_{\mathcal{O}} G(t,x,y)^p dy. \quad (130)$$

Theorem 21. *Let $\alpha \in (0,2)$, $\alpha \neq 1$. Assume that for any $T > 0$*

$$\lim_{h\to 0}\int_0^T \int_{\mathcal{O}} \left|G(t,x,y) - G(t+h,x,y)\right|^p dy\,dt = 0, \quad \forall x \in \mathcal{O}, \quad (131)$$

$$\lim_{|h|\to 0}\int_0^T \int_{\mathcal{O}} \left|G(t,x,y) - G(t,x+h,y)\right|^p dy\,dt = 0, \quad \forall x \in \mathcal{O}, \quad (132)$$

$$\int_0^T J_p(t)\,dt < \infty, \quad (133)$$

for some $p \in (\alpha,1)$ if $\alpha < 1$, or for some $p \in (\alpha,2]$ if $\alpha > 1$. Then (1) has a mild solution. Moreover, there exists a sequence

Table 1: Conditions for $I_K(X)(t,B)$ to be well defined.

	$\alpha < 1$	$\alpha > 1$
B is bounded	$X \in \mathscr{L}_\alpha$	$X \in \mathscr{L}_p$ for some $p \in (\alpha, 2]$
$B = \mathscr{O}$ is unbounded	$X \in \mathscr{L}_\alpha$ and X satisfies (83)	$X \in \mathscr{L}_p$ and X satisfies (116) for some $p \in (\alpha, 2]$

$(\tau_K)_{K\geq 1}$ *of stopping times with* $\tau_K \uparrow \infty$ *a.s. such that, for any* $T > 0$ *and* $K \geq 1$,

$$\sup_{(t,x)\in[0,T]\times\mathscr{O}} E\left(|u(t,x)|^p 1_{\{t\leq\tau_K\}}\right) < \infty. \quad (134)$$

Example 22 (heat equation). Let $L = \partial/\partial t - (1/2)\Delta$. Then $G(t,x,y) \leq \overline{G}(t,x-y)$ where $\overline{G}(t,x)$ is the fundamental solution of $Lu = 0$ on $\mathbb{R}^d$. Condition (133) holds if $p < 1 + 2/d$. If $\alpha < 1$, this condition holds for any $p \in (\alpha, 1)$. If $\alpha > 1$, this condition holds for any $p \in (\alpha, 1 + 2/d]$, as long as α satisfies (6). Conditions (131) and (132) hold by the continuity of the function G in t and x, by applying the dominated convergence theorem. To justify the application of this theorem, we use the trivial bound $(2\pi t)^{-dp/2}$ for both $G(t+h,x,y)^p$ and $G(t,x+h,y)^p$, which introduces the extra condition $dp < 2$. Unfortunately, we could not find another argument for proving these two conditions (In the case of the heat equation on $\mathbb{R}^d$, Lemmas A.2 and A.3 of [6] estimate the integrals appearing in (132) and (131), with $p = 1$ in (131). These arguments rely on the structure of $\overline{G}$ and cannot be used when $\mathscr{O}$ is a bounded domain.).

Example 23 (parabolic equations). Let $L = \partial/\partial t - \mathscr{L}$ where $\mathscr{L}$ is given by (31). Assuming (32), we see that (133) holds if $p < 1 + 2/d$. The same comments as for the heat equation apply here as well (Although in a different framework, a condition similar to (131) was probably used in the proof of Theorem 12.11 of [12] (page 217) for the claim $\lim_{s\to t} E|J_3(X)(s) - J_3(X)(t)|^p_{L^p(\mathscr{O})} = 0$. We could not see how to justify this claim, unless $dp < 2$.).

Example 24 (heat equation with fractional power of the Laplacian). Let $L = \partial/\partial t + (-\Delta)^\gamma$ for some $\gamma > 0$. By Lemma B.23 of [12], if $\alpha > 1$, then condition (133) holds for any $p \in (\alpha, 1+2\gamma/d)$, provided that α satisfies (36) (This condition is the same as in Theorem 12.19 of [12], which examines the same equation using the approach based on Hilbert-space valued solution.).

To verify conditions (131) and (132), we use the continuity of G in t and x and apply the dominated convergence theorem. To justify the application of this theorem, we use the trivial bound $C_{d,\gamma} t^{-dp/(2\gamma)}$ for both $G(t+h,x,y)^p$ and $G(t,x+h,y)^p$, which introduces the extra condition $dp < 2\gamma$. This bound can be seen from (33), using the fact that $\mathscr{G}(t,x,y) \leq \overline{\mathscr{G}}(t,x-y)$ where $\mathscr{G}$ and $\overline{\mathscr{G}}$ are the fundamental solutions of $\partial u/\partial t - \Delta u = 0$ on $\mathscr{O}$ and $\mathbb{R}^d$, respectively. (In the case of the same equation on $\mathbb{R}^d$, elementary estimates for the time and space increments of $\overline{G}$ can be obtained directly from (35), as on page 196 of [26]. These arguments cannot be used when $\mathscr{O}$ is a bounded domain.)

The remaining part of this section is dedicated to the proof of Theorem 21. The idea is to solve first the equation with the truncated noise Z_K (yielding a mild solution u_K) and then identify a sequence $(\tau_K)_{K\geq 1}$ of stopping times with $\tau_K \uparrow \infty$ a.s. such that, for any $t > 0$, $x \in \mathscr{O}$, and $L > K$, $u_K(t,x) = u_L(t,x)$ a.s. on the event $\{t \leq \tau_K\}$. The final step is to show that process u defined by $u(t,x) = u_K(t,x)$ on $\{t \leq \tau_K\}$ is a mild solution of (1). A similar method can be found in Section 9.7 of [12] using an approach based on stochastic integration of operator-valued processes, with respect to Hilbert-space-valued processes, which is different from our approach.

Since σ is a Lipschitz function, there exists a constant $C_\sigma > 0$ such that

$$|\sigma(u) - \sigma(v)| \leq C_\sigma |u - v|, \quad \forall u, v \in \mathbb{R}. \quad (135)$$

In particular, letting $D_\sigma = C_\sigma \vee |\sigma(0)|$, we have

$$|\sigma(u)| \leq D_\sigma (1 + |u|), \quad \forall u \in \mathbb{R}. \quad (136)$$

For the proof of Theorem 21, we need a specific construction of the Poisson random measure N, taken from [13]. We review briefly this construction.

Let $(\mathscr{O}_k)_{k\geq 1}$ be a partition of $\mathbb{R}^d$ with sets in $\mathscr{B}_b(\mathbb{R}^d)$ and let $(U_j)_{j\geq 1}$ be a partition of $\mathbb{R} \setminus \{0\}$ such that $\nu_\alpha(U_j) < \infty$ for all $j \geq 1$. We may take $U_j = \Gamma_{j-1}$ for all $j \geq 1$. Let $(E_i^{j,k}, X_i^{j,k}, Z_i^{j,k})_{i,j,k\geq 1}$ be independent random variables defined on a probability space $(\Omega, \mathscr{F}, P)$, such that

$$P\left(E_i^{j,k} > t\right) = e^{-\lambda_{j,k} t}, \qquad P\left(X_i^{j,k} \in B\right) = \frac{|B \cap \mathscr{O}_k|}{|\mathscr{O}_k|},$$
$$P\left(Z_i^{j,k} \in \Gamma\right) = \frac{|\Gamma \cap U_j|}{|U_j|}, \quad (137)$$

where $\lambda_{j,k} = |\mathscr{O}_k| \nu_\alpha(U_j)$. Let $T_i^{j,k} = \sum_{l=1}^i E_l^{j,k}$ for all $i \geq 1$. Then

$$N = \sum_{i,j,k\geq 1} \delta_{(T_i^{j,k}, X_i^{j,k}, Z_i^{j,k})} \quad (138)$$

is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d \times (\mathbb{R} \setminus \{0\})$ with intensity $dt dx \nu_\alpha(dz)$.

This section is organized as follows. In Section 6.1 we prove the existence of the solution of the equation with truncated noise Z_K. Sections 6.2 and 6.3 contain the proof of Theorem 21 when $\alpha < 1$ and $\alpha > 1$, respectively.

6.1. The Equation with Truncated Noise. In this section, we fix $K > 0$ and we consider the equation:

$$Lu(t,x) = \sigma(u(t,x)) \dot{Z}_K(t,x), \quad t > 0, \ x \in \mathscr{O} \quad (139)$$

with zero initial conditions and Dirichlet boundary conditions. A mild solution of (139) is a predictable process u which satisfies (2) with Z replaced by Z_K. For the next result, $\mathcal{O}$ can be a bounded domain in $\mathbb{R}^d$ or $\mathcal{O} = \mathbb{R}^d$ (with no boundary conditions).

Theorem 25. *Under the assumptions of Theorem 21,* (139) *has a unique mild solution* $u = \{u(t,x); t \geq 0, x \in \mathcal{O}\}$. *For any* $T > 0$,

$$\sup_{(t,x)\in[0,T]\times\mathcal{O}} E|u(t,x)|^p < \infty, \tag{140}$$

and the map $(t,x) \mapsto u(t,x)$ *is continuous from* $[0,T]\times\mathcal{O}$ *into* $L^p(\Omega)$.

Proof. We use the same argument as in the proof of Theorem 13 of [27], based on a Picard iteration scheme. We define $u_0(t,x) = 0$ and

$$u_{n+1}(t,x) = \int_0^t \int_{\mathcal{O}} G(t-s,x,y)\,\sigma(u_n(s,y))\,Z_K(ds,dy) \tag{141}$$

for any $n \geq 0$. We prove by induction on $n \geq 0$ that (i) $u_n(t,x)$ is well defined; (ii) $K_n(t) := \sup_{(t,x)\in[0,T]\times\mathcal{O}} E|u_n(t,x)|^p < \infty$ for any $T > 0$; (iii) $u_n(t,x)$ is $\mathcal{F}_t$-measurable for any $t > 0$ and $x \in \mathcal{O}$; (iv) the map $(t,x) \mapsto u_n(t,x)$ is continuous from $[0,T]\times\mathcal{O}$ into $L^p(\Omega)$ for any $T > 0$.

The statement is trivial for $n = 0$. For the induction step, assume that the statement is true for n. By an extension to random fields of Theorem 30, Chapter IV of [28], u_n has a jointly measurable modification. Since this modification is $(\mathcal{F}_t)_t$-adapted (in the sense of (iii)), it has a predictable modification (using an extension of Proposition 3.21 of [12] to random fields). We work with this modification, that we call also u_n.

We prove that (i)–(iv) hold for u_{n+1}. To show (i), it suffices to prove that $X_n \in \mathcal{L}_p$, where $X_n(s,y) = 1_{[0,t]}(s)G(t-s,x,y)\sigma(u_n(s,y))$. By (136) and (133),

$$\begin{aligned} &E\int_0^t \int_{\mathcal{O}} |X_n(s,y)|^p dy\,ds \\ &\quad\leq D_\sigma^p 2^{p-1}(1+K_n(t)) \int_0^t J_p(t-s)\,ds < \infty. \end{aligned} \tag{142}$$

In addition, if $\mathcal{O} = \mathbb{R}^d$, we have to prove that X_n satisfies (83) if $\alpha < 1$, or (116) if $\alpha > 1$ (see Table 1). If $\alpha < 1$, this follows as above, since $\alpha < p$ and hence $\sup_{(t,x)\in[0,T]\times\mathcal{O}} E|u(t,x)|^\alpha < \infty$; the argument for $\alpha > 1$ is similar.

Combined with the moment inequality (115) (or (129)), this proves (ii), since

$$\begin{aligned} &E|u_{n+1}(t,x)|^p \\ &\quad\leq C_{\alpha,p} K^{p-\alpha} D_\sigma^p 2^{p-1}(1+K_n(t)) \int_0^t J_p(t-s)\,ds, \end{aligned} \tag{143}$$

for any $x \in \mathcal{O}$. Property (iii) follows by the construction of the integral I_K.

To prove (iv), we first show the right continuity in t. Let $h > 0$. Writing the interval $[0,t+h]$ as the union of $[0,t]$ and $(t,t+h]$, we obtain that $E|u_{n+1}(t+h,x) - u_{n+1}(t,x)|^p \leq 2^{p-1}(I_1(h) + I_2(h))$, where

$$\begin{aligned} I_1(h) &= E\left|\int_0^t \int_{\mathcal{O}} \left(G(t+h-s,x,y) - G(t-s,x,y)\right)\right. \\ &\qquad \left.\times\, \sigma(u_n(s,y))\,Z_K(ds,dy)\right|^p, \\ I_2(h) &= E\left|\int_t^{t+h} \int_{\mathcal{O}} G(t+h-s,x,y)\,\sigma\right. \\ &\qquad \left.\times\,(u_n(s,y))\,Z_K(ds,dy)\right|^p. \end{aligned} \tag{144}$$

Using again (136) and the moment inequality (115) (or (129)), we obtain

$$\begin{aligned} I_1(h) &\leq D_\sigma^p 2^{p-1}(1+K_n(t)) \\ &\quad\times \int_0^t \int_{\mathcal{O}} |G(s+h,x,y) - G(s,x,y)|^p dy\,ds, \\ I_2(h) &\leq D_\sigma^p 2^{p-1}(1+K_n(t)) \\ &\quad\times \int_0^h \int_{\mathcal{O}} G(s,x,y)^p dy\,ds. \end{aligned} \tag{145}$$

It follows that both $I_1(h)$ and $I_2(h)$ converge to 0 as $h \to 0$, using (131) for $I_1(h)$ and the Dominated Convergence Theorem and (133) for $I_2(h)$, respectively. The left continuity in t is similar, by writing the interval $[0,t-h]$ as the difference between $[0,t]$ and $(t-h,t]$ for $h > 0$. For the continuity in x, similarly as above, we see that $E|u_{n+1}(t,x+h) - u_{n+1}(t,x)|^p$ is bounded by

$$\begin{aligned} &D_\sigma^p 2^{p-1}(1+K_n(t)) \\ &\quad\times \int_0^t \int_{\mathcal{O}} |G(s,x+h,y) - G(s,x,y)|^p dy\,ds, \end{aligned} \tag{146}$$

which converges to 0 as $|h| \to 0$ due to (132). This finishes the proof of (iv).

We denote $M_n(t) = \sup_{x\in\mathcal{O}} E|u_n(t,x)|^p$. Similarly to (143), we have

$$M_n(t) \leq C_1 \int_0^t (1 + M_{n-1}(s))\,J_p(t-s)\,ds, \quad \forall n \geq 1, \tag{147}$$

where $C_1 = C_{\alpha,p} K^{p-\alpha} D_\sigma^p 2^{p-1}$. By applying Lemma 15 of Erratum to [27] with $f_n = M_n$, $k_1 = 0$, $k_2 = 1$, and $g(s) = CJ_p(s)$, we obtain that

$$\sup_{n\geq 0} \sup_{t\in[0,T]} M_n(t) < \infty, \quad \forall T > 0. \tag{148}$$

We now prove that $\{u_n(t,x)\}_n$ converges in $L^p(\Omega)$, uniformly in $(t,x) \in [0,T]\times\mathcal{O}$. To see this, let $U_n(t) = \sup_{x\in\mathcal{O}} E|u_{n+1}(t,x) - u_n(t,x)|^p$ for $n \geq 0$. Using the moment inequality (115) (or (129)) and (135), we have

$$U_n(t) \leq C_2 \int_0^t U_{n-1}(s)\,J_p(t-s)\,ds, \tag{149}$$

where $C_2 = C_{\alpha,p}K^{p-\alpha}C_\sigma^p$. By Lemma 15 of Erratum to [27], $\sum_{n\geq 0} U_n(t)^{1/p}$ converges uniformly on $[0,T]$ (Note that this lemma is valid for all $p > 0$.).

We denote by $u(t,x)$ the limit of $u_n(t,x)$ in $L^p(\Omega)$. One can show that u satisfies properties (ii)–(iv) listed above. So u has a predictable modification. This modification is a solution of (139). To prove uniqueness, let v be another solution and denote $H(t) = \sup_{x\in\mathcal{O}} E|u(t,x) - v(t,x)|^p$. Then

$$H(t) \leq C_2 \int_0^t H(s) J_p(t-s)\,ds. \tag{150}$$

Using (133), it follows that $H(t) = 0$ for all $t > 0$. □

6.2. Proof of Theorem 21: Case α<1. In this case, for any $t > 0$ and $B \in \mathcal{B}_b(\mathbb{R}^d)$, we have (see (21))

$$Z(t,B) = \int_{[0,t]\times B\times(\mathbb{R}\setminus\{0\})} zN(ds,dx,dz). \tag{151}$$

The characteristic function of $Z(t,B)$ is given by

$$E\left(e^{iuZ(t,B)}\right) = \exp\left\{t|B|\int_{\mathbb{R}\setminus\{0\}}\left(e^{iuz}-1\right)\nu_\alpha(dz)\right\}, \quad \forall u \in \mathbb{R}. \tag{152}$$

Note that $\{Z(t,B)\}_{t\geq 0}$ is *not* a compound Poisson process since ν_α is infinite.

We introduce the stopping times $(\tau_K)_{K\geq 1}$, as on page 239 of [13]:

$$\tau_K(B) = \inf\{t > 0; |Z(t,B) - Z(t-,B)| > K\}, \tag{153}$$

where $Z(t-,B) = \lim_{s\uparrow t} Z(s,B)$. Clearly, $\tau_L(B) \geq \tau_K(B)$ for all $L > K$.

We first investigate the relationship between Z and Z_K and the properties of $\tau_K(B)$. Using construction (138) of N and definition (87) of Z_K, we have

$$\begin{aligned} Z(t,B) &= \sum_{i,j,k\geq 1} Z_i^{j,k} 1_{\{T_i^{j,k}\leq t\}} 1_{\{X_i^{j,k}\in B\}} =: \sum_{j,k\geq 1} Z^{j,k}(t,B), \\ Z_K(t,B) &= \sum_{i,j,k\geq 1} Z_i^{j,k} 1_{\{|Z_i^{j,k}|\leq K\}} 1_{\{T_i^{j,k}\leq t\}} 1_{\{X_i^{j,k}\in B\}}. \end{aligned} \tag{154}$$

We observe that $\{Z^{j,k}(t,B)\}_{t\geq 0}$ is a compound Poisson process with

$$E\left(e^{iuZ^{j,k}(t,B)}\right) = \exp\left\{t\left|\mathcal{O}_k\cap B\right|\int_{U_j}\left(e^{iuz}-1\right)\nu_\alpha(dz)\right\}, \quad \forall u\in\mathbb{R}. \tag{155}$$

Note that $\tau_K(B) > T$ means that all the jumps of $\{Z(t,B)\}_{t\geq 0}$ in $[0,T]$ are smaller than K in modulus; that is, $\{\tau_K(B) > T\} = \{\omega; |Z_i^{j,k}(\omega)| \leq K$ for all $i,j,k \geq 1$ for which $T_i^{j,k}(\omega) \leq T$ and $X_i^{j,k}(\omega) \in B\}$. Hence, on $\{\tau_K(B) > T\}$,

$$Z([0,t]\times A) = Z_K([0,t]\times A) = Z_L([0,t]\times A), \tag{156}$$

for any $L > K$, $t \in [0,T]$, and $A \in \mathcal{B}_b(\mathbb{R}^d)$ with $A \subset B$. Using an approximation argument and the construction of the integrals $I(X)$ and $I_K(X)$, it follows that, for any $X \in \mathcal{L}_\alpha$ and for any $L > K$, a.s. on $\{\tau_K(B) > T\}$, we have

$$I(X)(T,B) = I_K(X)(T,B) = I_L(X)(T,B). \tag{157}$$

The next result gives the probability of the event $\{\tau_K(B) > T\}$.

Lemma 26. *For any $T > 0$ and $B \in \mathcal{B}_b(\mathbb{R}^d)$,*

$$P(\tau_K(B) > T) = \exp\left(-T|B|K^{-\alpha}\right). \tag{158}$$

Consequently, $\lim_{K\to\infty} P(\tau_K(B) > T) = 1$ and $\lim_{K\to\infty}\tau_K(B) = \infty$ a.s.

Proof. Note that $\{\tau_K(B) > T\} = \bigcap_{j,k\geq 1}\{\tau_K^{j,k}(B) > T\}$, where

$$\tau_K^{j,k}(B) = \inf\left\{t > 0; \left|Z^{j,k}(t,B) - Z^{j,k}(t-,B)\right| > K\right\}. \tag{159}$$

Since $\nu_\alpha(\{z; |z| > K\}) = K^{-\alpha}$ and $(\tau_K^{j,k}(B))_{j,k\geq 1}$ are independent, it is enough to prove that, for any $j,k \geq 1$,

$$P\left(\tau_K^{j,k}(B) > T\right) = \exp\left\{-T\left|B\cap\mathcal{O}_k\right|\nu_\alpha\left(\{z;|z|>K\}\cap U_j\right)\right\}. \tag{160}$$

Note that $\{\tau_K^{j,k}(B) > T\} = \{\omega; |Z_i^{j,k}(\omega)| \leq K$ for all i for which $T_i^{j,k} \leq T$ and $X_i^{j,k} \in B\}$ and $(T_n^{j,k})_{n\geq 1}$ are the jump times of a Poisson process with intensity $\lambda_{j,k}$. Hence,

$$\begin{aligned} P\left(\tau_K^{j,k}(B) > T\right) &= \sum_{n\geq 0}\sum_{m=0}^{n}\sum_{I\subset\{1,\ldots,n\},\mathrm{card}(I)=m} P\left(T_n^{j,k} \leq T < T_{n+1}^{j,k}\right) \\ &\quad\times P\left(\bigcap_{i\in I}\left\{X_i^{j,k}\in B\right\}\right) \\ &\quad\times P\left(\bigcap_{i\in I}\left\{\left|Z_i^{j,k}\right|\leq K\right\}\right) \\ &\quad\times P\left(\bigcap_{i\in I^c}\left\{X_i^{j,k}\notin B\right\}\right) \\ &= \sum_{n\geq 0} e^{-\lambda_{j,k}T}\frac{\left(\lambda_{j,k}T\right)^n}{n!} \\ &\quad\times\left[1 - P\left(X_1^{j,k}\in B\right)P\left(\left|Z_1^{j,k}\right|>K\right)\right]^n \\ &= \exp\left\{-\lambda_{j,k}TP\left(X_1^{j,k}\in B\right)P\left(\left|Z_1^{j,k}\right|>K\right)\right\}, \end{aligned} \tag{161}$$

which yields (160).

To prove the last statement, let $A_k^{(n)} = \{\tau_K(B) > n\}$. Then $P(\overline{\lim}_K A_K^{(n)}) \geq \overline{\lim}_K P(A_K^{(n)}) = 1$ for any $n \geq 1$, and hence $P(\bigcap_{n\geq 1}\overline{\lim}_K A_K^{(n)}) = 1$. Hence, with probability 1, for any

n, there exists some K_n such that $\tau_{K_n} > n$. Since $(\tau_K)_K$ is nondecreasing, this proves that $\tau_K \to \infty$ with probability 1. □

Remark 27. The construction of $\tau_K(B)$ given above is due to [13] (in the case of a symmetric measure ν_α). This construction relies on the fact that B is a bounded set. Since $Z(t,\mathbb{R}^d)$ (and consequently $\tau_K(\mathbb{R}^d)$) is not well defined, we could not see why this construction can also be used when $B = \mathbb{R}^d$, as it is claimed in [13]. To avoid this difficulty, one could try to use an increasing sequence $(E_n)_n$ of sets in $\mathscr{B}_b(\mathbb{R}^d)$ with $\bigcup_n E_n = \mathbb{R}^d$. Using (157) with $B = E_n$ and letting $n \to \infty$, we obtain that $I(X)(t,\mathbb{R}^d) = I_K(t,\mathbb{R}^d)$ a.s. on $\{t \le \tau_K\}$, where $\tau_K = \inf_{n\ge1}\tau_K(E_n)$. But $P(\tau_K > t) \le P(\underline{\lim}_n\{\tau_K(E_n) > t\}) \le \underline{\lim}_n P(\tau_K(E_n) > t) = \lim_n \exp(-t|E_n|K^{-\alpha}) = 0$ for any $t > 0$, which means that $\tau_K = 0$ a.s. Finding a suitable sequence $(\tau_K)_K$ of stopping times which could be used in the case $\mathcal{O} = \mathbb{R}^d$ remains an open problem.

In what follows, we denote $\tau_K = \tau_K(\mathcal{O})$. Let u_K be the solution of (139), whose existence is guaranteed by Theorem 25.

Lemma 28. *Under the assumptions of Theorem 21, for any $t > 0$, $x \in \mathcal{O}$, and $L > K$,*

$$u_K(t,x) = u_L(t,x) \quad \text{a.s. on } \{t \le \tau_K\}. \tag{162}$$

Proof. By the definition of u_L and (157),

$$\begin{aligned} u_L(t,x) &= \int_0^t \int_{\mathcal{O}} G(t-s,x,y)\,\sigma(u_L(s,y))\,Z_L(ds,dy) \\ &= \int_0^t \int_{\mathcal{O}} G(t-s,x,y)\,\sigma(u_L(s,y))\,Z_K(ds,dy) \end{aligned} \tag{163}$$

a.s. on the event $\{t \le \tau_K\}$. Using the definition of u_K and Proposition C.1 (Appendix C), we obtain that, with probability 1,

$$\begin{aligned} &(u_K(t,x) - u_L(t,x))\,1_{\{t\le\tau_K\}} \\ &\quad = 1_{\{t\le\tau_K\}} \int_0^t \int_{\mathcal{O}} G(t-s,x,y) \\ &\qquad \times (\sigma(u_K(s,y)) - \sigma(u_L(s,y))) \\ &\qquad \times 1_{\{s\le\tau_K\}} Z_K(ds,dy). \end{aligned} \tag{164}$$

Let $M(t) = \sup_{x\in\mathcal{O}} E(|u_K(t,x) - u_L(t,x)|^p 1_{\{t\le\tau_K\}})$. Using the moment inequality (115) and the Lipschitz condition (135), we get

$$M(t) \le C \int_0^t J_p(t-s)\,M(s)\,ds, \tag{165}$$

where $C = C_{\alpha,p}K^{p-\alpha}C_\sigma^p$. Using (133), it follows that $M(t) = 0$ for all $t > 0$. □

For any $t > 0$ and $x \in \mathcal{O}$, let $\Omega_{t,x} = \bigcap_{L>K}\{t \le \tau_K(t), u_K(t,x) \ne u_L(t,x)\}$, where L and K are positive integers. Let $\Omega_{t,x}^* = \Omega_{t,x} \cap \{\lim_{K\to\infty}\tau_K = \infty\}$.

By Lemmas 26 and 28, $P(\Omega_{t,x}^*) = 1$.

The next result concludes the proof of Theorem 21.

Proposition 29. *Under the assumptions of Theorem 21, the process $u = \{u(t,x); t \ge 0, x \in \mathcal{O}\}$ defined by*

$$\begin{aligned} u(\omega,t,x) &= u_K(\omega,t,x), \quad \text{if } \omega \in \Omega_{t,x}^*,\ t \le \tau_K(\omega) \\ u(\omega,t,x) &= 0, \quad \text{if } \omega \notin \Omega_{t,x}^* \end{aligned} \tag{166}$$

is a mild solution of (1).

Proof. We first prove that u is predictable. Note that

$$u(t,x) = \lim_{K\to\infty} \left(u_K(t,x)\,1_{\{t\le\tau_K\}}\right) 1_{\Omega_{t,x}^*}. \tag{167}$$

The process $X(\omega,t,x) = 1_{\{t\le\tau_K\}}(\omega)$ is clearly predictable, being in the class $\mathscr{C}$ defined in Remark 11. By the definition of $\Omega_{t,x}$, since u_K, u_L are predictable, it follows that $(\omega,t,x) \mapsto 1_{\Omega_{t,x}^*}(\omega)$ is $\mathscr{P}$-measurable. Hence, u is predictable.

We now prove that u satisfies (2). Let $t > 0$ and $x \in \mathcal{O}$ be arbitrary. Using (157) and Proposition C.1 (Appendix C), with probability 1, we have

$$\begin{aligned} &1_{\{t\le\tau_K\}} u(t,x) \\ &\quad = 1_{\{t\le\tau_K\}} u_K(t,x) \\ &\quad = 1_{\{t\le\tau_K\}} \int_0^t \int_{\mathcal{O}} G(t-s,x,y)\,\sigma \\ &\qquad\qquad \times (u_K(s,y))\,Z_K(ds,dy) \\ &\quad = 1_{\{t\le\tau_K\}} \int_0^t \int_{\mathcal{O}} G(t-s,x,y)\,\sigma \\ &\qquad\qquad \times (u_K(s,y))\,Z(ds,dy) \\ &\quad = 1_{\{t\le\tau_K\}} \int_0^t \int_{\mathcal{O}} G(t-s,x,y)\,\sigma \\ &\qquad\qquad \times (u_K(s,y))\,1_{\{s\le\tau_K\}} Z(ds,dy) \\ &\quad = 1_{\{t\le\tau_K\}} \int_0^t \int_{\mathcal{O}} G(t-s,x,y)\,\sigma \\ &\qquad\qquad \times (u(s,y))\,1_{\{s\le\tau_K\}} Z(ds,dy) \\ &\quad = 1_{\{t\le\tau_K\}} \int_0^t \int_{\mathcal{O}} G(t-s,x,y)\,\sigma \\ &\qquad\qquad \times (u(s,y))\,Z(ds,dy). \end{aligned} \tag{168}$$

For the second last equality, we used the fact that processes $X(s,y) = 1_{[0,t]}(s)G(t-s,x,y)\sigma(u_K(s,y))1_{\{s\le\tau_K\}}$ and $Y(s,y) = 1_{[0,t]}(s)G(t-s,x,y)\sigma(u(s,y))1_{\{s\le\tau_K\}}$ are modifications of each other (i.e., $X(s,y) = Y(s,y)$ a.s. for all $s > 0$, $y \in \mathcal{O}$), and, hence, $[X-Y]_{\alpha,t,\mathcal{O}} = 0$ and $I(X)(t,\mathcal{O}) = I(Y)(t,\mathcal{O})$ a.s. The conclusion follows letting $K \to \infty$, since $\tau_K \to \infty$ a.s. □

6.3. Proof of Theorem 21: Case $\alpha>1$. In this case, for any $t>0$ and $B \in \mathscr{B}_b(\mathbb{R}^d)$, we have (see (22))

$$Z(t,B) = \int_{[0,t]\times B\times(\mathbb{R}\setminus\{0\})} z\widehat{N}(ds,dx,dz). \tag{169}$$

To introduce the stopping times $(\tau_K)_{K\geq 1}$ we use the same idea as in Section 9.7 of [12].

Let $M(t,B) = \sum_{j\geq 1}(L_j(t,B) - EL_j(t,B))$ and $P(t,B) = L_0(t,B)$, where $L_j(t,B) = L_j([0,t]\times B)$ was defined in Section 2. Note that $\{M(t,B)\}_{t\geq 0}$ is a zero-mean square-integrable martingale and $\{P(t,B)\}_{t\geq 0}$ is a compound Poisson process with $E[P(t,B)] = t|B|\mu$ where $\mu = \int_{|z|>1} z\nu_\alpha(dz) = \beta(\alpha/(\alpha-1))$. With this notation,

$$Z(t,B) = M(t,B) + P(t,B) - t|B|\mu. \tag{170}$$

We let $M_K(t,B) = P_K(t,B) - E[P_K(t,B)] = P_K(t,B) - t|B|\mu_K$, where

$$P_K(t,B) = \int_{[0,t]\times B\times(\mathbb{R}\setminus\{0\})} z1_{\{1<|z|\leq K\}} N(ds,dx,dz) \tag{171}$$

and $\mu_K = \int_{1<|z|\leq K} z\nu_\alpha(dz)$. Recalling definition (88) of Z_K, it follows that

$$Z_K(t,B) = M(t,B) + P_K(t,B) - t|B|\mu_K. \tag{172}$$

For any $K>0$, we let

$$\tau_K(B) = \inf\{t>0; |P(t,B) - P(t-,B)| > K\}, \tag{173}$$

where $P(t-,B) = \lim_{s\uparrow t} P(s,B)$.

Lemma 26 holds again, but its proof is simpler than in the case $\alpha<1$, since $\{P(t,B)\}_{t\geq 0}$ is a compound Poisson process. By (138),

$$\begin{aligned} P(t,B) &= \sum_{i,j,k\geq 1} Z_i^{j,k} 1_{\{|Z_i^{j,k}|>1\}} 1_{\{T_i^{j,k}\leq t\}} 1_{\{X_i^{j,k}\in B\}}, \\ P_K(t,B) &= \sum_{i,j,k\geq 1} Z_i^{j,k} 1_{\{1<|Z_i^{j,k}|\leq K\}} 1_{\{T_i^{j,k}\leq t\}} 1_{\{X_i^{j,k}\in B\}}. \end{aligned} \tag{174}$$

Hence, on $\{\tau_K(B) > T\}$, for any $L>K$, $t\in[0,T]$, and $A \in \mathscr{B}_b(\mathbb{R}^d)$ with $A\subset B$,

$$P([0,t]\times A) = P_K([0,t]\times A) = P_L([0,t]\times A). \tag{175}$$

Let $b_K = \mu - \mu_K = \int_{|z|>K} z\nu_\alpha(dz)$. Using (170) and (172), it follows that

$$\begin{aligned} Z([0,t]\times A) &= Z_K([0,t]\times A) - t|A|b_K \\ &= Z_L([0,t]\times A) - t|A|b_L \end{aligned} \tag{176}$$

for any $L>K$, $t\in[0,T]$, and $A\in\mathscr{B}_b(\mathbb{R}^d)$ with $A\subset B$. Let $p\in(\alpha,2]$ be fixed. Using an approximation argument and the construction of the integrals $I(X)$ and $I_K(X)$, it follows that, for any $X\in\mathscr{L}_\alpha$ and for any $L>K$, a.s. on $\{\tau_K(B)>T\}$, we have

$$\begin{aligned} I(X)(T,B) &= I_K(X)(T,B) - b_K\int_0^T\int_{\mathscr{O}} X(s,y)\,dy\,ds \\ &= I_L(X)(T,B) - b_L\int_0^T\int_{\mathscr{O}} X(s,y)\,dy\,ds. \end{aligned} \tag{177}$$

We denote $\tau_K = \tau_K(\mathscr{O})$. We consider the following equation:

$$\begin{aligned} Lu(t,x) &= \sigma(u(t,x))\dot{Z}_K(t,x) - b_K\sigma(u(t,x)), \\ &\qquad t>0,\ x\in\mathscr{O} \end{aligned} \tag{178}$$

with zero initial conditions and Dirichlet boundary conditions. A mild solution of (178) is a predictable process u which satisfies

$$\begin{aligned} u(t,x) &= \int_0^t\int_{\mathscr{O}} G(t-s,x,y)\sigma(u(s,y))Z_K(ds,dy) \\ &\quad - b_K\int_0^t\int_{\mathscr{O}} G(t-s,x,y)\sigma(u(s,y))\,dy\,ds \quad \text{a.s.} \end{aligned} \tag{179}$$

for any $t>0$, $x\in\mathscr{O}$. The existence and uniqueness of a mild solution of (178) can be proved similarly to Theorem 25. We omit these details. We denote this solution by v_K.

Lemma 30. *Under the assumptions of Theorem 21, for any $t>0$, $x\in\mathscr{O}$, and $L>K$,*

$$v_K(t,x) = v_L(t,x) \quad \text{a.s. on } \{t\leq\tau_K\}. \tag{180}$$

Proof. By the definition of v_L and (177), a.s. on the event $\{t\leq\tau_K\}$, $v_L(t,x)$ is equal to

$$\begin{aligned} &\int_0^t\int_{\mathscr{O}} G(t-s,x,y)\sigma(v_L(s,y))Z_L(ds,dy) \\ &\quad - b_L\int_0^t\int_{\mathscr{O}} G(t-s,x,y)\sigma(v_L(s,y))\,dy\,ds \\ &= \int_0^t\int_{\mathscr{O}} G(t-s,x,y)\sigma(v_L(s,y))Z_K(ds,dy) \\ &\quad - b_K\int_0^t\int_{\mathscr{O}} G(t-s,x,y)\sigma(v_L(s,y))\,dy\,ds. \end{aligned} \tag{181}$$

Using the definition of v_K and Proposition C.1 (Appendix C), we obtain that, with probability 1,

$$\begin{aligned} &(v_K(t,x) - v_L(t,x))1_{\{t\leq\tau_K\}} \\ &= 1_{\{t\leq\tau_K\}}\bigg(\int_0^t\int_{\mathscr{O}} G(t-s,x,y)(\sigma(v_K(s,y))) \\ &\qquad - \sigma(v_L(s,y)))1_{\{s\leq\tau_K\}}Z_K(ds,dy) \\ &\quad - \int_0^t\int_{\mathscr{O}} G(t-s,x,y)(\sigma(v_K(s,y))) \\ &\qquad - \sigma(v_L(s,y)))1_{\{s\leq\tau_K\}}\,dy\,ds\bigg). \end{aligned} \tag{182}$$

Letting $M(t) = \sup_{x\in\mathcal{O}} E(|v_K(t,x) - v_L(t,x)|^p 1_{\{t\le\tau_K\}})$, we see that $M(t) \le 2^{p-1}(E|A(t,x)|^p + E|B(t,x)|^p)$ where

$$\begin{aligned} A(t,x) = \int_0^t \int_{\mathcal{O}} & G(t-s,x,y)(\sigma(v_K(s,y))) \\ & - \sigma(v_L(s,y))\, 1_{\{s\le\tau_K\}} Z_K(ds,dy), \\ B(t,x) = \int_0^t \int_{\mathcal{O}} & G(t-s,x,y)(\sigma(v_K(s,y))) \\ & - \sigma(v_L(s,y))\, 1_{\{s\le\tau_K\}} dy\, ds. \end{aligned} \tag{183}$$

We estimate separately the two terms. For the first term, we use the moment inequality (129) and the Lipschitz condition (135). We get

$$\sup_{x\in\mathcal{O}} E|A(t,x)|^p \le C \int_0^t J_p(t-s) M(s)\, ds, \tag{184}$$

where $C = C_{\alpha,p} K^{p-\alpha} C_\sigma^p$. For the second term, we use Hölder's inequality $|\int f g d\mu| \le (\int |f|^p d\mu)^{1/p} (\int |g|^q d\mu)^{1/q}$ with $f(s,y) = G(t-s,x,y)^{1/p} \; (\sigma(v_K(s,y))) - \sigma(v_L(s,y)) 1_{\{s\le\tau_K\}}$ and $g(s,y) = G(t-s,x,y)^{1/q}$, where $p^{-1} + q^{-1} = 1$. Hence,

$$\begin{aligned} |B(t,x)|^p \le\; & C_\sigma^p K_t^{p/q} \\ & \times \int_0^t \int_{\mathcal{O}} G(t-s,x,y) \\ & \times |v_K(s,y) - v_L(s,y)|^p 1_{\{s\le\tau_K\}} dy\, ds, \end{aligned} \tag{185}$$

where $K_t = \int_0^t J_1(s) ds < \infty$ (Since $\mathcal{O}$ is a bounded set, $J_1(s) \le C J_p(s)^{1/p}$ where C is a constant depending on $|\mathcal{O}|$ and p. Since $p > 1$, $\int_0^t J_p(s)^{1/p} ds \le c_t (\int_0^t J_p(s) ds)^{1/p} < \infty$ by (133). This shows that $K_t < \infty$.). Therefore,

$$\sup_{x\in\mathcal{O}} E|B(t,x)|^p \le C_t \int_0^t J_1(t-s) M(s)\, ds, \tag{186}$$

where $C_t = C_\sigma^p K_t^{p/q}$. From (184) and (186), we obtain that

$$M(t) \le C_t' \int_0^t \left(J_p(t-s) + J_1(t-s) \right) M(s)\, ds, \tag{187}$$

where $C_t' = 2^{p-1}(C \vee C_t)$. This implies that $M(t) = 0$ for all $t > 0$. □

For any $t > 0$ and $x \in \mathcal{O}$, we let $\Omega_{t,x} = \bigcap_{L>K} \{t \le \tau_K, v_K(t,x) \neq v_L(t,x)\}$ where K and L are positive integers, and $\Omega_{t,x}^* = \Omega_{t,x} \cap \{\lim_{K\to\infty} \tau_K = \infty\}$. By Lemma 30, $P(\Omega_{t,x}^*) = 1$.

Proposition 31. *Under the assumptions of Theorem 21, the process $u = \{u(t,x); t \ge 0, x \in \mathcal{O}\}$ defined by*

$$\begin{aligned} u(\omega,t,x) &= v_K(\omega,t,x), \quad \text{if } \omega \in \Omega_{t,x}^*,\ t \le \tau_K(\omega), \\ u(\omega,t,x) &= 0, \quad \text{if } \omega \notin \Omega_{t,x}^* \end{aligned} \tag{188}$$

is a mild solution of (1).

Proof. We proceed as in the proof of Proposition 29. In this case, with probability 1, we have

$$\begin{aligned} & 1_{\{t\le\tau_K\}} u(t,x) \\ & = 1_{\{t\le\tau_K\}} \left(\int_0^t \int_{\mathcal{O}} G(t-s,x,y) \sigma(u(s,y)) Z(ds,dy) \right. \\ & \qquad \left. - b_K \int_0^t \int_{\mathcal{O}} G(t-s,x,y) \sigma(u(s,y))\, dy\, ds \right). \end{aligned} \tag{189}$$

The conclusion follows letting $K \to \infty$, since $\tau_K \to \infty$ a.s. and $b_K \to 0$. □

Appendices

A. Some Auxiliary Results

The following result is used in the proof of Theorem 13.

Lemma A.1. *If X has a $S_\alpha(\sigma,\beta,0)$ distribution then*

$$\lambda^\alpha P(|X| > \lambda) \le c_\alpha^* \sigma^\alpha, \quad \forall \lambda > 0, \tag{A.1}$$

where $c_\alpha^* > 0$ is a constant depending only on α.

Proof. Consider the following steps.

Step 1. We first prove the result for $\sigma = 1$. We treat only the right tail, with the left tail being similar. We denote X by X_β to emphasize the dependence on β. By Property 1.2.15 of [18], $\lim_{\lambda\to\infty} \lambda^\alpha P(X_\beta > \lambda) = C_\alpha((1+\beta)/2)$, where $C_\alpha = (\int_0^\infty x^{-\alpha} \sin x dx)^{-1}$. We use the fact that, for any $\beta \in [-1,1]$,

$$P\left(X_\beta > \lambda\right) \le P(X_1 > \lambda), \quad \forall \lambda > \lambda_\alpha \tag{A.2}$$

for some $\lambda_\alpha > 0$ (see Property 1.2.14 of [18] or Section 1.5 of [29]). Since $\lim_{\lambda\to\infty} \lambda^\alpha P(X_1 > \lambda) = C_\alpha$, there exists $\lambda_\alpha^* > \lambda_\alpha$ such that

$$\lambda^\alpha P(X_1 > \lambda) < 2C_\alpha, \quad \forall \lambda > \lambda_\alpha^*. \tag{A.3}$$

It follows that $\lambda^\alpha P(X_\beta > \lambda) < 2C_\alpha$ for all $\lambda > \lambda_\alpha^*$ and $\beta \in [-1,1]$. Clearly, for all $\lambda \in (0,\lambda_\alpha^*]$ and $\beta \in [-1,1]$, $\lambda^\alpha P(X_\beta > \lambda) \le \lambda^\alpha \le (\lambda_\alpha^*)^\alpha$.

Step 2. We now consider the general case. Since X/σ has a $S_\alpha(1,\beta,0)$ distribution, by Step 1, it follows that $\lambda^\alpha P(|X| > \sigma\lambda) \le c_\alpha^*$ for any $\lambda > 0$. The conclusion follows multiplying by σ^α. □

In the proof of Theorem 13 and Lemma A.3 below, we use the following remark, due to Adam Jakubowski (personal communication).

Remark A.2. Let X be a random variable such that $P(|X| > \lambda) \leq K\lambda^{-\alpha}$ for all $\lambda > 0$, for some $K > 0$ and $\alpha \in (0, 2)$. Then, for any $A > 0$,

$$\begin{aligned} E\left(|X| 1_{\{|X| \leq A\}}\right) &\leq \int_0^A P(|X| > t)\,dt \\ &\leq K \frac{1}{1-\alpha} A^{1-\alpha}, \quad \text{if } \alpha < 1, \\ E\left(|X| 1_{\{|X| > A\}}\right) &\leq \int_A^\infty P(|X| > t)\,dt + AP(|X| > A) \\ &\leq K \frac{\alpha}{\alpha - 1} A^{1-\alpha}, \quad \text{if } \alpha > 1, \\ E\left(X^2 1_{\{|X| \leq A\}}\right) &\leq 2\int_0^A tP(|X| > t)\,dt \\ &\leq K \frac{2}{2-\alpha} A^{2-\alpha}, \quad \text{for any } \alpha \in (0, 2). \end{aligned} \tag{A.4}$$

The next result is a generalization of Lemma 2.1 of [24] to the case of nonsymmetric random variables. This result is used in the proof of Lemma 15 and Proposition 18.

Lemma A.3. *Let $(\eta_k)_{k\geq 1}$ be independent random variables such that*

$$\sup_{\lambda > 0} \lambda^\alpha P(|\eta_k| > \lambda) \leq K, \quad \forall k \geq 1 \tag{A.5}$$

for some $K > 0$ and $\alpha \in (0, 2)$. If $\alpha > 1$, we assume that $E(\eta_k) = 0$ for all k, and, if $\alpha = 1$, we assume that η_k has a symmetric distribution for all k. Then for any sequence $(a_k)_{k\geq 1}$ of real numbers, we have

$$\sup_{\lambda > 0} \lambda^\alpha P\left(\left|\sum_{k\geq 1} a_k \eta_k\right| > \lambda\right) \leq r_\alpha K \sum_{k\geq 1} |a_k|^\alpha, \tag{A.6}$$

where $r_\alpha > 0$ is a constant depending only on α.

Proof. We consider the intersection of the event on the left-hand side of (A.6) with the event $\{\sup_{k\geq 1} |a_k \eta_k| > \lambda\}$ and its complement. Hence,

$$\begin{aligned} &P\left(\left|\sum_{k\geq 1} a_k \eta_k\right| > \lambda\right) \\ &\quad \leq \sum_{k\geq 1} P(|a_k \eta_k| > \lambda) + P\left(\left|\sum_{k\geq 1} a_k \eta_k 1_{\{|a_k \eta_k| \leq \lambda\}}\right| > \lambda\right) \\ &\quad =: I + II. \end{aligned} \tag{A.7}$$

Using (A.5), we have $I \leq K\lambda^{-\alpha} \sum_{k\geq 1} |a_k|^\alpha$. To treat II, we consider 3 cases.

Case 1 ($\alpha < 1$). By Markov's inequality and Remark A.2, we have

$$II \leq \frac{1}{\lambda} \sum_{k\geq 1} |a_k| E\left(|\eta_k| 1_{\{|a_k \eta_k| \leq \lambda\}}\right) \leq K \frac{1}{1-\alpha} \lambda^{-\alpha} \sum_{k\geq 1} |a_k|^\alpha. \tag{A.8}$$

Case 2 ($\alpha > 1$). Let $X = \sum_{k\geq 1} a_k \eta_k 1_{\{|a_k \eta_k| \leq \lambda\}}$. Since $E(\sum_{k\geq 1} a_k \eta_k) = 0$,

$$\begin{aligned} |E(X)| &= \left|E\left(\sum_{k\geq 1} a_k \eta_k 1_{\{|a_k \eta_k| > \lambda\}}\right)\right| \\ &\leq \sum_{k\geq 1} |a_k| E\left(|\eta_k| 1_{\{|a_k \eta_k| > \lambda\}}\right) \\ &\leq \frac{K\alpha}{\alpha - 1} \lambda^{1-\alpha} \sum_{k\geq 1} |a_k|^\alpha, \end{aligned} \tag{A.9}$$

where we used Remark A.2 for the last inequality. From here, we infer that

$$|E(X)| < \frac{\lambda}{2}, \quad \text{for any } \lambda > \lambda_\alpha, \tag{A.10}$$

where $\lambda_\alpha^\alpha = 2K(\alpha/(\alpha - 1)) \sum_{k\geq 1} |a_k|^\alpha$. By Chebyshev's inequality, for any $\lambda > \lambda_\alpha$,

$$\begin{aligned} II &= P(|X| > \lambda) \leq P(|X - E(X)| > \lambda - |E(X)|) \\ &\leq \frac{4}{\lambda^2} E|X - E(X)|^2 \leq \frac{4}{\lambda^2} \sum_{k\geq 1} a_k^2 E\left(\eta_k^2 1_{\{|a_k \eta_k| \leq \lambda\}}\right) \\ &\leq \frac{8K}{2-\alpha} \lambda^{-\alpha} \sum_{k\geq 1} |a_k|^\alpha, \end{aligned} \tag{A.11}$$

using Remark A.2 for the last inequality. On the other hand, if $\lambda \in (0, \lambda_\alpha]$,

$$II = P(|X| > \lambda) \leq 1 \leq \lambda_\alpha^\alpha \lambda^{-\alpha} = 2K \frac{\alpha}{\alpha - 1} \lambda^{-\alpha} \sum_{k\geq 1} |a_k|^\alpha. \tag{A.12}$$

Case 3 ($\alpha = 1$). Since η_k has a symmetric distribution, we can use the original argument of [24]. □

B. Fractional Power of the Laplacian

Let $\overline{G}(t, x)$ be the fundamental solution of $\partial u/\partial t + (-\Delta)^\gamma u = 0$ on $\mathbb{R}^d$, $\gamma > 0$.

Lemma B.1. *For any $p > 1$, there exist some constants $c_1, c_2 > 0$ depending on d, p, and γ such that*

$$c_1 t^{-(d/2\gamma)(p-1)} \leq \int_{\mathbb{R}^d} \overline{G}(t, x)^p dx \leq c_2 t^{-(d/2\gamma)(p-1)}. \tag{B.1}$$

Proof. The upper bound is given by Lemma B.23 of [12]. For the lower bound, we use the scaling property of the functions $(g_{t,\gamma})_{t>0}$. We have

$$\begin{aligned}
&\overline{G}(t,x)\\
&= \int_0^\infty \frac{1}{(4\pi t^{1/\gamma} r)^{d/2}} \exp\left(-\frac{|x|^2}{4t^{1/\gamma} r}\right) g_{1,\gamma}(r)\,dr\\
&\geq \int_1^\infty \frac{1}{(4\pi t^{1/\gamma} r)^{d/2}} \exp\left(-\frac{|x|^2}{4t^{1/\gamma} r}\right) g_{1,\gamma}(r)\,dr\\
&\geq \frac{1}{(4\pi t^{1/\gamma})^{d/2}} \exp\left(-\frac{|x|^2}{4t^{1/\gamma}}\right) C_{d,\gamma}
\end{aligned} \tag{B.2}$$

with $C_{d,\gamma} := \int_1^\infty r^{-d/2} g_{1,\gamma}(r)\,dr < \infty$,

and hence

$$\begin{aligned}
&\int_{\mathbb{R}^d} \overline{G}(t,x)^p dx \geq c'_{d,\gamma,p} t^{-dp/2\gamma}\\
&\quad\times \int_{\mathbb{R}^d} \exp\left(-\frac{p|x|^2}{4t^{1/\gamma}}\right) dx = c_{d,p,\gamma} t^{-(d/2\gamma)(p-1)}.
\end{aligned} \tag{B.3}$$

□

C. A Local Property of the Integral

The following result is the analogue of Proposition 8.11 of [12].

Proposition C.1. *Let $T > 0$ and $\mathcal{O} \subset \mathbb{R}^d$ be a Borel set. Let $X = \{X(t,x); t \geq 0, x \in \mathbb{R}^d\}$ be a predictable process such that $X \in \mathcal{L}_\alpha$ if $\alpha < 1$, or $X \in \mathcal{L}_p$ for some $p \in (\alpha, 2]$ if $\alpha > 1$. If $\mathcal{O}$ is unbounded, assume in addition that X satisfies* (83) *if $\alpha < 1$, or X satisfies* (116) *for some $p \in (\alpha, 2)$, if $\alpha > 1$. Suppose that there exists an event $A \in \mathcal{F}_T$ such that*

$$X(\omega, t, x) = 0, \quad \forall \omega \in A,\ t \in [0,T],\ x \in \mathcal{O}. \tag{C.1}$$

Then for any $K > 0$, $I(X)(T,\mathcal{O}) = I_K(X)(T,\mathcal{O}) = 0$ a.s. on A.

Proof. We only prove the result for $I(X)$, with the proof for $I_K(X)$ being the same. Moreover, we include only the argument for $\alpha < 1$; the case $\alpha > 1$ is similar. The idea is to reduce the argument to the case when X is a simple process, as in the proof Proposition of 8.11 of [12].

Step 1. We show that the proof can be reduced to the case of a bounded set $\mathcal{O}$. Let $X_n(t,x) = X(t,x)1_{\mathcal{O}_n}(x)$ where $\mathcal{O}_n = \mathcal{O} \cap E_n$ and $(E_n)_n$ is an increasing sequence of sets in $\mathcal{B}_b(\mathbb{R}^d)$ such that $\bigcup_n E_n = \mathbb{R}^d$. Then $X_n \in \mathcal{L}_\alpha$ satisfies (C.1). By the dominated convergence theorem,

$$E\int_0^T \int_{\mathcal{O}} |X_n(t,x) - X(t,x)|^\alpha \longrightarrow 0. \tag{C.2}$$

By the construction of the integral, $I(X_{n_k})(T,\mathcal{O}) \to I(X)(T,\mathcal{O})$ a.s. for a subsequence $\{n_k\}$. It suffices to show that $I(X_n)(T,\mathcal{O}) = 0$ a.s. on A for all n. But $I(X_n)(T,\mathcal{O}) = I(X_n)(T,\mathcal{O}_n)$ and $\mathcal{O}_n$ is bounded.

Step 2. We show that the proof can be reduced to the case of a bounded processes. For this, let $X_n(t,x) = X(t,x)1_{\{|X(t,x)|\leq n\}}$. Clearly, $X_n \in \mathcal{L}_\alpha$ is bounded and satisfies (C.1) for all n. By the dominated convergence theorem, $[X_n - X]_\alpha \to 0$, and hence $I(X_{n_k})(T,\mathcal{O}) \to I(X)(T,\mathcal{O})$ a.s. for a subsequence $\{n_k\}$. It suffices to show that $I(X_n)(T,\mathcal{O}) = 0$ a.s. on A for all n.

Step 3. We show that the proof can be reduced to the case of bounded continuous processes. Assume that $X \in \mathcal{L}_\alpha$ is bounded and satisfies (C.1). For any $t > 0$ and $x \in \mathbb{R}^d$, we define

$$X_n(t,x) = n^{d+1} \int_{(t-1/n)\vee 0}^t \int_{(x-1/n,x]\cap\mathcal{O}} X(s,y)\,dy\,ds, \tag{C.3}$$

where $(a,b] = \{y \in \mathbb{R}^d; a_i < y_i \leq b_i \text{ for all } i = 1,\ldots,d\}$. Clearly, X_n is bounded and satisfies (C.1). We prove that $X_n \in \mathcal{L}_\alpha$. Since X_n is bounded, $[X_n]_\alpha < \infty$. To prove that X_n is predictable, we consider

$$F(t,x) = \int_0^t \int_{(0,x]\cap\mathcal{O}} X(s,y)\,dy\,ds. \tag{C.4}$$

Since X is predictable, it is progressively measurable; that is, for any $t > 0$, the map $(\omega,s,x) \mapsto X(\omega,s,x)$ from $\Omega \times [0,t] \times \mathbb{R}^d$ to $\mathbb{R}$ is $\mathcal{F}_t \times \mathcal{B}([0,t]) \times \mathcal{B}(\mathbb{R}^d)$-measurable. Hence, $F(t,\cdot)$ is $\mathcal{F}_t \times \mathcal{B}(\mathbb{R}^d)$-measurable for any $t > 0$. Since the map $t \mapsto F(\omega,t,x)$ is left continuous for any $\omega \in \Omega, x \in \mathbb{R}^d$, it follows that F is predictable, being in the class $\mathcal{C}$ defined in Remark 11. Hence, X_n is predictable, being a sum of 2^{d+1} terms involving F.

Since F is continuous in (t,x), X_n is continuous in (t,x). By Lebesgue differentiation theorem in $\mathbb{R}^{d+1}$, $X_n(\omega,t,x) \to X(\omega,t,x)$ for any $\omega \in \Omega$, $t > 0$, and $x \in \mathcal{O}$. By the bounded convergence theorem, $[X_n - X]_\alpha \to 0$. Hence, $I(X_{n_k})(T,\mathcal{O}) \to I(X)(T,\mathcal{O})$ a.s. for a subsequence $\{n_k\}$. It suffices to show that $I(X_n)(T,\mathcal{O}) = 0$ a.s. on A for all n.

Step 4. Assume that $X \in \mathcal{L}_\alpha$ is bounded, continuous, and satisfies (C.1). Let $(U_j^{(n)})_{j=1,\ldots,m_n}$ be a partition of $\mathcal{O}$ in Borel sets with Lebesgue measure smaller than $1/n$. Let $x_j^n \in U_j^{(n)}$ be arbitrary. Define

$$X_n(t,x) = \sum_{k=0}^{n-1} \sum_{j=1}^{m_n} X\left(\frac{kT}{n}, x_j^n\right) 1_{(kT/n,(k+1)T/n]}(t)\, 1_{U_j^{(n)}}(x). \tag{C.5}$$

Since X is continuous in (t,x), $X_n(t,x) \to X(t,x)$. By the bounded convergence theorem, $[X_n - X]_\alpha \to 0$, and hence

$I(X_{n_k})(T,\mathcal{O}) \to I(X)(T,\mathcal{O})$ a.s. for a subsequence $\{n_k\}$. Since on the event A,

$$I(X_n)(T,\mathcal{O}) = \sum_{k=0}^{n-1}\sum_{j=1}^{m_n} X\left(\frac{kT}{n}, x_j^n\right) Z\left(\left(\frac{kT}{n}, \frac{(k+1)T}{n}\right] \times U_j^{(n)}\right) = 0, \tag{C.6}$$

it follows that $I(X)(T,\mathcal{O}) = 0$ a.s. on A. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The author is grateful to Robert Dalang for suggesting this problem. This research is supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

References

[1] E. Giné and M. B. Marcus, "The central limit theorem for stochastic integrals with respect to Lévy processes," *The Annals of Probability*, vol. 11, no. 1, pp. 58–77, 1983.

[2] K. Itô, "Stochastic integral," *Proceedings of the Imperial Academy*, vol. 20, pp. 519–524, 1944.

[3] K. Itô, "On a stochastic integral equation," *Proceedings of the Japan Academy*, vol. 22, no. 2, pp. 32–35, 1946.

[4] G. Kallianpur, J. Xiong, G. Hardy, and S. Ramasubramanian, "The existence and uniqueness of solutions of nuclear space-valued stochastic differential equations driven by Poisson random measures," *Stochastics and Stochastics Reports*, vol. 50, no. 1-2, pp. 85–122, 1994.

[5] S. Albeverio, J. L. Wu, and T. S. Zhang, "Parabolic SPDEs driven by Poisson white noise," *Stochastic Processes and their Applications*, vol. 74, no. 1, pp. 21–36, 1998.

[6] E. S. L. Bié, "Étude d'une EDPS conduite par un bruit poissonnien," *Probability Theory and Related Fields*, vol. 111, no. 2, pp. 287–321, 1998.

[7] D. Applebaum and J. L. Wu, "Stochastic partial differential equations driven by Lévy space-time white noise," *Random Operators and Stochastic Equations*, vol. 8, no. 3, pp. 245–259, 2000.

[8] C. Mueller, "The heat equation with Lévy noise," *Stochastic Processes and Their Applications*, vol. 74, no. 1, pp. 67–82, 1998.

[9] L. Mytnik, "Stochastic partial differential equation driven by stable noise," *Probability Theory and Related Fields*, vol. 123, no. 2, pp. 157–201, 2002.

[10] D. A. Dawson, "Infinitely divisible random measures and superprocesses," in *Stochastic Analysis and Related Topics*, H. Körezlioğlu and A. Üstünel, Eds., vol. 31, pp. 1–129, Birkhäuser Boston, Boston, Mass, USA, 1992.

[11] C. Mueller, L. Mytnik, and A. Stan, "The heat equation with time-independent multiplicative stable Lévy noise," *Stochastic Processes and Their Applications*, vol. 116, no. 1, pp. 70–100, 2006.

[12] S. Peszat and J. Zabczyk, *Stochastic Partial Differential Equations with Lévy Noise*, vol. 113 of *Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, UK, 2007.

[13] S. Peszat and J. Zabczyk, "Stochastic heat and wave equations driven by an impulsive noise," in *Stochastic Partial Differential Equations and Applications VII*, G. Da Prato and L. Tubaro, Eds., vol. 245, pp. 229–242, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2006.

[14] S. Albeverio, V. Mandrekar, and B. Rüdiger, "Existence of mild solutions for stochastic differential equations and semilinear equations with non-Gaussian Lévy noise," *Stochastic Processes and Their Applications*, vol. 119, no. 3, pp. 835–863, 2009.

[15] C. Marinelli and M. Röckner, "Well-posedness and asymptotic behavior for stochastic reaction-diffusion equations with multiplicative Poisson noise," *Electronic Journal of Probability*, vol. 15, pp. 1528–1555, 2010.

[16] E. Priola and J. Zabczyk, "Structural properties of semilinear SPDEs driven by cylindrical stable processes," *Probability Theory and Related Fields*, vol. 149, no. 1-2, pp. 97–137, 2011.

[17] B. Øksendal, "Stochastic partial differential equations driven by multi-parameter white noise of Lévy processes," *Quarterly of Applied Mathematics*, vol. 66, no. 3, pp. 521–537, 2008.

[18] G. Samorodnitsky and M. S. Taqqu, *Stable Non-Gaussian Random Processes*, Chapman & Hall, New York, NY, USA, 1994.

[19] W. Feller, *An Introduction to Probability Theory and Its Applications*, vol. 2, John Wiley & Sons, New York, NY, USA, 2nd edition, 1971.

[20] B. S. Rajput and J. Rosiński, "Spectral representations of infinitely divisible processes," *Probability Theory and Related Fields*, vol. 82, no. 3, pp. 451–487, 1989.

[21] J. B. Walsh, "An introduction to stochastic partial differential equations," in *École d'Été de Probabilités de Saint-Flour XIV*, vol. 1180 of *Lecture Notes in Mathematics*, pp. 265–439, Springer, Berlin, Germany, 1986.

[22] S. I. Resnick, *Heavy-Tail Phenomena: Probabilistic and Statistical Modelling*, Springer Series in Operations Research and Financial Engineering, Springer, New York, NY, USA, 2007.

[23] P. Billingsley, *Probability and Measure*, Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons, New York, NY, USA, 3rd edition, 1995.

[24] E. Giné and M. B. Marcus, "Some results on the domain of attraction of stable measures on $C(K)$," *Probability and Mathematical Statistics*, vol. 2, no. 2, pp. 125–147, 1982.

[25] A. Truman and J. L. Wu, "Fractal Burgers' equation driven by Lévy noise," in *Stochastic Partial Differential Equations and Applications VII*, G. Da Prato and L. Tubaro, Eds., vol. 245, pp. 295–310, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2006.

[26] P. Azerad and M. Mellouk, "On a stochastic partial differential equation with non-local diffusion," *Potential Analysis*, vol. 27, no. 2, pp. 183–197, 2007.

[27] R. C. Dalang, "Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.'s," *Electronic Journal of Probability*, vol. 4, no. 6, 29 pages, 1999, Erratum in *Electronic Journal of Probability*, vol. 6, 5 pages, 2001.

[28] C. Dellacherie and P. A. Meyer, *Probabilités et Potentiel*, vol. 1, Hermann, Paris, France, 1975.

[29] J. P. Nolan, *Stable Distributions: Models for Heavy Tailed Data*, chapter 1, Birkhäauser, Boston, Mass, USA, 2013, http://academic2.american.edu/~jpnolan/stable/chap1.pdf.